# Methods for multi-loop computations

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# Lecture 3

## Differential Equations for MIs

- **Introduction:**
  1. Back to the MIs
  2. Choice of basis $\rightarrow$ *physical cuts*
- **Differential Equations method (DE):**
  1. Derive the equations
  2. Decoupling
  3. Choice of variables
- **Canonical (Henn-like) differential equations**

# Introduction

- We have seen how a **physical amplitude** can be reduced to **MIs**
- **Two-loop 4-point** functions: from $\approx$ **1000 Ints** $\rightarrow$ $\approx$ **10 MIs**

**Problems remain:**

1. **How do we compute them?**
2. MIs form a **basis**, choice is **not** unique !
3. $\rightarrow$ How to **choose** them?
4. Are there basis choices that are **better than others**?

Different methods have been developed for computing Feynman Integrals:

1. **Feynman Parameters**
2. Mellin Barnes representations
3. **Dispersion relations** *see g-2 of electron at 3 loops !*

4. **Differential Equations (DE).**

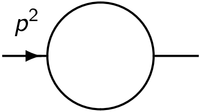
We will focus our attention on Differential Equations!

# **Differential Equations** (for Feynman integrals!)

**Example:**

Let us consider the case of the 1-loop massive Sunrise

$$S(d; p^2, m^2) =$$




$$= \int \mathfrak{D}^d k \frac{1}{(k^2 + m^2)\left((k-p)^2 + m^2\right)}$$

It must be a **scalar function** only of the ratio $p^2/m^2$

we could put for simplicity: $m^2 = 1\,,\;\; p^2 = z$, but it is clearer to keep all variables for the moment

**Idea:**

- Since I know the integral must depend only on $p^2$, can I *"by-pass"* the direct loop-integration?
- In other words, can I write a **dispersion relation** for $S(d;p^2)$ in $p^2$? *(sort of...)*

If I had:

$$\frac{d}{d\,p^2} S(d;p^2) = f(p^2) \qquad \text{plus} \qquad S(d;p_0^2) = N_d\,,$$

then I could write

$$S(d;p^2) = N_d + \int_{p_0^2}^{p^2} dt\, f(t) \quad \rightarrow \quad \text{Bingo!!}$$

Differentiating respect to $p^2$ amounts to differentiating respect to $p^\mu$ !

$$p^2 = p^\mu\, p_\mu \longrightarrow \frac{\partial\, p^2}{\partial\, p_\mu} = 2\, p^\mu\,,$$

$$p_\mu\, \frac{\partial}{\partial\, p_\mu} = p_\mu\, \frac{\partial\, p^2}{\partial\, p_\mu}\, \frac{\partial}{\partial\, p^2} \;\; \longrightarrow \;\; \frac{\partial}{\partial\, p^2} = \frac{1}{2\, p^2}\, \left( p_\mu\, \frac{\partial}{\partial\, p_\mu} \right)$$

This differential operator contains only the external momentum!
I can apply it directly on the **integral representation** of the sunrise !!

$$\frac{d}{d\, p^2} S(d; p^2) = \frac{1}{2\, p^2}\, \left( p_\mu\, \frac{\partial}{\partial\, p_\mu} \right) \int \mathfrak{D}^d k\, \frac{1}{(k^2 + m^2)\,((k-p)^2 + m^2)}$$

Using the fact that in dimensional regularisation integrals are **always convergent** I can act with the operator directly on the integrand!

$$\frac{d}{d\,p^2} S(d;p^2) = \frac{1}{2\,p^2} \int \mathfrak{D}^d k \, \left( p_\mu \, \frac{\partial}{\partial\, p_\mu} \right) \frac{1}{(k^2+m^2)\,((k-p)^2+m^2)}$$

