Local realism, macrorealism and non-contextuality: Unified approach

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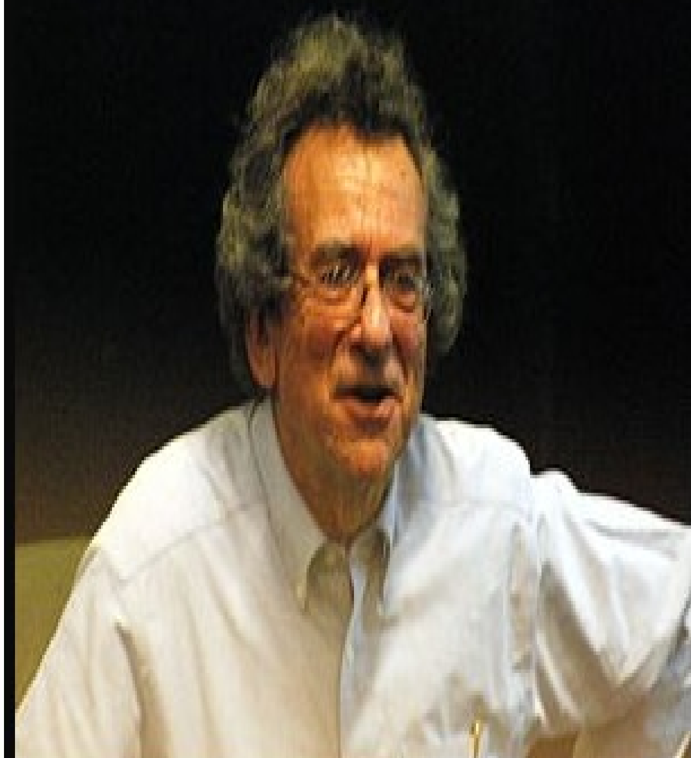

If I were forced to sum up in one sentence what the Copenhagen interpretation says to me, it would be 'Shut up and calculate!

(David Mermin)

izquotes.com

"Different generations of physicists differed in the degree to which they thought that the interpretation of quantum mechanics remains a serious problem! I declared myself to be among those who feel uncomfortable when asked to articulate what we really think about the quantum theory, adding that, If I were forced to sum up in one sentence what the Copenhagen interpretation says to me, it would be *Shut up and calculate!*"

"..my professors – whom I viewed as agents of Copenhagen – when I was first learning quantum mechanics as a graduate student at Harvard, a mere 30 years after the birth of the subject said '*You'll never get a PhD if you allow yourself to be distracted by such frivolities,*' they kept advising me, '*so get back to serious business and produce some results.*' 'Shut up,' in other words, 'and calculate.' And so I did ...."

– David Mermin

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A snapshot of foundational attitudes toward quantum mechanics

Maximilian Schlosshauer [a,*], Johannes Kofler [b], Anton Zeilinger [c,d]

