Algebraic formulation of Quantum theory

Particle identity and Entanglement

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February, 2014

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Motivations: Operator algebra and Quantum thoery

- Quantum theory as formulated in conventional framework using statevectors in Hilbert space misses statistical nature of the underlying Quantum physics. Formulation using operators, $\mathcal{C}^*$ algebra and and density matrices appropriately captures this nature and leads to correct formulation of particle identity.

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- Quantum theory as formulated in conventional framework using statevectors in Hilbert space misses statistical nature of the underlying Quantum physics. Formulation using operators, $\mathcal{C}^*$ algebra and and density matrices appropriately captures this nature and leads to correct formulation of particle identity.
- We will explore how Hilbert space and classical configuration space emerges in this framework. We will also examine how particle identity is correctly characterised here. Also we explore how topology and anomalies lead to incoherent mixture of Hilbert spaces.

Introduction

- In studies of foundations of quantum theory, it is of interest to study mixed states and their origins.
- Focus has been on separable states and entropy created by partial tracing.
- But this method is not appropriate for identical particles as we will show.
- A much more universal construction is based on restrictions of states to subalgebras and the GNS construction.
- Based on: Entanglement and Particle Identity: A Unifying Approach A. P. Balachandran, T. R. Govindarajan, Amilcar R. de Queiroz, A. F. Reyes-Lega, **PRL 110**, 080503 (2013), arXiv:1303.0688,1205.2882.
- Algebraic Approach to Entanglement and Entropy, arXiv:1301.1300.

Algebraic Quantum theory

- In algebraic quantum theory we have a state ω and an algebra $\mathcal{A}$.
- The state ω on an observable $\alpha \in \mathcal{A}$ is generally representable in terms of a density matrix ρ_ω and an operator $\pi_\omega(\alpha)$ representing α on a Hilbert space.
- The mean value of the observable α is then

$$\omega(\alpha) = \mathrm{Tr}\left(\rho_\omega \alpha\right) \equiv \mathrm{Tr}(\rho_\omega \pi_\omega(\alpha)) \,.$$

- The state and its density matrix are normalised:

$$\omega(\mathbb{1}) = \mathrm{Tr}\,\rho_\omega = 1.$$

No need for Hilbert space!

- A state ω need not be presented using a density matrix.
- ω is a linear map from $\mathcal{A}$ to $\mathbb{C}$ with the properties

$$\omega(\mathbb{1}) = 1, \quad \omega(\alpha^*) = \overline{\omega(\alpha)}, \quad \omega(\alpha^*\alpha) \geq 0 \qquad \text{for all } \alpha \text{ in } \mathcal{A},$$

  where "$*$" is a hermitean conjugation.
- State vectors and Hilbert spaces play no role at this point.
- Gel'fand, Naimark and Segal described the reconstruction of the Hilbert space $\mathcal{H}_\omega$ from the data $(\mathcal{A}, \omega)$.
- The algebra $\mathcal{A}$ acts by a representation π_ω on $\mathcal{H}_\omega$.

GNS construction

- This reconstruction, known as the GNS construction, has played foundational role in the theory of operator algebras.
- GNS construction is the proper framework for the study of entanglement.
- It presents the state ω as density matrix ρ in terms of orthogonal rank 1 density matrices ρ_i:

$$\rho_i \rho_j = \delta_{ij} \rho_i, \quad \mathrm{Tr}\, \rho_i = 1,$$

- Hence

$$\rho = \sum_i \lambda_i \rho_i, \;\; \omega = \sum_i \lambda_i \omega_i, \;\; \lambda_i > 0, \;\; \sum_i \lambda_i = 1,$$

- The von Neumann entropy for ω is then

$$S(\omega) = -\,\mathrm{Tr}\;\rho \log \rho = -\sum_i \lambda_i \log \lambda_i .$$

- We can associate an entropy to a pair $(\mathcal{A}, \omega)$ of a state and an algebra of observables.

What is Entanglement?

- For a system of *non-identical* constituents A_i with Hilbert spaces $\mathcal{H}_i$, 'entanglement' can be understood in terms of 'partial trace'.
- Consider a bipartite system and the Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2.$$

- Then a density matrix $\rho_{12} = |\psi\rangle\langle\psi|$ with a normalised vector state $|\psi\rangle \in \mathcal{H}$ is entangled when the density matrix

$$\rho_i = \mathrm{Tr}_j\, \rho_{12} \quad (j, i = 1, 2, \quad i \neq j)$$

  obtained by partial tracing has non-zero von Neumann entropy $S(\rho_i)$:

$$S(\rho_i) = -\,\mathrm{Tr}\, \rho_i \log \rho_i \neq 0.$$

- There is no entanglement if $S(\rho_i) = 0$.

