# Methods for multi-loop computations

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# Lecture II

## Special Functions

- **Introduction:**
  1. Analytic properties of the Scattering amplitude
- **Special Functions 1.** Iterated integrals
  1. From **polylogarithms** to **GHPLs**
  2. Chen Iterated integrals
  3. Trascendentality in repeated integrations
- **Special Functions 2.** Elliptic Functions
  1. How to introduce a concept of trascendentality ?
  2. More questions than answers :-)

# Introduction

- Scattering amplitudes (SA) are **analytic functions** on the complex plane
- Analytical structure of SA is dictated by interplay of:
  1. Number of **independent scales**
  2. **Kinematical constraints**
- This goes into the functions needed to describe the result !
- Let's see what happens with **Vector Boson Pair Production**

Vector boson pair (VBP) production - in **massless QCD**:

- $q\,\bar{q} \to \gamma\,\gamma$
  1. **2** independent scales: $s + t + u = 0$
- $q\,\bar{q} \to Z\,\gamma\,/\,W^{\pm}\,\gamma$
  1. **3** independent scales: $s + t + u = m^2 \to$ with linear kinematics!
- $q\,\bar{q} \to Z\,Z\,/\,W^{\pm}\,W^{\pm}$
  1. **3** independent scales: $s + t + u = 2\,m^2 \to$ with non-linear kinematics!
- $q\,\bar{q} \to Z\,W$
  1. **4** independent scales: $s + t + u = m_Z^2 + m_W^2 \to \ldots$

## **VBP-production** - What determines the complexity?

- $q\,\bar{q} \to \gamma\,\gamma$ **( All MIs computed in $\approx$ 2000 )**
- $q\,\bar{q} \to Z\,\gamma\,/\,W^{\pm}\,\gamma$ **( All MIs computed in $\approx$ 2001 )**
- $q\,\bar{q} \to Z\,Z\,/\,W^{\pm}\,W^{\pm}$ **( Planar MIs computed in 2013 )**
- $q\,\bar{q} \to Z\,W$ **( Planar MIs computed in 2014 )**

note that:

1. "Discovery" of **HPLs** came in 1999
2. Extension to **2d-HPLs** in 2001 $\to$ needed for 1 more scale in $V\,\gamma$ !
3. **12 years** to "add no more scales" $\to$ non-linear kinematics !

Fundamental step in order to complete a **multi-loop** computation:

Understand the analytical properties of functions that express the result!

**Special Functions:**

1. Logarithms
2. Polylogarithms
3. Generalised Harmonic-Polylogarithms (GHPLs)
4. Chen iterated integrals
5. Elliptic functions
6. *Elliptic Polylogarithms (??)*

- Functions needed for VBP-production are the so-called **GHPLs**.
- GHPLs are a special class of **iterated integrals**.
- More scales or more complicated kinematical constraints influence the **analyticity structure** of these iterated integrals.
- As long as they are GHPLs we can "handle them"...
- "Experience" shows that at some point iterated integrals are not enough.
  1. "too many" **internal masses**
  2. "too many" **loops**
  3. "more complicated cut-structure" of **non-planar integrals**

$\rightarrow$ **Elliptic Functions...** very little is known...

But let us go step by step and start with what we can do!

# Special Functions 1.

Iterated integrals (and mainly GHPLs !)

- Many classes of Feynman integrals, once expanded in $(d-4)$, seem to be naturally expressed in terms of iterated integrals $\rightarrow$ (see Lecture 3).

- This is true in particular when there are **no masses** in the loops $\rightarrow$ large range of applicability in **massless QCD**!

- Simplest example of iterated integrals are:
  **Multiple Polylogarithms (MPLs)** or
  **Generalised Harmonic Polylogarithms (GHPLs)**.

