P R A Y A S
Students' Journal
of Physics

Dimerized Ising chain: two point correlation

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Abstract. The free energy of an elastic Ising chain, with a dimerizing distortion, in an external magnetic field has been calculated earlier. The free energy exhibits a tricritical point. In this paper, we calculate the spin-spin correlation in this model.

Communicated by: L. Satpathy

1. Introduction

One of the most influential model of a system capable of a phase transition is the Ising model. This was invented by Wilhelm Lentz. He gave his student, Ernst Ising, this model as a problem to solve. For the one dimensioal case, Ising solved it in 1925. The solution of this model in two dimensions is due to Lars Onsager. By solving it exactly, in the absence of external magnetic field, he studied the behaviour of various correlations near the phase transition point. This was in 1944. Unlike the first order Ising model, for the two dimensional one, phase transition occurs at finite temperature. Despite years of intensive effort, an exact solution of the Ising model in three dimensions or in two dimensions with external magnetic field is still lacking. Our focus, in this paper, will be a distorted version of one dimensional Ising model. But before going to it, let us introduce the original Ising model in one dimension.

Consider a one dimensional lattice or chain. We will consider a very long chain and identify the two ends of this chain. At each lattice sites spin variables S_is are sitting which can be up or down. The other way of parametrizing this is to assign $+1$ for spins pointing up and -1 for those pointing down. The subscript i is the index identifying the lattice site at which S_i is sitting. The Hamiltonian of the system is given by

$$H = -J \sum_i J S_i S_{i+1} - h \sum_i S_i, \tag{1}$$

with $J > 0$. Clearly there are only interactions between neighbouring spins and the strength of the interaction is controlled by the coupling J. In the above equation, h represents an external magnetic field. Computing the partition function and the free energy exactly, it can be shown that the sistem shows a critical behaviour near $T = 0$. There are excellent discussions of this model in many introductory text books on statistical mechanics. We refer to the reader the book by Baxter [1]–the calculational techniques of which we will follow throughout.

Though the Ising model in one dimension does not show a phase transition at finite temperature, it is possible to distort the model inducing interesting phase structures. Transitions between them are then controlled by temperature and other parameters of the model. One such model, which will be the focus of this article, is the dimerized Ising chain. Interestingly, this model has a tricritical point in its phase diagram [2]. A simple description of this model was provided in [3]. Our aim in this note is to provide a calculation of the two point correlator of this model. Further, analyzing the behaviour of the correlator, we identify the critical point where correlation length diverges.

We begin by introducing the model following [3].

1.1 The model

The Hamiltonian is given by:

$$H = -J_0 \sum_i (S_i S_{i+1}) - J_1 \epsilon \sum_i ((-1)^i S_i S_{i+1}) - h \sum_i S_i + N\omega_0 \epsilon^2. \tag{2}$$

Here

- N is the number of spin variables. These variables take values $S_i = \pm 1$. The sum is over the chain sites.
- J_0 is the exchange constant.
- J_1 is the first derivative of J_0 with respect to the distance between the spins.
- ω_0 is the frequency of dimerized distortion.
- ϵ is the lattice distortion resulting in long and short bond lengths between adjacent spins. The term $N\omega_0\epsilon^2$ has been introduced to stabilize the model.

The lattice distortion parameter ϵ causes alternating long and short bonds between neighbouring spins. This results in alternating nearest-neighour coupling constants $J_0 \pm \epsilon J_1$. This distortion is known as the dimerizing lattice distortion. By introducing two matrices,

$$F = \begin{pmatrix} e^{\beta(J_0+\epsilon J_1)+\beta h} & e^{-\beta(J_0+\epsilon J_1)} \\ e^{-\beta(J_0+\epsilon J_1)} & e^{\beta(J_0+\epsilon J_1)-\beta h} \end{pmatrix} \tag{3}$$

and

$$G = \begin{pmatrix} e^{\beta(J_0-\epsilon J_1)+\beta h} & e^{-\beta(J_0-\epsilon J_1)} \\ e^{-\beta(J_0-\epsilon J_1)} & e^{\beta(J_0-\epsilon J_1)-\beta h} \end{pmatrix} \tag{4}$$

the partition function

$$Z = \sum_{S_i} e^{-\beta H} \tag{5}$$

can be brought to a form

$$Z = \mathrm{Tr}[(FG)^{N/2}]. \tag{6}$$

Since this has been already discussed in [3], we have been very brief here. The explicit form of FG follows from (3) and (4) and we record it here for later use.