Brief Introduction to GNS construction

- A $*$-algebra $\mathcal{A}$ is an associative algebra over $\mathbb{C}$ with a hermitean conjugation:

$$* : \mathcal{A} \to \mathcal{A}, \qquad *^2 \quad = \mathrm{id}.$$

- A C^*-norm $\|\cdot\|$ on such an algebra $\mathcal{A}$ is a norm fulfilling the property

$$\|\alpha^*\alpha\| = \|\alpha\|^2, \quad \forall \alpha \in \mathcal{A}.$$

  If it exists, it is unique.
- The data is : $(\mathcal{A}, \omega)$. We can construct $(\mathcal{H}_\omega, \pi_\omega(\mathcal{A}))$.
- For $\alpha \in \mathcal{A}, \alpha \to |\alpha\rangle$ in a complex vector space $\hat{\mathcal{A}}$ with

$$\begin{aligned} |\lambda\alpha + \mu\beta\rangle \quad &= \lambda|\alpha\rangle + \mu|\beta\rangle, \\ \lambda, \mu \in \mathbb{C} \qquad &; \alpha, \beta \in \mathcal{A}. \end{aligned}$$

  with the inner product using ω with all the usual properties:

$$\langle\beta|\alpha\rangle = \omega(\beta^*\alpha).$$

Hilbert space and scalar product

- It is not yet a scalar product in a Hilbert space, as there may be $|\alpha\rangle$ of zero norm: $\langle\alpha|\alpha\rangle = 0$.
- Let $\mathcal{N}_\omega$ denote the subspace of $\mathcal{A}$ whose image $\hat{\mathcal{N}_\omega} \subset \hat{\mathcal{A}}$ are vectors of zero norm: $\mathcal{N}_\omega = \{\alpha \in \mathcal{A} \mid \langle\alpha|\alpha\rangle = 0\}$.
- Consider the vector space: $\hat{\mathcal{A}}/\hat{\mathcal{N}_\omega} = \{|[a]\rangle := |a + \mathcal{N}_\omega\rangle, \;\; a \in \mathcal{A}\}$.
- Now $\hat{\mathcal{A}}/\hat{\mathcal{N}_\omega}$ has a well-defined scalar product $\langle\cdot|\cdot\rangle$ given by

$$\langle[a]|[b]\rangle = \omega(a^* b).$$

  The vector $|\mathcal{N}_\omega\rangle$ is the only 'zero' vector.
- The Hilbert space $\mathcal{H}_\omega$ is obtained from $\hat{\mathcal{A}}/\hat{\mathcal{N}_\omega}$.
- $\mathcal{H}_\omega$ carries a representation π_ω of $\mathcal{A}$:

$$\pi_\omega(a)|[b]\rangle := |[ab]\rangle.$$

- We have now obtained $(\mathcal{H}_\omega, \pi_\omega(\mathcal{A}))$ from $(\mathcal{A}, \omega)$.

Irreducibility and entropy - 1

- The state ω can be represented as a density matrix ρ_ω,

$$\rho_\omega = |[\mathbb{1}_{\mathcal{A}}]\rangle\langle[\mathbb{1}_{\mathcal{A}}]|.$$

- The representation π_ω may not be irreducible.
- It can be reduced to a direct sum of irreducible representations (IRR's) $\pi_\omega^{(\alpha)}$:

$$\pi_\omega = \oplus_\alpha \pi_\omega^{(\alpha)}.$$

  That is because $\mathcal{A}$ is a $*$-algebra.
- We can decompose ρ_ω into a convex sum of orthogonal rank 1 density matrices:
- We write $|[\mathbb{1}_{\mathcal{A}}]\rangle = \sum_\alpha |[\mathbb{1}_{\mathcal{A}}^{(\alpha)}]\rangle, \quad |[\mathbb{1}_{\mathcal{A}}^{(\alpha)}]\rangle \in \mathcal{H}_\omega^{(\alpha)}$.
- We set $\lambda_\alpha = \langle[\mathbb{1}_{\mathcal{A}}^{(\alpha)}]|[\mathbb{1}_{\mathcal{A}}^{(\alpha)}]\rangle$ and define $|\chi^{(\alpha)}\rangle = \frac{1}{\sqrt{\lambda_\alpha}}|[\mathbb{1}_{\mathcal{A}}^{(\alpha)}]\rangle$