## What is an iterated integral ?

Given a *set* of integration kernels $K_j(t)$ we can define:

$$\mathcal{I}(i\,;x) = \int_{x_0}^{x} K_i(t)\, dt\,,$$

$$\mathcal{I}(j,i\,;x) = \int_{x_0}^{x} K_j(t)\, \mathcal{I}(i\,;t)\, dt$$

...

$$\mathcal{I}(i_n,...,i_1\,;x) = \int_{x_0}^{x} K_{i_n}(t)\, \mathcal{I}(i_{n-1},...,i_1\,;t)\, dt$$

This objects appear as the "*natural choice*" to represent solution of Feynman integrals once expanded in $d - 4$.

## Step 1. The Logarithm

The logarithm is a trivial example of iterated integral:

$$\log(x) = \int_1^x \frac{dt}{t}\,, \qquad \log\left(1 - \frac{x}{a}\right) = \int_0^x \frac{dt}{t-a}\,, \quad \forall a \neq 0\,.$$

With the obvious consequence:

$$\frac{d}{dx}\log(x) = \frac{1}{x}\,, \qquad \frac{d}{dx}\log\left(1 - \frac{x}{a}\right) = \frac{1}{x-a}, \quad \forall a \neq 0\,.$$

- Important lesson: differentiating the log we get *something easier!*

## Step 2. The Di-Logarithm (Spence's function)

Already at 1-loop it is clear that logs are not enough.

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \log(1-t) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad \forall x \in \mathbb{C} - [1,\infty).$$

With the obvious consequence:

$$\frac{d}{dx}\mathrm{Li}_2(x) = -\frac{1}{x}\log(1-x).$$

- Important lessons:
  1. Differentiating the $\mathrm{Li}_2$ we get *something easier* $\rightarrow$ the log !
  2. The $\mathrm{Li}_2$ is an iterated integral with kernel $1/t$ !

Soon the idea has been generalised to the so-called **classical polylogarithms**

$$\mathrm{Li}_{n+1}(x) = \int_0^x \frac{dt}{t}\,\mathrm{Li}_n(t)\,, \qquad \forall x \in \mathbb{C} - [1,\infty)$$
$$\mathrm{Li}_1(x) = -\log\left(1-x\right).$$

With the obvious consequence:

$$\frac{d}{dx}\,\mathrm{Li}_n(x) = \frac{1}{x}\mathrm{Li}_{n-1}(x)\,.$$

- Important lesson:
  1. Differentiating the $\mathrm{Li}_n$ we get *something easier* $\rightarrow$ the $\mathrm{Li}_{n-1}$ !
  2. Is something missing??

How are the $\mathrm{Li}_n(x)$ built ?

1. We start with an **integration kernel**:

$$\mathrm{Li}_1(x) = -\log(1-x) = \int_0^x dt\, K(t)\,, \quad \text{with} \quad K(t) = -\frac{1}{t-1}\,.$$

2. We proceed then integrating on a **different** kernel

$$\mathrm{Li}_{n+1}(x) = \int_0^x dt\, \hat{K}(t)\, \mathrm{Li}_n(t)\,, \quad \text{with} \quad \hat{K}(t) = \frac{1}{t}\,.$$

3. What happens **mixing up** the two kernels?

$$K_0(t) = \frac{1}{t}\,, \quad K_1(t) = \frac{1}{t-1}\,.$$

1. It makes sense to “mix” **all 3 possibilities:**

$$K_0(t) = \frac{1}{t}\,, \quad K_{+1}(t) = \frac{1}{t-1}\,, \quad K_{-1}(t) = \frac{1}{t+1}\,.$$

2. And define the following functions:

$$G(0,x) = \log(x) = \int_1^x dt\, K_0(t)\,,$$
$$G(\pm 1,x) = \log(1 \mp x) = \int_0^x dt\, K_{\pm 1}(t)\,.$$

3. And finally

$$G(a,\vec{n},x) = \int_0^x dt\, K_a(t)\, G(\vec{n},t)\,, \quad \text{with} \quad a = \{0,1,-1\}\,.$$

These are the so-called **Harmonic Polylogarithms** (HPLs).