This is **very similar to an IBP !**

1. Acts in the same way as IBPs $\rightarrow$ doesn't change the topology!
2. The result can be reduced to MIs again !!

Performing the derivatives we find

$$\frac{\partial}{\partial p_\mu}\frac{1}{((k-p)^2+m^2)} = -\frac{1}{((k-p)^2+m^2)^2}\,\left[2(p^\mu - k^\mu)\right]$$

so that

$$\left(p_\mu\,\frac{\partial}{\partial p_\mu}\right)\frac{1}{((k-p)^2+m^2)} = \frac{2\,k\cdot p - 2p^2}{((k-p)^2+m^2)^2}$$

and using

$$2\,k\cdot p = (k^2+m^2) - ((k-p)^2+m^2) + p^2$$

we finally find

$$\frac{d}{d\,p^2}\frac{1}{((k-p)^2+m^2)} = \frac{(k^2+m^2)}{((k-p)^2+m^2)^2} - \frac{1}{((k-p)^2+m^2)} - \frac{p^2}{((k-p)^2+m^2)^2}$$

Let us introduce this notation:

$$\mathcal{I}(n_1, n_2) = \int \mathfrak{D}^d k \, \frac{1}{(k^2 + m^2)^{n_1} \, ((k-p)^2 + m^2)^{n_2}} \,, \qquad S(d; p^2) = \mathcal{I}(1,1) \,.$$

Then the **derivative** reads

$$\frac{d}{d\,p^2} \mathcal{I}(1,1) = \frac{1}{2p^2} \, \left( \mathcal{I}(0,2) - \mathcal{I}(1,1) \right) - \frac{1}{2} \, \mathcal{I}(1,2) \,.$$

But now these integrals can be **reduced** to the two **MIs**!! (see Lecture 1)

$$\mathcal{I}(1,0) = T(d; m) \,, \qquad \mathcal{I}(1,1) = S(d; p^2)$$

The **reduction identities** read:

$$\mathcal{I}(0,2) = \mathcal{I}(2,0) = -\frac{(d-2)}{2m^2}\, T(d;m)$$
$$\mathcal{I}(1,2) = -\frac{(d-2)}{2m^2(p^2+4m^2)}\, T(d;m) - \frac{(d-3)}{p^2+4m^2}\, S(d;p^2)$$

with which the **differential equation** becomes:

$$\frac{d}{d\,p^2} S(d;p^2) = \frac{1}{2}\left(\frac{(d-3)}{p^2+4m^2} - \frac{1}{p^2}\right) S(d;p^2) - \frac{(d-2)}{p^2(p^2+4m^2)}\, T(d;m)$$

Linear First Order differential equation for $S(d,p^2)$ !

The homogeneous equation reads

$$\frac{d}{d\,p^2} S_H(d;p^2) = \frac{1}{2}\left(\frac{(d-3)}{p^2+4m^2} - \frac{1}{p^2}\right) S_H(d;p^2)$$

which has solution

$$S_H(d;p^2) = \sqrt{\frac{(p^2+4m^2)^{d-3}}{p^2}}$$

which finally gives for the solution by **quadrature**

$$S(d;p^2) = -(d-2)\,T(d;m^2)\sqrt{\frac{(p^2+4m^2)^{d-3}}{p^2}} \times \int_{p_0^2}^{p^2} d\,t\,\frac{t^{-1/2}}{(t+4m^2)^{(d-1)/2}} \;+\; S(d;p_0^2)$$

This is a *dispersion relation* for the sunrise !

Given an initial condition this equation can be integrated easily!

- Note that

$$S(d;\, p^2 \to 0) \to -\left(\frac{d-2}{2m^2}\right) T(d, m^2)$$

- And the quadrature formula becomes:

$$S(d; p^2) = -(d-2)\, T(d; m^2) \sqrt{\frac{(p^2+4m^2)^{d-3}}{p^2}}$$

$$\times \int_0^{p^2} d\,t\, \frac{t^{-1/2}}{(t+4m^2)^{(d-1)/2}} \;+\; S(d;\, 0)$$

- And rescaling of $4\, m^2$

$$\int_0^{p^2} d\,t\, \frac{t^{-1/2}}{(t+4m^2)^{(d-1)/2}} = (4m^2)^{(2-d)/2} \int_0^{p^2/4m^2} x^{-1/2}\, (x+1)^{(1-d)/2}\, dx$$

Two points still have to be discussed:

1. How did I get the **boundary condition**?
2. What Happens if we expand in $d \to 4$ ?

$\to$ for point 2. see Exercises...