Irreducibility and entropy - 2

- Then ρ_ω in terms of pure states as

$$\rho_\omega = \sum_\alpha \lambda_\alpha \rho_\omega^{(\alpha)}, \quad \lambda_\alpha > 0, \quad \sum_\alpha \lambda_\alpha = 1, \quad \rho_\omega^{(\alpha)} \rho_\omega^{(\beta)} = \delta_{\alpha\beta} \rho_\omega^{(\alpha)}.$$

- The von Neumann entropy of ρ_ω is:

$$S(\rho_\omega) = -\operatorname{Tr} \rho_\omega \log \rho_\omega = -\sum_\alpha \lambda_\alpha \log \lambda_\alpha.$$

- There are important issues related to the uniqueness of the decomposition and hence of the entropy of ω (R. Sorkin)

Example $M_2(\mathcal{C})$

- The choice $\mathcal{A} = M_2(\mathbb{C})$ of 2×2 is a simple example to illustrate the GNS construction.
- $\mathcal{A}$ acts on $\mathbb{C}^2$. Let
$$\{|i\rangle : \ i = 1, 2, \ \ \langle i|j\rangle = \delta_{ij}\} \tag{4.1}$$
be an orthonormal basis. Then the matrix units $e_{ij} = |i\rangle\langle j|$ span $M_2(\mathbb{C})$. Note that $e_{ij}e_{kl} = \delta_{jk}e_{il}$.
- An element α of $\mathcal{A}$ can be expanded as $\alpha = \sum_{i,j} \alpha_{ij}e_{ij}$.
- For the state ω we choose
$$\omega(\alpha) = \lambda\alpha_{11} + (1-\lambda)\alpha_{22}, \quad 0 \leq \lambda \leq 1. \tag{4.2}$$

Example $M_2(\mathcal{C})$

- For our choice for ω we obtain:

$$\omega(\alpha^*\alpha) = \lambda(|\alpha_{11}|^2 + |\alpha_{21}|^2) + (1-\lambda)(|\alpha_{12}|^2 + |\alpha_{22}|^2). \qquad (4.3)$$

  The null space depends on λ. We consider 3 cases.

- **Case 1**: $\lambda = 0$. Null space is:

$$\mathcal{N}_\omega = \left\{ \begin{pmatrix} \alpha_{11} & 0 \\ \alpha_{21} & 0 \end{pmatrix} \;:\; \alpha_{11}, \alpha_{21} \in \mathbb{C} \right\} \cong \mathbb{C}^2. \qquad (4.4)$$

- Since $\hat{\mathcal{A}} \cong \mathbb{C}^4$, we obtain $\mathcal{H}_\omega = \hat{\mathcal{A}}/\hat{\mathcal{N}_\omega} \cong \mathbb{C}^2$, with basis $\{|[e_{k2}]\rangle\}_{k=1,2}$.
- The representation π_ω of $\mathcal{A}$ on $\mathcal{H}_\omega$ is

$$\pi_\omega(e_{ij})|[e_{k2}]\rangle = \delta_{jk}|[e_{i2}]\rangle. \qquad (4.5)$$

  It is irreducible. So we conclude that ρ_ω is a rank 1 projector and has vanishing entropy: $S(\rho_\omega) = 0$.

Example $M_2(\mathcal{C})$

- **Case 2**: $\lambda = 1$. This is similar to the case $\lambda = 0$.
- **Case 3**: $0 < \lambda < 1$. There are no non-zero null vectors in this case: Hence $\mathcal{H}_\omega = \hat{\mathcal{A}}/\hat{\mathcal{N}_\omega} \cong \mathbb{C}^4$,
- The representation π_ω is given by

$$\pi_\omega(e_{ij})|[e_{kl}]\rangle = |[e_{ij}e_{kl}]\rangle = \delta_{jk}|[e_{il}]\rangle \tag{4.6}$$

  This representation is reducible into two two-dimensional irreducible ones: $\mathcal{H}_\omega = \mathbb{C}^2 \oplus \mathbb{C}^2$. They have bases $\{e_{a1}\}_{a=1,2}$ and $\{e_{a2}\}_{a=1,2}$
- We express $|[\mathbb{1}_\mathcal{A}]\rangle$ in terms of its components in these subspaces:

$$|[\mathbb{1}_\mathcal{A}]\rangle = |[e_{11}]\rangle + |[e_{22}]\rangle. \tag{4.7}$$