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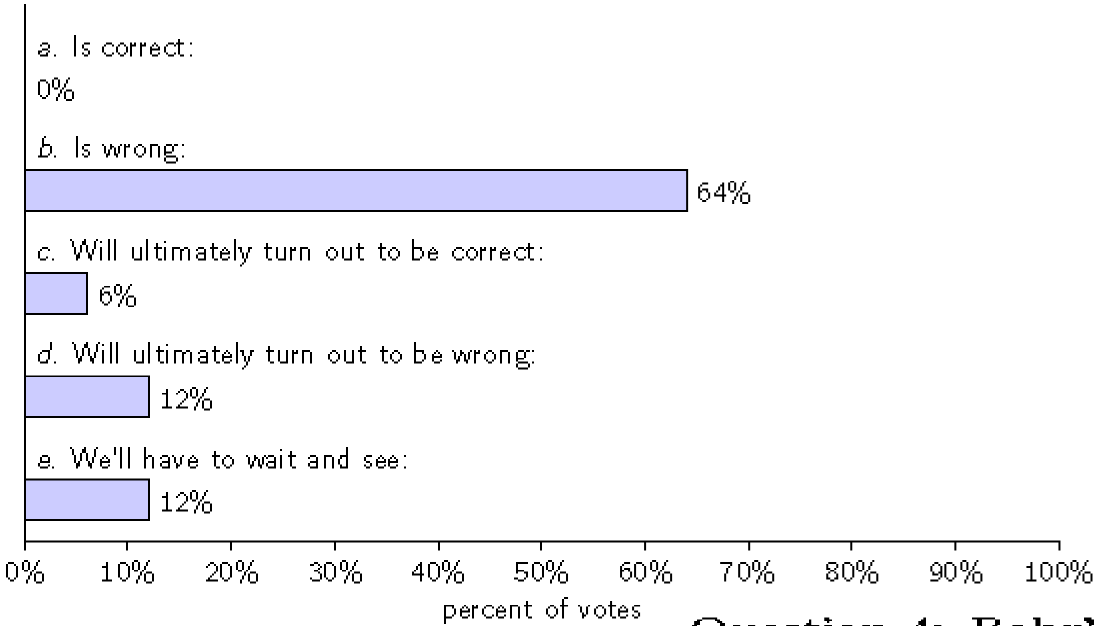
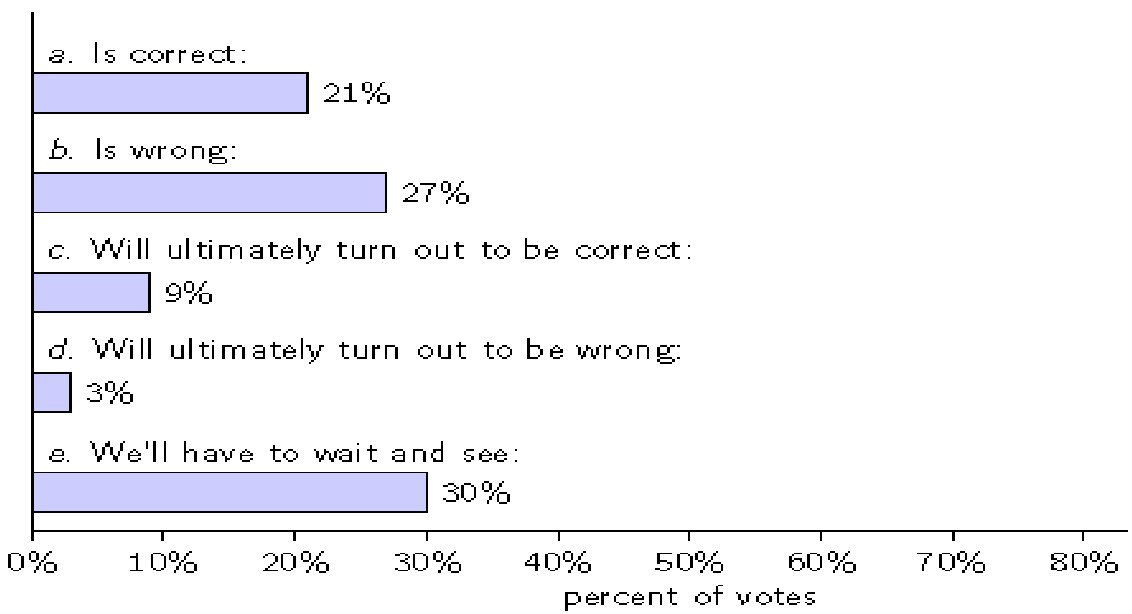
A survey probing respondents' views on various foundational issues in quantum mechanics was recently created by Schlosshauer, Kofler, and Zeilinger and then given to 33 participants at a quantum foundations conference. The participants completed a questionnaire containing 16 multiple-choice questions probing opinions on quantum- foundational issues. Participants included physicists, philosophers, and mathematicians.