- The **Boundary condition** is given by the value of the integral in a specific kinematical point.
- This is in general easier to compute than the **orginal integral**:

$$S(d;\, p^2) = \int \frac{\mathfrak{D}^d k}{(k^2+m^2)((p-k)^2+m^2)}$$

$$= \int_0^1 dx \int \frac{\mathfrak{D}^d k}{\left[(k^2+m^2)(1-x)+((p-k)^2+m^2)x\right]^2}$$

with the usual algebra

$$= \int_0^1 dx \int \frac{\mathfrak{D}^d k}{\left(k^2+m^2+p^2x(1-x)\right)^2}$$

Now it is trivial to take the limit $p^2 \to 0$

$$S(d;\, p^2 \to 0) = \int_0^1 dx \int \frac{\mathfrak{D}^d k}{\left(k^2 + m^2\right)^2} = \mathcal{I}(2,0)$$

which can be *reduced to MIs* giving:

$$= -\frac{(d-2)}{2\, m^2}\; T(d; m^2)$$

In this case integration in '$dx$' becomes **completely trivial**!

Very often direct computation of the boundary condition is **not needed**!

- Let us go back to differential equation:

$$\frac{d}{d\,p^2} S(d;p^2) = \frac{1}{2}\left(\frac{(d-3)}{p^2+4m^2} - \frac{1}{p^2}\right) S(d;p^2) - \frac{(d-2)}{p^2(p^2+4m^2)}\, T(d;m^2)$$

- There are two **denominators**: $1/p^2$, and $1/(p^2+4m^2)$.

- $1/(p^2+4m^2)$ represents the **threshold** $p^2 \to -4m^2$
  it is a real discontinuity of the function!

- $1/p^2$ is instead a **pseudo-threshold**
  the sunrise must be regular in that point!!

We can use **regularity** in $p^2 \to 0$ in order to **infer** boundary condition:

$$\lim_{p^2 \to 0} \left( p^2 \frac{d}{d\,p^2} S(d;p^2) \right) \to 0$$

$$0 = \lim_{p^2 \to 0} \left[ \frac{1}{2} \left( \frac{(d-3)p^2}{p^2+4m^2} - 1 \right) S(d;p^2) - \frac{(d-2)}{(p^2+4m^2)} T(d;m^2) \right]$$

$$0 = -\frac{1}{2} S(d;0) - \frac{(d-2)}{4m^2} T(d;m^2) \quad \to \quad S(d;0) = -\frac{(d-2)}{2m^2} T(d;m^2)\,.$$

This easy example shows already *(almost)* all **main features** of the differential equation method.

What changes in a general, **multi-loop** case?

Everything works in the exact same way except for one thing:

A general two-(multi-)loop Feynman graph can have **more than 1 MI**!

Quite in general, given some **l-loop** topology, i.e.

$$\mathcal{I}(a_1, ..., a_\sigma;\ b_1, ..., b_t) = \int \prod_{i=1}^{l} \mathfrak{D}^d k_i \, \frac{S_1^{a_1} ... S_\sigma^{a_\sigma}}{D_1^{b_1} ... D_t^{b_t}}$$

1. Identify the **external invariants** '$s_j$' the integrals depend on .
2. Use **IBPs**, **LIs** and **SRs** to reduce it to **<u>N MIs</u>** $M_i(s_j)$, $i = 1, ..., N$ .
3. Express derivatives $d/ds_j$ as combinations of $d/dp_i^\mu$
4. Applying $d/ds_k$ on the masters $M_i(s_j)$ we obtain again a combination of integrals of the form $\mathcal{I}(a_1, ..., a_\sigma;\ b_1, ..., b_t)$
5. **Reduce** the *r.h.s* to **MIs**.