Example $M_2(\mathcal{C})$

- ω_λ is not pure and can be expressed as the following density matrix:

$$\rho_{\omega_\lambda} = |[e_{11}]\rangle\langle[e_{11}]| + |[e_{22}]\rangle\langle[e_{22}]|. \tag{4.8}$$

- $$\rho_{\omega_\lambda} = \lambda\rho_{11} + (1-\lambda)\rho_{22}, \tag{4.9}$$

  where ρ_{11} and ρ_{22} are the rank 1 density matrices

$$\begin{aligned} \rho_{11} &= \frac{1}{\lambda}|[e_{11}]\rangle\langle[e_{11}]|, \\ \rho_{22} &= \frac{1}{1-\lambda}|[e_{22}]\rangle\langle[e_{22}]|. \end{aligned} \tag{4.10}$$

- We can read off the entropy of ρ_{ω_λ} to be

$$S(\rho_{\omega_\lambda}) = -\lambda\log\lambda - (1-\lambda)\log(1-\lambda). \tag{4.11}$$

Example Simple harmonic oscillator

- Consider annihilation and creation operators $a, a^\dagger$, the vacuum $|0\rangle$ and the state $|n\rangle$ given by: $|n\rangle = \frac{a^\dagger}{\sqrt{n!}}|0\rangle$.
- Consider the mixed state:

$$\omega = \lambda|0\rangle\langle 0| + (1-\lambda)|1\rangle\langle 1|, 0 \leq \lambda \leq 1$$

- The null space is: $\alpha|0\rangle = \alpha|1\rangle = 0$.
- If $P = |0\rangle\langle 0| + |1\rangle\langle 1| = P_0 + P_1$ is the projector to $|0\rangle$ and $|1\rangle$. Then $\mathcal{N} = \mathcal{A}(1-P)$
- We can now complete the GNS construction and find the representation is reducible given by

$$\mathcal{H}_\omega = \mathcal{H}_\omega^{(0)} + \mathcal{H}_\omega^{(1)} = \mathcal{A}[|[|0\rangle]\rangle\langle[\langle 0|]| + |[|1\rangle]\rangle\langle[\langle 1|]]$$

We get the entropy as:

$$S_\omega = -\lambda\log\lambda - (1-\lambda)\log(1-\lambda)$$

Subalgebra and Entanglement

- For a bipartite system of non-identical particles A and B with $\mathcal{H}_A$ and $\mathcal{H}_B$, a vector state $|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, of the form

$$|\psi\rangle = \sum_{i,j} C_{ij} |\chi_{A,i}\rangle \otimes |\eta_{B,j}\rangle,$$

- It is entangled if it cannot be reduced to the form $|\psi\rangle = |\chi'_A\rangle \otimes |\eta'_B\rangle$ by a change of basis.
- A measure of entanglement is the von Neumann entropy of the reduced density matrix

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B} |\psi\rangle\langle\psi|.$$

- The vector $|\psi\rangle$ is entangled if and only if $S(\rho_A) = -\rho_A \log \rho_A \neq 0$.
- The physical meaning of partial trace is it maps a density matrix ρ and a state ω on $\mathcal{A}$ to their restrictions ρ_A, ω_A on $\mathcal{A}_0 \equiv K_A \otimes \mathbb{1}_B$ where K_A is an observable acting on $\mathcal{H}_\mathcal{A}$.

Identical particles

- There are many cases where partial trace cannot be interpreted this way and has no physical meaning. A known example is that of identical fermions.
- An N-particle vector of identical fermions is a linear combination of vectors of the form: $|\psi\rangle = |\psi_1\rangle \wedge |\psi_2\rangle \wedge \ldots \wedge |\psi_N\rangle$. It lives in the N-fold antisymmetric product $\mathcal{H}$ of the one-particle Hilbert space $\mathcal{H}^{(1)}$: $\mathcal{H} = \bigwedge^{N} \mathcal{H}^{(1)}, \quad |\psi\rangle \in \mathcal{H}$.
- The algebra $\mathcal{A}$ of *observables* must necessarily leave $\mathcal{H}$ invariant. The observables must be permutation invariant. An operator $K_1 \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$ is not permutation invariant and not an observable.
- Hence *partial traces do not correspond to restrictions to subalgebras of observables on* $\mathcal{H}$.
- But the restriction of a state ω on $\mathcal{A}$ to a subalgebra $\mathcal{A}_0$ is always sensible. What we need is a criterion to select $\mathcal{A}_0$ appropriately for a physical question.