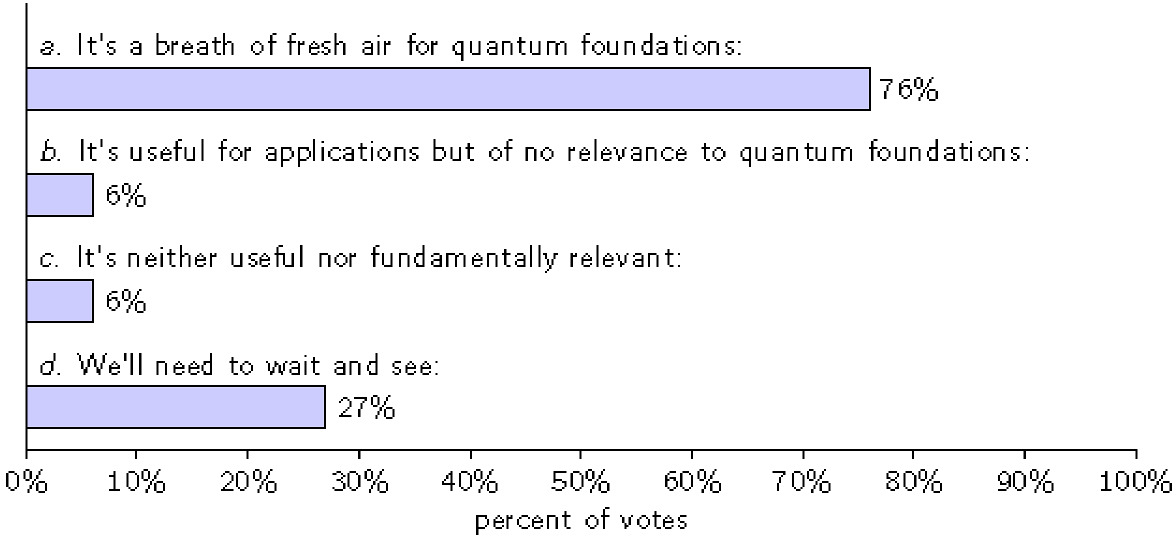
Question 3: Einstein's view of quantum mechanics



Question 4: Bohr's view of quantum mechanics

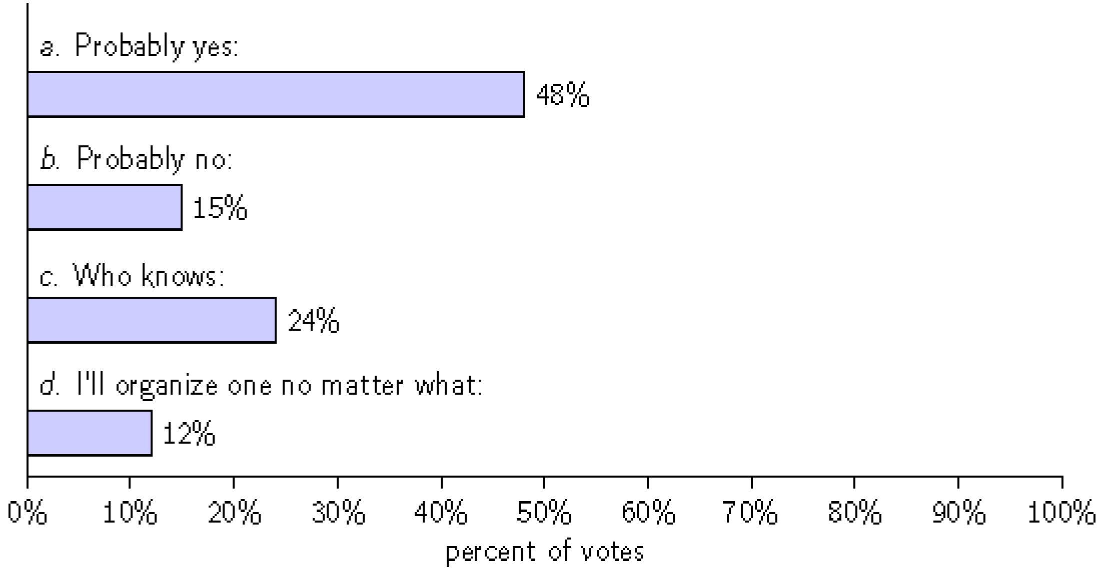


Question 7: What about quantum information?