**Generalisation** $\rightarrow$ **Generalised Harmonic Polylogarithms (GHPLs)**

The **GHPLs** are defined allowing for any linear rational factor as Kernel !

1.

$$G(0;x) = \log(x)\,, \qquad G(a;x) = \log\left(1 - \frac{x}{a}\right), \quad \forall a \neq 0\,.$$

2.

$$G(\vec{0}_n;x) = \frac{1}{n!}\log^n(x)\,, \qquad G(a,\vec{n};x) = \int_0^x \frac{dt}{t-a}\,G(\vec{n};t)$$

3. Note that '$a$' can also be a *function of other variables* ...

## Definitions

Given a GHPL $G(\vec{n}; x)$ :

1. $\vec{n}$ is said **index vector**. $G(1, 0, -1, 1; x) \to \vec{n} = (1, 0, -1, 1)$
2. Number of elements of $\vec{n}$ is said **weight** $w$.
   $G(1, 0, -1, 1; x)$ has weight $w = 4$.
3. The weight is often called **degree of transcendentality** of the GHPLs.
   $w = 4 \to$ transcendality 4.
4. Set of all indices is said **Alphabet.**
   Alphabet of HPLs is $\{1, 0, -1\}$

## Important:

1. The index vector contains the **analytical structure** of the GHPLs.
2. The analytical structure of the **S-Matrix** goes into the **index vector**!
3. The more complicated is the **cut structure** the more complicated will be the **indices** of the GHPLs.

Many HPLs can be written as classical Polylogarithms:

$$G(1,1;x) = \frac{1}{2}\log(1-x)^2\,, \quad G(0,1;x) = -\mathrm{Li}_2(x)\,,$$

$$G(0,1,0;x) = 2\,\mathrm{Li}_3(x) - \log(x)\,\mathrm{Li}_2(x)\,, \ldots$$

But obviously not all of them. First examples at **weight 4**:

$$G(-1,0,0,1;x) = \int_0^x \frac{dt}{t+1}\int_0^t \frac{du}{u}\int_0^u \frac{dv}{v}\int_0^v \frac{dw}{w-1}$$

$$= -\int_0^x \frac{dt}{t+1}\,\mathrm{Li}_3(t)\,.$$

All GHPLs up to weight 3 can be **always** written as **classical polylogarithms** !

## Properties of GHPLs:

1. **Shuffle algebra** *(true for iterated integrals)*:

$$G(a;x)G(b,c;x) = G(a,b,c;x) + G(b,a,c;x) + G(b,c,a;x)$$

2. **Scale invariance**:

$$G(a_1,...,a_n;x) = G(\lambda\, a_1,...,\lambda\, a_n;\lambda x)\,, \quad \forall\lambda \in \mathbb{C}\,,\ a_n \neq 0$$

3. **Cut structure**:
   Whenever the variable $x$ becomes **larger** than any of the indices the GHPLs develop an **imaginary part!**

$$G(a;x) = \ln\left(1 - x/a\right)\ \in\ \mathbb{R}\,, \quad \forall x \leq a\,.$$

**Two important values:**

1.
$$\lim_{x\to 0} G(\vec{n};\, x) = 0\,, \qquad \forall \vec{n} \neq \vec{0}_n$$

2.
$$\lim_{x\to a} G(a, \vec{n};\, x) \to \infty\,, \qquad \forall \vec{n} \in \mathbb{C}^n$$

- **HPLs** have been found to be the *right set of functions* to express Feynman integrals depending on **two independent scales**. *(with $x$ some appropriate dimensionless ratio of the two...)*

- This is true *almost independently* on the **number of loops**.

**Examples:**

1. 1-,2-,3- and 4-loop **massive 2-point functions** in *special kinematical configurations*: $\{p^2, m^2\}$ .

2. 1- and 2-loop **QED form-factor**: $\{p^2, m_e^2\}$

3. 1-, 2-, 3-loop **4-point functions** in massless QCD with *on-shell legs*: $\{t, u\}$ with $s = -t - u$ .

4. many others...