Identical particles

- We illustrate GNS construction for entanglement entropy in identical particles by identifying subalgebras of one-particle observables through coproduct.
- We consider three examples. The general setting is: one-particle Hilbert space $\mathcal{H}^{(1)} \cong \mathbb{C}^d$. The full one-particle observable algebra is the group algebra of $U(d)$, $\mathbb{C}U(d)$.
- The two-particle Hilbert space is then the subspace of $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$ consisting of either *symmetric* (bosonic statistics) or *antisymmetric* (fermionic statistics) tensors.
- The coproduct is a homomorphism $\Delta : \mathbb{C}U(d) \to \mathbb{C}U(d) \otimes \mathbb{C}U(d)$ that allows us to map one-particle observables to the two-particle sector.
- The map Δ is not fixed *a priori*. The conventional choice $\Delta(g) = g \otimes g$, for $g \in U(d)$.
- The crucial property is *coassociativity*:

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta.$$

Two fermions

- Caution: we consider algebras $\mathcal{A}$ and $\mathcal{A}_0$ with unity.
- We consider measurements where a subset of one-particle observables is considered. For example from among d levels only d' are observed.
- In our example $d = 4, d' = 2$. The 2-fermion space $\Lambda^2\mathcal{H}^1$ is 6 dimensional. This can be seen from decomposition of $4 \otimes 4 = 10 \oplus 6$ into symmetric and antisymmetric tensors.
- Consider an orthonormal basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle, |e_4\rangle\}$ for $\mathcal{H}^1$.
- A basis for 2-fermion space $\Lambda^2\mathcal{H}^1$ is given by $\{|e_i\rangle\Lambda|e_j\rangle\}_{1\leq i<j\leq 4}$.
- We consider *only* one-particle observables containing $\mathbb{1}_4$ and causing transitions between the states $|e_1\rangle$ and $|e_2\rangle$. The relevant algebra of observables is $\mathbb{C}U(2) \otimes \mathbb{1}_4$.
- These observables are generated by operators of the form $M_{ij} = |e_i\rangle\langle e_j|$, with $1 \leq i, j \leq 2$ and $\mathbb{1}_4$.

Two fermions - contd.,

- Basis vectors of $\Lambda^2\mathcal{H}^{(1)}$ are:

$$|a\rangle = |e_1\rangle \wedge |e_2\rangle, |b\rangle = |e_3\rangle \wedge |e_4\rangle,\ |\alpha_1\rangle = |e_1\rangle \wedge |e_3\rangle,$$
$$|\alpha_2\rangle = |e_2\rangle \wedge |e_3\rangle,\ |\beta_1\rangle = |e_1\rangle \wedge |e_4\rangle,\ |\beta_2\rangle = |e_2\rangle \wedge |e_4\rangle$$

- It is easy to obtain the matrix representations of the one-particle observables. As an illustration, we compute:

$$\Delta(M_{12})|\alpha_2\rangle \ = \ \Delta(M_{12})|e_2\rangle \wedge |e_3\rangle = |\alpha_1\rangle$$

- The four matrices $A_{ij} \equiv \Delta(M_{ij})$ (for $i, j = 1, 2$):

$$\begin{aligned} A_{11} &= \text{diag}\{1, e_{11}, e_{11}, 0\},\ \ A_{22} = \text{diag}\{1, e_{22}, e_{22}, 0\}, \\ A_{12} &= \text{diag}\{0, e_{12}, e_{12}, 0\},\ \ A_{21} = \text{diag}\{0, e_{21}, e_{21}, 0\}, \end{aligned}$$

- e_{ij} denote matrix units on $M_2(\mathbb{C})$, i.e., $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. To this must be added the unit matrix $\mathbb{1}_6$.

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Two fermions - contd.,

- Consider a θ-dependent state vector, given by

$$|\psi_\theta\rangle = \cos\theta|\beta_1\rangle + \sin\theta|\alpha_2\rangle.$$

- **Case 1**: $0 < \theta < \frac{\pi}{2}$
- The null space $\mathcal{N}_\theta$ is generated by $B = \text{diag}\{1, 0_2, 0_2, 0\}$ and $\mathbb{1}_6 - A_{11} - A_{22}$.
- The GNS Hilbert space $\mathcal{H}_\theta$ is thus four-dimensional.
- A straightforward computation shows that the subspace spanned by $|[A_{12}]\rangle$ and $|[A_{22}]\rangle$, and by $|[A_{11}]\rangle$ and $|[A_{21}]\rangle$, are irreducible. The two representations are isomorphic.
- From $|[\mathbb{1}_6]\rangle = |[A_{11} + A_{22}]\rangle$ we obtain Projections:

$$P_1|[\mathbb{1}_6]\rangle = |[A_{11}]\rangle, \;\; P_2|[\mathbb{1}_6]\rangle = |[A_{22}]\rangle.$$