$$FG = \begin{pmatrix} \frac{1}{A} + AB^2 & B\sqrt{C} + \frac{1}{B\sqrt{C}} \\ \frac{B}{\sqrt{C}} + \frac{\sqrt{C}}{B} & \frac{1}{A} + \frac{A}{B^2} \end{pmatrix}. \tag{7}$$

Here we have defined

$$A = e^{2\beta J_0}, \;\; B = e^{\beta h}, \;\; C = e^{4\epsilon\beta J_1}. \tag{8}$$

The eigenvalues of the matrix (7) are

$$\begin{aligned} \lambda_1 &= \Big(\frac{1}{A} + \frac{A}{2B^2} + \frac{AB^2}{2} - \sqrt{\frac{A^2C + 4B^2C + 4B^6C + A^2B^8C + B^4(4 - 2A^2C + 4C^2)}{4B^4C}}\Big) \\ \lambda_2 &= \Big(\frac{1}{A} + \frac{A}{2B^2} + \frac{AB^2}{2} + \sqrt{\frac{A^2C + 4B^2C + 4B^6C + A^2B^8C + B^4(4 - 2A^2C + 4C^2)}{4B^4C}}\Big). \end{aligned} \tag{9}$$

Note that $\lambda_1 < \lambda_2$. The matrix which diagonalizes FG is given by

$$D = \begin{pmatrix} D_1 & D_2 \\ 1 & 1 \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned} D_1 &= -\frac{A\sqrt{C} - AB^4\sqrt{C} + \sqrt{4B^4 + A^2C + 4B^2C - 2A^2B^4C + 4B^6C + A^2B^8C + 4B^4C^2}}{2B(B^2 + C)}, \\ D_2 &= -\frac{A\sqrt{C} - AB^4\sqrt{C} - \sqrt{4B^4 + A^2C + 4B^2C - 2A^2B^4C + 4B^6C + A^2B^8C + 4B^4C^2}}{2B(B^2 + C)}. \end{aligned} \tag{11}$$

The partition function then becomes

$$Z = \lambda_1^{\frac{N}{2}} + \lambda_2^{\frac{N}{2}} \tag{12}$$

Then the free energy per site in the thermodynamic limit is given by

$$F = -\frac{1}{\beta} \lim_{N\to\infty} \ln Z = -\frac{1}{2\beta} \ln\lambda_2. \tag{13}$$

up to some additive ϵ^2 term [3].

In the next section, we present our computation of two point spin-spin correlation in this model. This follows the technique of [1].

2. Calculation of the Two-point correlation function

Let us consider the two point correlation $< S_1 S_3 >$ for example. This is given by

$$< S_1 S_3 >= Z^{-1}\mathrm{Tr}[\mathcal{S}(FG)\mathcal{S}(FG)^{\frac{N}{2}-1}] \tag{14}$$

where we have introduced $\mathcal{S}$ as a two by two matrix

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{15}$$

Therefore, in general, we have (for $j-i$ even, $j>i$, the case where $j-i$ odd will be considered later.)

$$< S_i S_j >= Z^{-1}\mathrm{Tr}[\mathcal{S}(FG)^{\frac{j-i}{2}}\mathcal{S}(FG)^{\frac{N}{2}-\frac{j-i}{2}}] \tag{16}$$

Denoting $j-i=\alpha$ with α even, we finally have

$$< S_i S_j >= Z^{-1}\mathrm{Tr}[\mathcal{S}(FG)^{\frac{\alpha}{2}}\mathcal{S}(FG)^{\frac{N-\alpha}{2}}]. \tag{17}$$

Also it easily follows that the one point function is given by

$$< S_i >= Z^{-1}\mathrm{Tr}[\mathcal{S}(FG)^{\frac{N}{2}}]. \tag{18}$$

2.1 When j-i is even

In this case, we write (17) as

$$\begin{aligned} < S_i S_j >= Z^{-1}\mathrm{Tr}[\quad & DD^{-1}\mathcal{S}D\,\underbrace{D^{-1}FGD...D^{-1}FGD}\,D^{-1}\mathcal{S}D \\ & \underbrace{D^{-1}FGD...D^{-1}FGD}\,D^{-1}], \end{aligned} \tag{19}$$

where the first underbrace contains $\alpha/2$ terms of $D^{-1}FGD$ and the second one contains $(N-\alpha)/2$ terms. To simplify things further, we define S_1, S_2, S_3 and S_4 such that

$$D^{-1}SD = \begin{pmatrix} \mathcal{S}_1 & \mathcal{S}_2 \\ \mathcal{S}_3 & \mathcal{S}_4 \end{pmatrix}. \tag{20}$$