Since the sector contains $N$ MIs we expect to find a linear system of **N coupled first-order differential equations.**

For every external invariant $s_k$ we will have:

$$\frac{\partial}{\partial s_k}\left(\begin{array}{c} M_1(s_j) \\ ... \\ M_N(s_j) \end{array}\right) = \left(\begin{array}{ccc} C_{11}(d,s_i) & ... & C_{1N}(d,s_i) \\ & ... & \\ C_{N1}(d,s_i) & ... & C_{NN}(d,s_i) \end{array}\right)\left(\begin{array}{c} M_1(s_j) \\ ... \\ M_N(s_j) \end{array}\right)$$

$+$ Sub-topologies

Sub-topologies are assumed to be known $\rightarrow$ **bottom-up** approach!

# Euler scaling relation

- Feynman integrals are **homogeneous functions** of the external invariants

$$\mathcal{I}(s_1, ..., s_k) \to \mathcal{I}(\lambda\, s_1, ..., \lambda\, s_k) = \lambda^{\alpha} \mathcal{I}(s_1, ..., s_k)\,.$$

- This means that they satisfy the Euler scaling relations:

$$\left( s_1 \frac{\partial}{\partial s_1} + ... + s_k \frac{\partial}{\partial s_k} \right) \mathcal{I}(s_1, ..., s_k) = \alpha\, \mathcal{I}(s_1, ..., s_k)\,,$$

so that one derivative *is not independent* from the others...

How do we solve a coupled linear system?

A coupled linear system of $N$ equations is equivalent to a $N$-**th order differential equation** for one of the MIs.

This reflects the huge **jump in complexity** that there is going from 1 loop $\longrightarrow$ **2** or **more** loops.

As long as there is 1 MI $\rightarrow$ linear first ord Diff. Eq.
$\rightarrow$ can always be solved **by quadrature!**

$$\frac{d}{dx} f(x) = H(x)\, f(x) + g(x)$$

$$f(x) = F(x) \int_{x_0}^{x} dt\, \frac{g(t)}{F(t)} + f(x_0)$$

where $F(x)$ solves the **homogeneous equation:**

$$\frac{d}{dx} F(x) = H(x)\, F(x) \quad \rightarrow \quad F(x) = \exp\left( \int^{x} H(t)\, dt \right)$$

- If we have a system, the only chance to obtain the same simplicity is to **decouple** (or at least *triangularize*) its matrix..
- Find a new basis of MIs, $m_i(s_j)$, such that

$$\frac{\partial}{\partial s_k}\begin{pmatrix} m_1(s_j) \\ \dots \\ m_N(s_j) \end{pmatrix} = \begin{pmatrix} c_{11}(d,s_i) & c_{12}(d,s_i) & \dots & c_{1N}(d,s_i) \\ 0 & c_{22}(d,s_i) & \dots & c_{2N}(d,s_i) \\ \dots & \dots & \dots & \\ 0 & 0 & \dots & c_{NN}(d,s_i) \end{pmatrix}$$

$$\times \begin{pmatrix} m_1(s_j) \\ \dots \\ m_N(s_j) \end{pmatrix} + \text{Sub-topologies}$$

- Unfortunately it is (almost) **impossible** to achieve this decoupling for generic values of the dimensions $d$.
- On the contrary experience shows that this can be much more easily done in the limit $d \to 4$ (or in general $d \to 2n$ ...)
- Taylor expanding the differential equations:

$$\frac{\partial}{\partial s_k} \begin{pmatrix} m_1(s_j) \\ \ldots \\ m_N(s_j) \end{pmatrix} = \begin{pmatrix} c_{11}(4, s_i) & c_{12}(d, s_i) & \ldots & c_{1N}(4, s_i) \\ 0 & c_{22}(4, s_i) & \ldots & c_{2N}(4, s_i) \\ \ldots & \ldots & \ldots & \\ 0 & 0 & \ldots & c_{NN}(4, s_i) \end{pmatrix}$$

$$\times \begin{pmatrix} m_1(s_j) \\ \ldots \\ m_N(s_j) \end{pmatrix} + \mathcal{O}(d-4)$$

- Choosing the **right basis** for which this happens is matter of luck and maybe some experience...
- If this happens, the equations can be integrated one after the other **by quadrature**.
- Given $N$ **initial conditions** we can then obtain the results in **closed form** order by order in $(d-4)$.
- $\rightarrow$ Expanding in $d \rightarrow 4$ is also **necessary** in order to recover the poly-logarithmic structure of the final result *(if any...)*.