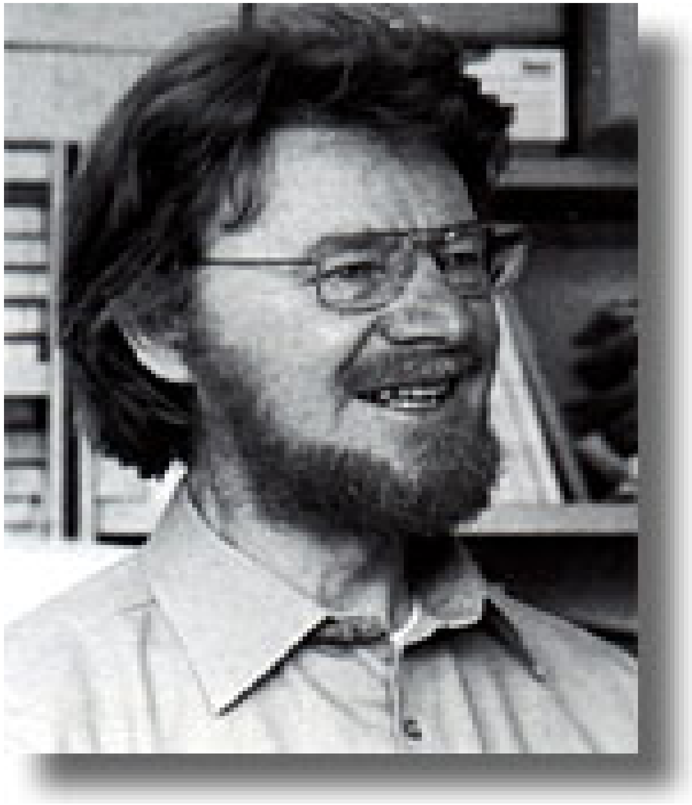
Evidently, there is broad enthusiasm—or at least open-mindedness—about quantum information, with three in four respondents regarding quantum information as "a breath of fresh air for quantum foundations." Indeed, it it hard to deny the impact quantum information theory has had on the field of quantum foundations over the past decade. It has inspired new ways of thinking about quantum theory and has produced information-theoretic derivations (reconstructions) of the structure of the theory. On the practical side, the quantum-information boom has helped fund numerous foundational research projects. Last but not least, quantum information has given foundational pursuits new legitimacy.

Question 16: In 50 years, will we still have conferences devoted to quantum foundations?



Should those who answered "probably yes" be proven right, then it would be fascinating to conduct another such poll 50 years from now. Notable write-ins included "I won't be here," and "I hope not."

Local realism and Bell's Inequality

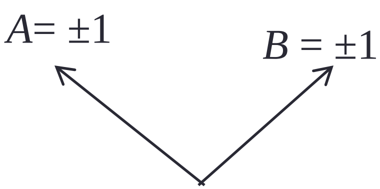


John Bell

Predictions of quantum mechanics cannot be squared with the belief, called local realism that physical systems have realistic properties whose pre-existing values are revealed by measurements. The predictions of quantum mechanics for spatially separated systems are at odds with any version of local realism

Local Realism

- Realism is a worldview according to which external reality is assumed to exist and have definite properties, whether or not they are observed by someone.
- Locality demands that ”if two measurements are made at places remote from one another the setting of one measurement device does not influence the result obtained with the other.”
- Joint assumption local realism (LR) :



$$\text{LR:} \quad P(A, B|a, b) = \sum\nolimits_{\lambda} \rho(\lambda)\, P(A|a, \lambda)\, P(B|b, \lambda)$$

- *Local realism restricts correlations in the form of Bell’s inequality (BI)*

J. S. Bell, Physics 1, 195 (1964).

Bell's Inequality

CHSH version of Bell's Inequality:

$$|C(A,B)-C(A,B')|+|C(A',B)+C(A',B')| \leq 2$$

where

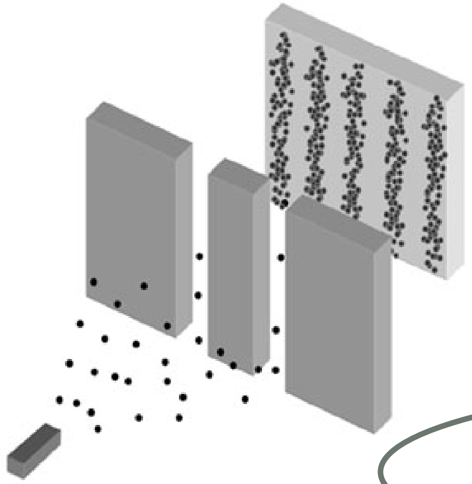
$C(A,B)= \int \rho(\lambda)A(a,\lambda)B(b,\lambda)d\lambda,$