More explicitly,

$$\begin{aligned} \mathcal{S}_1 &= \frac{-A(-1+B^4)\sqrt{C}}{\sqrt{A^2C+4B^2C+4B^6C+A^2B^8C+B^4(4-2A^2C+4C^2)}}, \\ \mathcal{S}_2 &= \frac{-A(-1+B^4)\sqrt{C}-\sqrt{A^2C+4B^2C+4B^6C+A^2B^8C+B^4(4-2A^2C+4C^2)}}{\sqrt{A^2C+4B^2C+4B^6C+A^2B^8C+B^4(4-2A^2C+4C^2)}}, \end{aligned}$$

$$\mathcal{S}_3 = \frac{A(-1+B^4)\sqrt{C} - \sqrt{A^2C+4B^2C+4B^6C+A^2B^8C+B^4(4-2A^2C+4C^2)}}{\sqrt{A^2C+4B^2C+4B^6C+A^2B^8C+B^4(4-2A^2C+4C^2)}},$$

$$\mathcal{S}_4 = \frac{A(-1+B^4)\sqrt{C}}{\sqrt{A^2C+4B^2C+4B^6C+A^2B^8C+B^4(4-2A^2C+4C^2)}}. \quad (21)$$

Then it follows that

$$< S_iS_j >= Z^{-1}\mathrm{Tr}\Big[D^{-1}\mathcal{S}D \begin{pmatrix} \lambda_1^{\frac{\alpha}{2}} & 0 \\ 0 & \lambda_2^{\frac{\alpha}{2}} \end{pmatrix} D^{-1}\mathcal{S}D \begin{pmatrix} \lambda_1^{\frac{N-\alpha}{2}} & 0 \\ 0 & \lambda_2^{\frac{N-\alpha}{2}} \end{pmatrix}\Big]. \quad (22)$$

Now using (20), after carrying out the matrix multiplications we reach at

$$< S_iS_j >= Z^{-1}\mathrm{Tr}\Big[\begin{pmatrix} \mathcal{S}_1^2\lambda_1^{\frac{N}{2}} + \mathcal{S}_2\mathcal{S}_3\lambda_1^{\frac{N-\alpha}{2}}\lambda_2^{\frac{\alpha}{2}} & \mathcal{S}_1\mathcal{S}_2\lambda_1^{\frac{\alpha}{2}}\lambda_2^{\frac{N-\alpha}{2}} + \mathcal{S}_2\mathcal{S}_4\lambda_2^{\frac{N}{2}} \\ \mathcal{S}_1\mathcal{S}_3\lambda_1^{\frac{N}{2}} + \mathcal{S}_3\mathcal{S}_4\lambda_2^{\frac{\alpha}{2}}\lambda_1^{\frac{N-\alpha}{2}} & \mathcal{S}_2\mathcal{S}_3\lambda_1^{\frac{\alpha}{2}}\lambda_2^{\frac{N-\alpha}{2}} + \mathcal{S}_4^2\lambda_2^{\frac{N}{2}} \end{pmatrix}\Big]. \quad (23)$$

Taking the trace, we get,

$$< S_iS_j >= Z^{-1}\mathcal{S}_1^2\lambda_1^{\frac{N}{2}} + \mathcal{S}_2\mathcal{S}_3\lambda_1^{\frac{N-\alpha}{2}}\lambda_2^{\frac{\alpha}{2}} + \mathcal{S}_2\mathcal{S}_3\lambda_1^{\frac{\alpha}{2}}\lambda_2^{\frac{N-\alpha}{2}} + \mathcal{S}_4^2\lambda_2^{\frac{N}{2}}. \quad (24)$$

Now in the thermodynamic limit (with $N \to \infty$), using the fact that $\lambda_2 > \lambda_1$, we get

$$< S_iS_j > = \frac{1}{\lambda_2^{\frac{N}{2}}}\Big[\mathcal{S}_4^2\lambda_2^{\frac{N}{2}} + \mathcal{S}_2\mathcal{S}_3\Big(\frac{\lambda_1}{\lambda_2}\Big)^{\frac{\alpha}{2}}\lambda_2^{\frac{N}{2}}\Big] \quad (25)$$

$$= \mathcal{S}_4^2 + \mathcal{S}_2\mathcal{S}_3\Big(\frac{\lambda_1}{\lambda_2}\Big)^{\frac{\alpha}{2}}. \quad (26)$$