Let us go back to our easy **1-loop example** (put $m = 1$):

$$\frac{d}{d\,p^2} S(d;p^2) = \frac{1}{2}\left(\frac{(d-4)}{p^2+4} + \frac{1}{p^2+4} - \frac{1}{p^2}\right) S(d;p^2) - \frac{(d-2)}{p^2(p^2+4)}\, T(d;1)$$

**Expand** everything in $d \to 4$:

$$(d-2)\, T(d;1) = \frac{1}{(d-4)}$$

$$S(d;p^2) = \frac{1}{d-4}\, S^{(-1)}(4;p^2) + S^{(0)}(4;p^2) + \mathcal{O}(d-4)$$

Note that:

$$T(d;m) = \int \frac{\mathfrak{D}^d k}{k^2+m^2} = \frac{(4\pi)^{(d-4)/2}}{\Gamma\left(3-\frac{d}{2}\right)} \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{k^2+m^2} = \frac{m^{d-2}}{(d-2)(d-4)}\,.$$

Plugging all expansions in and **collecting** order by order in $(d-4)$ we get a **chained set of differential equations:**

$$\frac{d}{d\,p^2} S^{(-1)}(4;p^2) = \frac{1}{2}\left(\frac{1}{p^2+4} - \frac{1}{p^2}\right) S^{(-1)}(4;p^2)$$

$$-\frac{1}{p^2(p^2+4)}$$

$$\frac{d}{d\,p^2} S^{(0)}(4;p^2) = \frac{1}{2}\left(\frac{1}{p^2+4} - \frac{1}{p^2}\right) S^{(0)}(4;p^2)$$

$$+\frac{1}{2}\frac{1}{p^2+4} S^{(-1)}(4;p^2)$$

$+$ higher orders...

They need to be solved one after the order $\rightarrow$ **bottom-up**

1. Homogeneous part is the same **at every order**

$$\frac{d}{d\,p^2} f(p^2) = \frac{1}{2}\left(\frac{1}{p^2+4} - \frac{1}{p^2}\right) f(p^2)$$

2. Solution of homogeneous equation gives the **integration kernel!**
   At order $(-1)$:

$$F(p^2) = F(p_0^2) - f(p^2)\int_{p_0^2}^{p^2} \frac{dt}{f(t)}\,\frac{1}{t(t+4)}$$

3. Problem: solving it we get a **square-root**

$$f(p^2) = \sqrt{\frac{p^2+4}{p^2}} \quad \rightarrow \quad \int_{p_0^2}^{p^2} dt\,\sqrt{\frac{t}{t+4}}\,\frac{1}{t(t+4)}$$

   This doesn't give trivially polylogs !

Change of variable to **Landau variable**

$$t = \frac{(1-x)^2}{x} \quad \to \quad x = \frac{\sqrt{t+4}-\sqrt{t}}{\sqrt{t+4}+\sqrt{t}}, \quad \text{and} \quad dt = -\frac{(1-x^2)}{x^2}\, dx$$

so that finally

$$\int_{p_0^2}^{p^2} dt \sqrt{\frac{t}{t+4}}\, \frac{1}{t(t+4)} \quad \to \quad -\int_{x_0^2}^{x_p^2} \frac{dx}{(1+x)^2} = \frac{1}{1+x}\Big|_{x_0}^{x_p}$$