- It is easy to see:$\||P_1|[\mathbb{1}_6]\rangle\|^2 = \cos^2\theta, \;\; \|P_2|[\mathbb{1}_6]\rangle\|^2 = \sin^2\theta$. The entropy is:

$$S(\theta) = -\cos^2\theta\log\cos^2\theta - \sin^2\theta\log\sin^2\theta.$$

Two fermions - contd.,

- **Case 2**: $\theta = 0$.
- In this case we have $|\psi_0\rangle = |\beta_1\rangle$.
- The null vectors are given by the four-dimensional space

$$\mathcal{N}_{0,0} = \text{Span}\left\{|B\rangle, |[\mathbb{1}_6 - A_{11}]\rangle, |[A_{22}]\rangle, |[A_{12}]\rangle\right\}.$$

- This means $\mathcal{H}_{\theta=0} \cong \mathbb{C}^2$. Hence, the representation is irreducible so that the corresponding entropy vanishes.
- The situation is completely equivalent for the case $\theta = \dfrac{\pi}{2}$
- Thus, $\mathcal{H}_\theta$ decomposes into irreducible subspaces:

$$\mathcal{H}_\theta \cong \begin{cases} \mathbb{C}^2, & \theta = 0, \pi/2 \\ \mathbb{C}^4 \cong \mathbb{C}^2 \oplus \mathbb{C}^2, & \theta \in (0, \pi/2). \end{cases}$$

The significant aspect: Slater rank of $|\psi_\theta\rangle = 1$ for $\theta = 0, \dfrac{\pi}{2}$, we get exactly zero for the entropy instead of $log 2$ by others.

Topology and anomaly

- A classical symmetry is generated by Q if its Poisson bracket with the Hamiltonian H is zero: $\{Q, H\} = 0$.
- At the quantum level we need a self adjoint Hamiltonian. For unbounded (semi bounded) Hamiltonian we need the domain $\mathcal{D}(H)$ of the operator also.
- The symmetry generator Q may not preserve the domain $\mathcal{D}(H)$. Then Q becomes anomalous.
- J. Esteve Phys Rev **D34** (1986), N S Manton Ann. Phys **159** (1985).
- Explain this by simple model.

Particle on a circle-classical

- The circle is parametrised by ϕ with $0 < \phi \leq 2\pi$.
- The dynamics of a 'free partcle' on a circle is given by:

$$\frac{d^2\phi(t)}{dt^2} = 0$$

- Parity: $P : \phi \longrightarrow -\phi$ and Time reversal: $T : e^{i\phi(t)} \longrightarrow e^{-i\phi(-t)}$
- Both P and T are symmetries of the classical equations of motion.

Particle on a circle

- The Hilbert space $\mathcal{H} = \mathcal{L}^2(S^1, d\phi)$ with inner product:

$$\langle \chi, \psi \rangle \;=\; \int_0^{2\pi} \; d\phi \; \bar{\chi}\psi$$

The essentially self adjoint Hamiltonian is parametrised by $e^{i\theta}$

$$\begin{aligned} H &= -\frac{d^2}{d\phi^2}, \\ \mathcal{D}_\theta &= \{\psi \in \mathcal{H}, \quad \psi(\phi + 2\pi) = e^{i\theta}\psi(\phi)\} \end{aligned}$$

along with differentiability conditions.

- Now Parity acting some state is: $P\psi(\phi) \;=\; \psi(-\phi)$. But in $\mathcal{D}_\theta$ we have:

$$P\mathcal{D}_\theta = \mathcal{D}_{-\theta}$$

- Hence P is anomalous unless $\theta = -\theta \;\; modulo \;\; 2\pi, \;\; i.e \;\; \theta = 0, \;\; \pi$.
- Similarly T is also anomalous. Note interestingly PT is not anomalous.