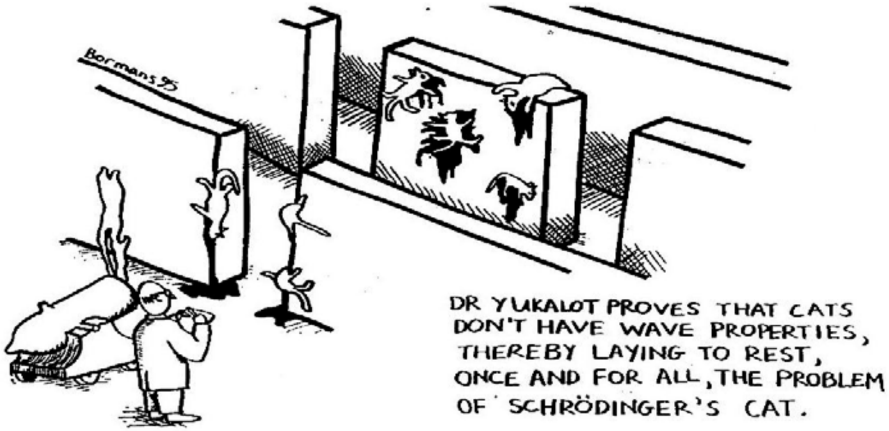
$A(a, \lambda)= \int \rho(\lambda)P(a|\lambda), B(b, \lambda)= \int \rho(\lambda) B(b|\lambda),$

is the correlation in the outcomes $A=\pm 1$, $B=\pm 1$ *of the observables a, b on two spatially separated systems.*

Macro-realism

When and how do physical systems stop behaving quantumly and begin to behave classically? How to distinguish quantum and classical behavior in a testable way?



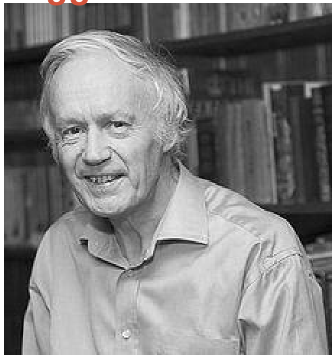


In the macroscopic realm
do superpositions survive?

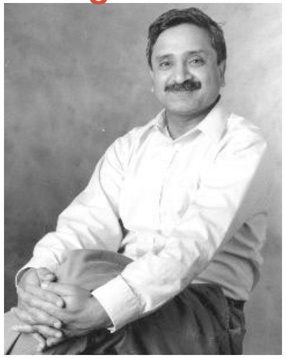
Leggett-Garg (1985)

A. J. Leggett and A. Garg, PRL **54**, 857 (1985)

Sir Anthony James Leggett



Prof. Anupam Garg



Macrorealism

Macrorealism per se

``Physical properties of a macroscopic object exist independent of the act of observation"

Non-invasive measurability

"The measurement of an observable at any instant of time does not influence its subsequent evolution"

Leggett-Garg Correlation Inequality
(Temporal Bell inequality)

Consider a *dynamic* system with a *dichotomic* quantity Q(t)

Dichotomic ⟹ Q(t) = ± 1 at any given time

t=0 $\quad t_1 \quad t_2 \quad t_3 \quad \ldots \quad t_k \quad \ldots \quad t_n$

$\longrightarrow$ t

$Q_1 \quad Q_2 \quad Q_3 \quad \ldots \quad Q_k \quad \ldots \quad Q_n$

$$Q_k \equiv Q(t_k)$$

A. J. Leggett and A. Garg, PRL 54, 857 (1985)

PhD Thesis, Johannes Kofler, 2004

Two-Time Correlation Coefficient

Temporal Correlation: $\Longrightarrow$ $C_{ij} = \langle Q(t_i)Q(t_j)\rangle \equiv \langle Q_i Q_j\rangle$

$$\left.\begin{array}{rcl} C_{ij} & = & +1 \longrightarrow \text{perfect correlation} \\ C_{ij} & = & -1 \longrightarrow \text{perfect anticorrelation} \\ C_{ij} & = & 0 \longrightarrow \text{No correlation} \end{array}\right\} \Longrightarrow -1 \leq C_{ij} \leq +1$$