This suggests that from the beginning we derive differential equations in a new variable $x$ such that

$$p^2 = \frac{(1-x)^2}{x} \quad \to \quad x = \frac{\sqrt{p^2+4}-\sqrt{p^2}}{\sqrt{p^2+4}+\sqrt{p^2}}, \qquad \frac{d}{dx} = -\frac{(1-x^2)}{x^2}\,\frac{d}{dp^2}$$

Differential equation becomes

$$\frac{d}{d\,x}S(d;x) = \left[\left(\frac{1}{1+x}+\frac{1}{1-x}\right)+(d-4)\left(\frac{1}{1+x}-\frac{1}{2\,x}\right)\right]\,S(d;x)$$

$$+\frac{1}{2\,(d-4)}\,\left(\frac{1}{1+x}+\frac{1}{1-x}\right)$$

And expanded order by order in $(d-4)$

$$\frac{d}{d\,x}S^{(-1)}(4;x)=\left(\frac{1}{1+x}+\frac{1}{1-x}\right)S^{(-1)}(4;x)+\frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right)$$

$$\frac{d}{d\,x}S^{(0)}(4;x)=\left(\frac{1}{1+x}+\frac{1}{1-x}\right)S^{(0)}(4;x)+\left(\frac{1}{1+x}-\frac{1}{2\,x}\right)S^{(-1)}(4;x)$$

$+$ higher orders...

$$\frac{d}{d\,x}S^{(n)}(4;x)=\left(\frac{1}{1+x}+\frac{1}{1-x}\right)S^{(n)}(4;x)+\left(\frac{1}{1+x}-\frac{1}{2\,x}\right)S^{(n-1)}(4;x)$$

Now **homogeneous equation** doesn't have any more square-roots

$$\frac{d}{d\,x} f(x) = \left(\frac{1}{1+x} + \frac{1}{1-x}\right) f(x) \quad \to \quad f(x) = \frac{1+x}{1-x}$$

Define then $\forall n\,,\; S^{(n)}(4;x) = f(x)\,M^{(n)}(x)$, new equations become

$$\frac{d}{d\,x} M^{(-1)}(x) = \frac{1}{(1+x)^2}$$

$$\frac{d}{d\,x} M^{(n)}(x) = \left(\frac{1}{1+x} - \frac{1}{2\,x}\right)\, M^{(n-1)}(x)\,,$$

Looking closely it is already clear that these are **HPLs with alphabet** $\{0,-1\}$ !

**Integrating** and **imposing the boundary condition** we get:

$$S^{(-1)}(4;x) = -\frac{1}{2}\,, \qquad S^{(0)}(4;x) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{1-x}\right) G(0,x)$$

$$S^{(1)}(4;x) = -\frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{1-x}\right)\left(\frac{\zeta_2}{2} - G(0,x) - \frac{1}{2}G(0,0,x) + G(-1,0,x)\right)$$

**When do we get generalised poly-logarithms?**

1. We need only **linear rational factors** in the equation
2. Solution of **homogeneous equation** is again only linear rational functions
3. $\rightarrow d \ln(x-a) \approx 1/(x-a)$

Can we be more precise?

## Canonical Form by J. Henn

Suppose we are able to find a **basis of Master Intergrals** such that the system of differential equations takes the following form:

$$\frac{\partial}{\partial s_k}\begin{pmatrix} m_1(s_j) \\ \ldots \\ m_N(s_j) \end{pmatrix} = (d-4)\begin{pmatrix} c_{11}(s_i) & \ldots & c_{1N}(s_i) \\ c_{21}(s_i) & \ldots & c_{2N}(s_i) \\ \ldots & \ldots & \ldots \\ c_{N1}(s_i) & \ldots & c_{NN}(s_i) \end{pmatrix}\begin{pmatrix} m_1(s_j) \\ \ldots \\ m_N(s_j) \end{pmatrix}$$

So that <u>the dependence from the kinematics is **factorised** from $d$</u>.

If now every function $c_{jk}(s_i) = d\ \log a$ <u>they all become obviously poly-logs</u>!