Restoring P and T

- For $\psi \in \mathcal{D}_\theta$ consider the mixed states:

$$\Omega = |\psi\rangle\langle\psi| + P\,|\psi\rangle\langle\psi|\,P,$$

- $Tr\,\Omega > 0$, Hence $\omega = \frac{\Omega}{Tr\Omega}$, is a well defined state on all observables.
- These are invaraiant under P and T.
- If $\mathcal{O}$ is an observable, Its mean value on the state ω is given by:

$$\omega(\mathcal{O}) = Tr(\mathcal{O}\omega) = \frac{1}{Tr\Omega}\left[\langle\psi|\mathcal{O}|\psi\rangle + \langle\psi|P\mathcal{O}P|\psi\rangle\right]$$

- Observe $\omega(\mathcal{O}) = 0$ for P odd $\mathcal{O}$, that is if: $P\mathcal{O}P = -\mathcal{O}$.

Formalism - Brief

- There is a formal way of arriving at this within Algebraic QFT. This is achieved by considering subalgebra and Gelfand-Naimark-Segal construction.
- APB,TRG,A Reyes-Lega,AQ - Phys.Rev.Lett.110,080503(2013).
- Given an algebra of observables $\mathcal{A}$ which is:

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_+ + \mathcal{A}_- \\ \mathcal{A}_\pm &= a_\pm, \quad Pa_\pm P = \pm a_\pm \end{aligned}$$

- Parity even subalgebra is :

$$\mathcal{A}_0 = \mathcal{A}_+ \oplus \mathcal{C}\mathbb{1}_-$$

where $\mathbb{1}_- = \mathbb{1} - \mathbb{1}_+$ and $\mathbb{1}_+$ satisfies:

$$\mathbb{1}_+ a_+ = a_+ \mathbb{1}_+ = a_+, \quad \mathbb{1}_+ a_- = a_- \mathbb{1}_+ = 0$$

Restriction to subalgebra

- We can consider a pure state ω_Θ and reduce it to parity even subalgebra $\mathcal{A}_0$ following GNS construction.
- $P\omega_\Theta P = \omega_{-\Theta}$. Now $\omega_\Theta = \omega_\Theta^+ + \omega_\Theta^-$. Here $\omega_\Theta^\pm$ acts on $\mathcal{A}_\pm$.
- Restriction for $a = a_+ + \lambda P_- \in \mathcal{A}_0$, with $\lambda \in \mathcal{C}$.
- This restriction will automatically produce:

$$\omega_\Theta|_{\mathcal{A}_0} = \frac{\omega_\Theta + \omega_{-\Theta}}{2}$$

  thereby restoring Parity. Similarly for Time reversal.
- For details of subalgebra reduction and GNS construction: see APB,TRG,A Reyes-Lega,AQ arXiv:1301.1300; 1205.2882

Electric dipole moment of neutron

- A non-zero electric dipole moment (EDM) of a nucleon implies parity (P-) and time-reversal (T-) violations. In the conventional approach to QCD, its θ-term in the action, which violates P and T, induces an electric dipole moment d_N of the neutron.
- The current experimental bound on d_N is

$$|d_N| < 6 \cdot 10^{-26} \text{ cm } (90\% \text{ confidence level}), \tag{8.1}$$

- This implies $|\theta| \lesssim 10^{-10}$ radians, Crewther, R. J., Di Vecchia, P., Veneziano, G., Witten, E., Phys. Lett., 1979, B88, 123
- From a novel effective Lagrangian approach for the pseudo scalar meson η' we get

$$d_N = \frac{e^2}{4\pi^2} D(M_N) \sin\Theta \tag{8.2}$$

where $D(M_N)$ is a finite even fuction of the mass of the nucleon mass (APB, TRG, AQ, **JHEP 05**(2012)012.
- We propose a mechanism for the vanishing of d_N via use of mixed states for any Θ.

Anomalies and restrictions

- We saw mixed states emerge from restrictions of pure states ω on an algebra $\mathcal{A}$ to a subalgebra $\mathcal{A}_0$.
- In a series of recent papers mixed states were introduced to eliminate anomalies.
  Mixed States from Anomalies A.P. Balachandran, Amilcar R. de Queiroz, Phys.Rev. D85 (2012) 025017; Electric Dipole Moment from QCD and How It Vanishes for Mixed States A.P. Balachandran, T.R. Govindarajan,Amilcar R. de Queiroz Eur.Phys.J.Plus 127 (2012) 118. It was proposed that anomalies can be eliminated by averaging a pure state ω over the anomalous group.
- We will mention that the averaged state case can also be regarded as the restriction of ω to a subalgebra.
- This happens because: *restriction and averaging give same answer*.