In all these cases one can find an appropriate dimensionless combination of the two variables which transforms the result in only HPLs:

- $x = p^2/m^2$
- $x = (\sqrt{p^2 + 4m^2} - \sqrt{p^2})/(\sqrt{p^2 + 4m^2} + \sqrt{p^2})$
- $x = t/u$
- ...

What happens when there are **more independent scales**?

**2d-HPLs** are easiest example of GHPLs,
introduced for dealing with **three-scale** process:

$$\gamma^*(p_4) \to q(p_1) + \bar{q}(p_2) + g(p_3)\,, \qquad s + t + u = p_4^2$$

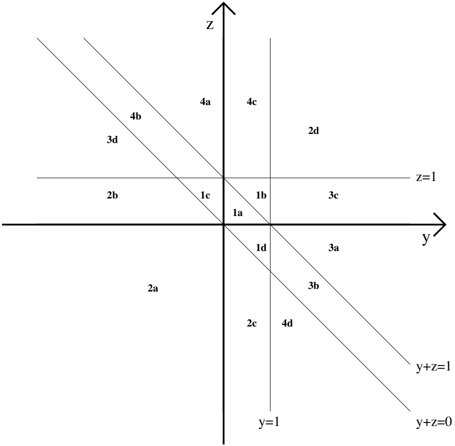
Depends on two dimensionless variables:

$$y = \frac{t}{p_4^2}\,, \qquad z = \frac{u}{p_4^2}$$

1. We need **HPLs** of 1 variables: $G(\{1, 0, -1\}; z)$
2. Plus **2d-HPLs** of the other, with **Alphabet**

$$G(\{1, 0, 1-z, -z\}; y)$$

The indices represent the different **kinematical cuts:** $\gamma^* \to q\bar{q}g$




In this case all cuts are **linear functions!**

Consider now **one more** external mass:

$$q(p_1) + \bar{q}(p_2) \to W(q_1) \, + \, W(q_2)$$

Where

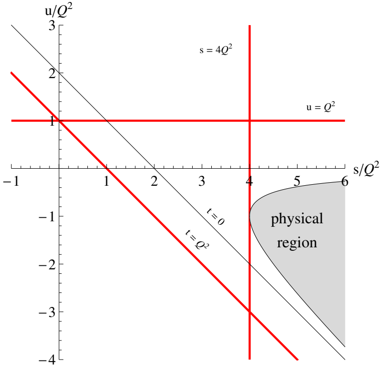
$$p_1^2 = p_2^2 = 0 \, , \quad q_1^2 = q_2^2 = m^2$$

and the **kinematics** is:

$$s = (p_1 + p_2)^2 > 4m^2 \, , \quad t = (p_1 - q_1)^2 < 0 \, , \quad u = (p_2 - q_1)^2 < 0$$

$$s + t + u = 2m^2 \, .$$

The two masses generate a more complicated cut structure
*(even if their value is the same!)*:




Same number of scales but cuts are together **linear** and **non-linear**

## Linearity + non-linearity $\rightarrow$

1. It is not possible to find a set of variables where all cuts are **linear functions**.

2. Parametrizing with

$$s = m^2 \frac{(1+x)^2}{x}\,, \qquad u = -m^2 z\,, \quad \rightarrow \quad 0 < x < 1\,,\;\; x < z < \frac{1}{x}\,.$$

   One can nevertheless write everything in terms of **GHPLs!**

3. **Alphabet** is more complicated:

$$G(\vec{v}\,;x)\,, \quad \text{with} \quad \vec{v} = \left\{0, 1, -1, i, -i, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\right\}\,,$$

$$G(\vec{f}(x)\,;z)\,, \quad \text{with} \quad \vec{f}(x) = \left\{0, -1, x, \frac{1}{x}, \frac{1+x^2}{x}, \frac{1+x+x^2}{x}, \frac{x}{1+x+x^2}\right\}$$

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$$G(\vec{f}(x)\,; z)\,, \quad \text{with} \quad \vec{f}(x) = \left\{0, -1, x, \frac{1}{x}, \frac{1+x^2}{x}, \frac{1+x+x^2}{x}, \frac{x}{1+x+x^2}\right\}$$

- Presence of **non-linear** indices connected with **complex** indices in the other variable.