Further, using (18), we therefore have

$$< S_iS_j > - < S_i >< S_j >= \mathcal{S}_2\mathcal{S}_3\Big(\frac{\lambda_1}{\lambda_2}\Big)^{\frac{\alpha}{2}}. \quad (27)$$

2.2 When $j-i$ is odd

In the case when $j-i$ is odd, the correlator takes the form:

$$< S_iS_j >= Z^{-1}\mathrm{Tr}[\quad DD^{-1}\mathcal{S}D\,\underbrace{D^{-1}FGD...D^{-1}FGD}\,D^{-1}F\mathcal{S}GD$$
$$\underbrace{D^{-1}FGD...D^{-1}FGD}\,D^{-1}], \quad (28)$$

where the first underbrace contains $(j-i-1)/2$ terms of $D^{-1}FGD$ and the one in the second underbrace has $(N-(j-i-1)-2)/2$ terms. Here also we will continue to write $j-i$ as α. Now, denoting

$$D^{-1}F\mathcal{S}GD = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \quad (29)$$

we can write the correlator as

$$\begin{aligned} < S_i S_j > &= Z^{-1}\mathrm{Tr}\Big[D^{-1}\mathcal{S}D\begin{pmatrix}\lambda_1^{\frac{\alpha-1}{2}} & 0\\ 0 & \lambda_2^{\frac{\alpha-1}{2}}\end{pmatrix}[D^{-1}F\mathcal{S}GD]\begin{pmatrix}\lambda_1^{\frac{N-\alpha-1}{2}} & 0\\ 0 & \lambda_2^{\frac{N-\alpha-1}{2}}\end{pmatrix}\Big] \\ &= Z^{-1}\mathrm{Tr}\Big[D^{-1}\mathcal{S}D\begin{pmatrix}\lambda_1^{\frac{\alpha-1}{2}} & 0\\ 0 & \lambda_2^{\frac{\alpha-1}{2}}\end{pmatrix}\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}\begin{pmatrix}\lambda_1^{\frac{N-\alpha-1}{2}} & 0\\ 0 & \lambda_2^{\frac{N-\alpha-1}{2}}\end{pmatrix}\Big] \end{aligned} \tag{30}$$

Note that since $\mathcal{S}, F, G$ and D are explicitly known, x_1, x_2, x_3, x_4 are calculable quantities. We can now progress as before to calculate (30). Since the manipulations are simple, we will skip the details. The final result, in the thermodynamic limit, turns out to be:

$$< S_i S_j >= \frac{1}{\lambda_2}\Big[\mathcal{S}_3 x_4 + \Big(\frac{\lambda_1}{\lambda_2}\Big)^{\frac{\alpha-1}{2}}\mathcal{S}_4 x_2\Big], \tag{31}$$

where $\mathcal{S}_3$ and $\mathcal{S}_4$ are given in (21). Further,

$$< S_i S_j > - < S_i >< S_j >= \frac{1}{\lambda_2}\Big(\frac{\lambda_1}{\lambda_2}\Big)^{\frac{\alpha-1}{2}}\mathcal{S}_4 x_2. \tag{32}$$

From (27) and (32), we see that the correlation dies off with the distance between the spins α since $\lambda_2 > \lambda_1$. However, it follows from (9) that if A, B, C satisfy

$$A^2C + 4B^2C + 4B^6C + A^2B^8C + B^4(4 - 2A^2C + 4C^2) = 0, \tag{33}$$

then $\lambda_2 = \lambda_1$. In this special case, correlation no longer falls with the distance (between the spins). This represents the appearence of long range correlation in the system. This critical point is normally described by introducing a *correlation length* ξ as

$$\xi = \frac{1}{\ln(\frac{\lambda_2}{\lambda_1})}. \tag{34}$$

This diverges when $\lambda_2 = \lambda_1$.

3. Discussions

We have shown that the spin-spin correlator for the dimerized Ising chain can be explicitly calculated. From the divergence of the correlation lenth, we have isolated the critical point of the system. Are the higher point correlators calculable? We leave this for future.

Note added:

After completing the work, we were informed by Goutam Tripathy that these results were presented in [4]. We thank him for bringing this paper into our notice.

Acknowledgements

This work is done under the supervision of Sudipta Mukherji, Institute of Physics, Bhubaneswar. We thank Somen Bhattacharjee for his instructive comments on the draft.

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