LG correlation inequality with 3 measurements

Define

$$\begin{aligned} K_3 &= C_{12} + C_{23} - C_{13} \\ &= \langle Q_1 Q_2 \rangle + \langle Q_2 Q_3 \rangle - \langle Q_1 Q_3 \rangle \end{aligned}$$

Notice that

When $Q_1 = Q_2$, $\quad Q_1 Q_2 + (Q_2 - Q_1) Q_3 = +1$

When $Q_1 \neq Q_2$, $\quad Q_1 Q_2 + (Q_2 - Q_1) Q_3 = -1 + (\pm 2) = +1 \text{ or } -3$

$$\Downarrow$$

$$-3 \leq \langle Q_1 Q_2 \rangle + \langle Q_2 Q_3 \rangle - \langle Q_1 Q_3 \rangle \leq 1$$

$$\Downarrow$$

Leggett-Garg Inequality (LGI)

$$-3 \leq K_3 \leq 1$$

LG correlation inequality with 4 measurements

Define
$$\begin{aligned} K_4 &= C_{12} + C_{23} + C_{34} - C_{14} \\ &= \langle Q_1 Q_2 \rangle + \langle Q_2 Q_3 \rangle + \langle Q_3 Q_4 \rangle - \langle Q_1 Q_4 \rangle \end{aligned}$$

$$\text{When } Q_2 = Q_4, \qquad Q_1(Q_2 - Q_4) + Q_3(Q_2 + Q_4) = 0 + (\pm 2) = \pm 2$$
$$\text{When } Q_2 \neq Q_4, \qquad Q_1(Q_2 - Q_4) + Q_3(Q_2 + Q_4) = \pm 2 + 0 = \pm 2$$

$$\Downarrow$$

$$-2 \leq \langle Q_1 Q_2 \rangle + \langle Q_2 Q_3 \rangle + \langle Q_3 Q_4 \rangle - \langle Q_1 Q_4 \rangle \leq 2$$

$$\Downarrow$$

LG correlation inequality

$$-2 \leq K_4 \leq 2$$

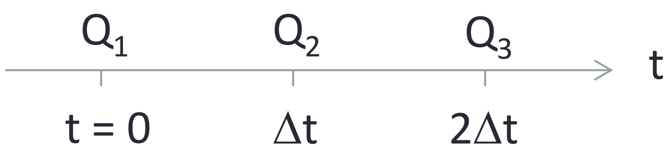
LGI with 3 measurements for a spin ½ particle