- Notice that they are solutions of the equations

$$1 + x^2 = 0 \,, \qquad 1 + x + x^2 = 0 \,.$$

- These indices make the **numerical evaluation** of these GHPLs much more complicated.

$$G(x, z) = \ln\left(1 - \frac{z}{x}\right) = \ln\left(\frac{z}{x} - 1\right) \pm i\pi \,, \quad \forall z > x$$

- We will need to take **limits** on these functions!

## "Golden" properties of GHPLs:

- They become easier under differentiation!
- If we differentiate enough times they become a **rational function**!
- Any properties of rational functions are **trivial**!

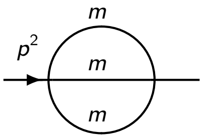
  1. If I know rational functions $\rightarrow$ I know Logs
  2. If I know Logs $\rightarrow$ I know di-Logs
  3. If I know do-logs $\rightarrow$ I know tri-Logs...

- Any property of GHPLs can be proved **by differentiating enough times**

# Special Functions 2.

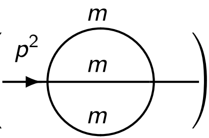
Elliptic functions

GHPLs are not the end of the story !




1. Massive two-loop Sunrise with **equal masses**
2. **two-scales:** $p^2$, $m^2$
3. It should be function of **one variable**, say $z = -p^2/m^2$ .
4. **HPLs** are unfortunately **not enough** !

**Imaginary part** of this graph comes from **Cutkosky-Veltman** rule:




$$\mathrm{Im}\left( \cdots \right) \approx K(w^2) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-w^2x^2)}} \,.$$

where $K(w^2)$ is the **complete elliptic integral of the first kind** and

$$w^2 = \frac{(E+m)^3(E-3m)}{(E-m)^3(E+3m)}\,, \quad E = \sqrt{p^2}$$

*(Exactly true in $d=2$, almost the same in $d=4$...)*

Given the imaginary part we can write a dispersion relation:

$$S(p^2) \approx \int_{s_0}^{\infty} \frac{du}{u - p^2 - i\epsilon} \, \mathrm{Im}(S(u))$$

$$\approx \int_{s_0}^{\infty} \frac{du}{u - p^2 - i\epsilon} K(w^2(u)) \quad \rightarrow \quad ???$$

There are 3 kinds of **complete elliptic integrals**

1. $$K(w^2) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-w^2x^2)}}$$

2. $$E(w^2) = \int_0^1 dx\, \frac{\sqrt{(1-w^2x^2)}}{\sqrt{(1-x^2)}}$$

3. $$\Pi(n; w^2) = \int_0^1 \frac{dx}{(1-n\,x^2)\,\sqrt{(1-x^2)(1-w^2x^2)}}$$

with

$$0 < w^2 < 1\,, \qquad 0 < n < 1\,.$$

It is easy to show that any integral of the form:

$$\mathcal{I}(a_0, a_1, a_2, a_3) = \int_0^1 dx \, \frac{x^{a_0}}{(1 - n\, x^2)^{a_1} \sqrt{(1 - x^2)^{a_2} (1 - w^2 x^2)^{a_3}}} \,,$$

can be written as linear combination of the three **master integrals:**

$$K(w^2)\,, \qquad E(w^2)\,, \qquad \Pi(n; w^2)\,.$$

plus *Elementary Functions*...

Problem with elliptic functions is that
**they do not get easier under differentiation!**

$$\frac{d}{dw^2}K(w^2) = \frac{1}{2w^2}\left[\frac{E(w^2)}{1-w^2} - K(w^2)\right]$$

*differentiating an elliptic function we get again elliptic functions!*

Because of this reason a **iterated-integral** representation is not known...

Still much to do on elliptic functions...