Time evolution and positive maps

- If a unitary time evolution $U(t)$ of a pure state ω on an algebra $\mathcal{A}$ is given, then its restriction $\omega|_{\mathcal{A}_0} = \omega_0$ is determined by

$$\omega_0 \to \ \omega_0(t) = [U(t)\,\omega]\,|_{\mathcal{A}_0}, \quad \omega_0(0) = \omega_0.$$

  The evolution of ω_0 is in general by positive maps: The Stinespring-Choi theorem.
- Even when $U(t)$ gives a unitary evolution on ω with Hamiltonian H:

$$U(t)\,\omega = e^{iHt}\,\omega\,e^{-iHt}.$$

- Note that *the rank of $\omega_0(t)$ need not be continuous in t even if that of $U(t)\,\omega$ is continuous in t. It can change discontinuously*.
- The case of a fermion with 3 internal degrees is a simple example. The single particle Hilbert space $\mathcal{H}^{(1)}$ was $\mathbb{C}^3$ with basis $\{|e_i\rangle\}_{i=1,2,3}$. Two-particle space was $\bar{3} = \bigwedge^2 \mathcal{H}^{(1)} \equiv \mathcal{H}^{(2)}$, with basis $\{|f^i\rangle = \varepsilon^{ijk}|e_j \wedge e_k\rangle\}_{i=1,2,3}$

Time evolution and positive maps - contd

- We chose the pure state

$$\omega_\theta = |\psi_\theta\rangle\langle\psi_\theta|, \quad |\psi_\theta\rangle = \cos\theta|f^1\rangle + \sin\theta|f^2\rangle,$$

  in the two-particle sector.
- The subalgebra $\mathcal{A}_0$ was the image under the coproduct of the algebra on $\mathcal{H}^{(1)}$ acting on $|e_1\rangle$ and $|e_2\rangle$.
- Our results pertinent for the discussion of time evolution were $\omega_{\theta,0} = \cos^2\theta\ \rho_\theta^{(1)} + \sin^2\theta\ \rho_\theta^{(3)}$, where

$$\rho_\theta^{(1)} = \frac{1}{\cos^2\theta}|[M^{11}]\rangle\langle[M^{11}]|, \rho_\theta^{(3)} = \frac{1}{\sin^2\theta}|[E^3]\rangle\langle[M^3]|$$

  and H_θ^{GNS}:

$$\mathcal{H}_\theta^{\text{GNS}} = \begin{cases} \mathbb{C}^2, & \theta = 0, \\ \mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}, & 0 < \theta < \pi/2, \\ \mathbb{C}, & \theta = \pi/2. \end{cases}$$

Time evolution and positive maps - contd

- Thus the rank of $\omega_{\theta,0} = \omega_\theta|_{\mathcal{A}_0}$ jumps from 2 to 1 as θ approaches 0 or $\pi/2$.
- Consider the unitary evolution $|\psi_\theta\rangle$ and ω_θ through Hamiltonian

$$H = -i|f^2\rangle\langle f^1| + i|f^1\rangle\langle f^2|.$$

  generating rotations:

$$e^{itH}|\psi_\theta\rangle = \cos(\theta+t)|f^1\rangle + \sin(\theta+t)|f^2\rangle.$$

- The restriction of this to $\omega_{\theta,0}$ is $U(t) : \omega_{\theta,0} \to \omega_{\theta+t,0}$. It is not unitary. It does not preserve the rank of $\omega_{\theta,0}$: jumps from 2 to 1 and back as t increases.
- We can write time evolution as positive maps so long as the rank of the density matrix stays constant or decreases.
- Positive maps cannot increase the rank of a state. Hence we cannot write evolution starting from $\theta = 0$ or $\pi/2$ in terms of positive maps.

Conclusions

- There is a natural formulation of quantum physics dispensing with the use of Hilbert space as initial data which is well-adapted to the study of entanglement and entropy.
- Hilbert space is an emergent concept.
- A state ω on an algebra $\mathcal{A}$ can be restricted to a subalgebra $\mathcal{A}_0$. The new state $\omega|_{\mathcal{A}_0}$ may not be pure even if ω is. Its entropy is a measure of entanglement of $\mathcal{A}_0$ with $\mathcal{A}$.
- This new approach to entanglement lets us treat identical particles obeying Bose, Fermi or even braid statistics with ease. Particle identity has posed severe problems in conventional approaches.
- We have also shown how time evolution by positive maps for $\omega|_{\mathcal{A}_0}$ emerges when ω evolves unitarily.
- We have considered quantum anomalies and their elimination by restricting states to subalgebras. In this manner, we can understand the use of mixed states to eliminate anomalies