Equation for sunrise is **not in the right form**:

$$\frac{d}{d\,x} S(d;x) = \left[ \left( \frac{1}{1+x} + \frac{1}{1-x} \right) + (d-4) \left( \frac{1}{1+x} - \frac{1}{2\,x} \right) \right] S(d;x)$$

$$+ \frac{1}{2\,(d-4)} \left( \frac{1}{1+x} + \frac{1}{1-x} \right)$$

Write differential equation for new basis:

$$m_1(d;x) = \mathcal{I}(2,0) = \int \frac{\mathfrak{D}^d k}{(k^2+m^2)^2}$$

$$m_2(d;x) = \frac{(1-x)(1+x)}{x} \mathcal{I}(2,1) = \frac{(1-x)(1+x)}{x} \int \frac{\mathfrak{D}^d k}{(k^2+m^2)^2\,((k-p)^2+m^2)}$$

It is trivial using the IBPs...

Differential equations for this basis become:

$$\frac{d}{dx}\begin{pmatrix} m_1(d,x) \\ m_2(d,x) \end{pmatrix} = (d-4)\begin{pmatrix} 0 & 0 \\ \frac{1}{2x} & \left(\frac{1}{1+x}-\frac{1}{2x}\right) \end{pmatrix}\begin{pmatrix} m_1(d,x) \\ m_2(d,x) \end{pmatrix}$$

The second master represents the sunrise, its equation is

$$\frac{d\, m_2(d,x)}{dx} = (d-4)\left[\frac{m_1(d,x)}{2x} + \left(\frac{1}{1+x}-\frac{1}{2x}\right) m_2(d,x)\right]$$

Whose integration is now completely elementary, **once expanded in $d-4$ !**

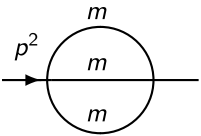
Decoupling in $d \to n$ and direct integration in poly-logarithms

What when they really **don't decouple**, not even in $d = 4$?

<u>Then we are in trouble!</u>

First case when this happens is the **massive two-loop sunrise**, *(see Lecture 2)*.

## Two-loop sunrise with equal masses




- It has two MIs

$$S_1(d;p^2) = \int \frac{\mathfrak{D}^d k \mathfrak{D}^d l}{D_1\, D_2\, D_3}\,, \qquad S_2(d;p^2) = \int \frac{\mathfrak{D}^d k \mathfrak{D}^d l}{D_1^2\, D_2\, D_3}\,.$$

- They respect two **coupled** differential equations ($m = 1\,, p^2 = z$)

$$z \frac{d}{dz} S_1(d;z) = (d-3) S_1(d;z) + 3 S_2(d;z)$$

$$\begin{aligned} z(z+1)(z+9)\frac{d}{dz} S_2(d;z) &= \frac{1}{2}(d-3)(8-3d)(z+3) S_1(d;z) \\ &+ \frac{1}{2}\left[(d-4)z^2 + 10(2-d)z + 9(8-3d)\right] S_2(d;z) \\ &+ \frac{1}{2}(d-2)^2\, z\, T(d)\,. \end{aligned}$$

- There exists no general algorithm to solve a **coupled system**.
- Best thing is usually **rewrite it** as **second order differential equation** for one of the two MIs, and try to solve that one.
- The second order differential equation can be solved only in terms of **Elliptic functions...** $\rightarrow$ ?
- Here still **more questions** than **answers**...

## Something to read...:

- Differential Equations for Feynman Graph Amplitudes, **E. Remiddi**, [hep-th/9711188]
- Differential Equations for Two-Loop Four-Point Functions, **T. Gehrmann, E. Remiddi**, [hep-ph/9912329]
- Feynman Diagrams and Differential Equations, **M. Argeri, P. Mastrolia**, [arXiv:0707.4037]
- Harmonic Polylogarithms, **E. Remiddi, J. Vermaseren**, [hep-ph/9905237]
- From polygons to symbols to polylogarithmic functions, **C. Duhr, H. Gangl, J. Rhodes**, [arXiv:1110.0458]

# Thanks !!