A spin ½ particle precessing about y axis

Hamiltonian : $H = \frac{1}{2}\,\omega\sigma_y$

Initial State : highly mixed state : $\rho_0 = \frac{1}{2}\,\mathbb{1}$

Dichotomic observable: σ_z → eigenvalues ± 1

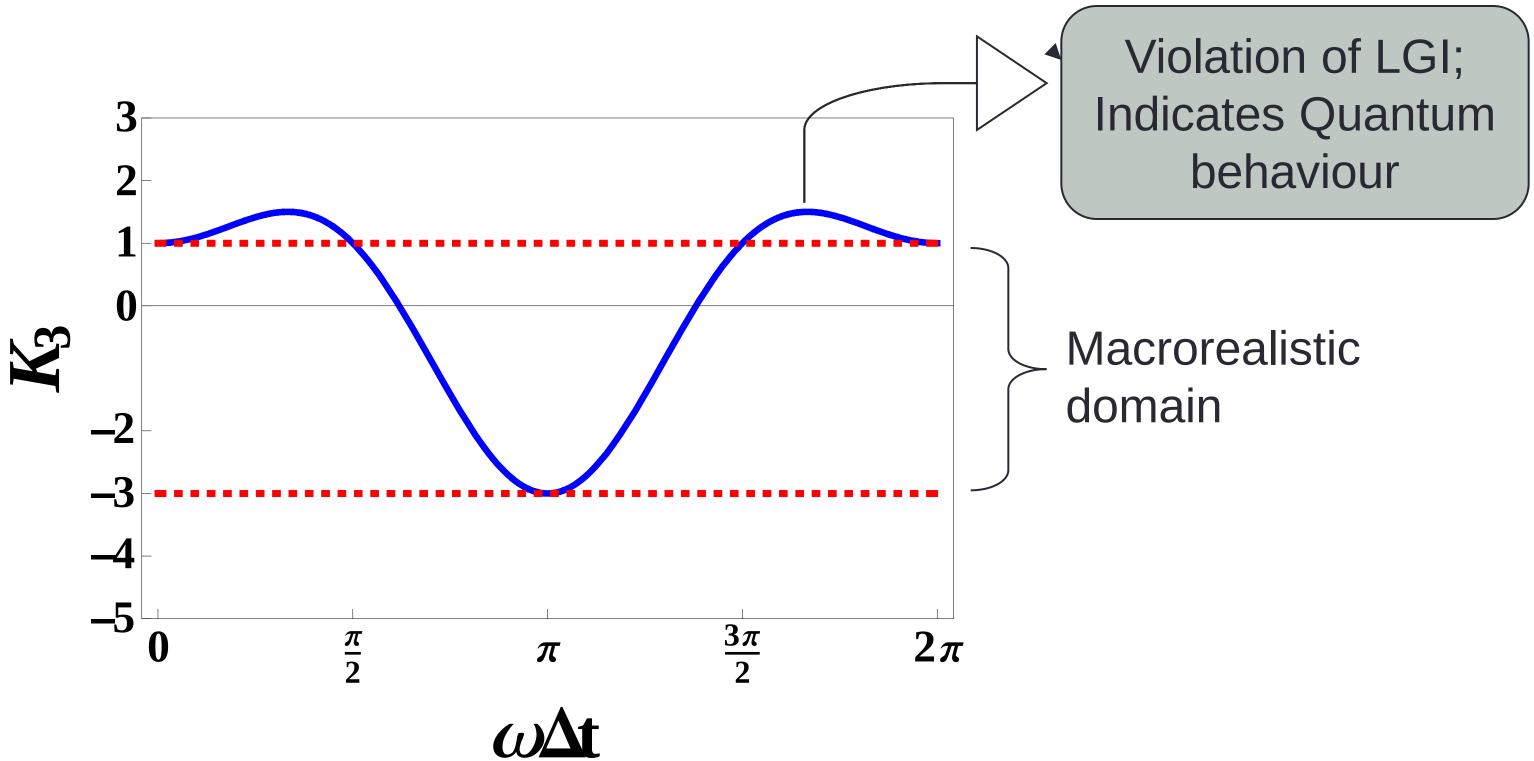


$$
\begin{aligned}
C_{12} &= \langle \sigma_z(0)\sigma_z(\Delta t)\rangle = \langle \sigma_z e^{-iH\Delta t}\sigma_z e^{iH\Delta t}\rangle \\
&= \langle \sigma_z\left[\sigma_z \cos(\omega\Delta t) + \sigma_x(\sin\omega\Delta t)\right]\rangle \\
&\equiv \cos(\omega\Delta t)
\end{aligned}
$$

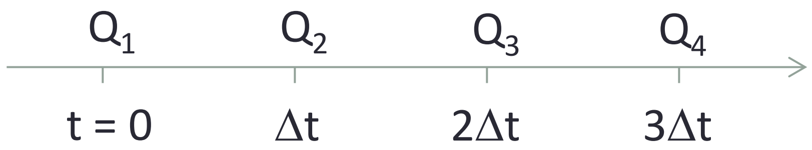
$$
\begin{aligned}
C_{23} &= \langle \sigma_z(\Delta t)\sigma_z(2\Delta t)\rangle \equiv \cos(\omega\Delta t) \\
C_{13} &= \langle \sigma_z(\Delta t)\sigma_z(3\Delta t)\rangle \equiv \cos(2\omega\Delta t)
\end{aligned}
$$

LGI violation

$$K_3 = C_{12} + C_{23} - C_{13} = 2\cos(\omega\Delta t) - \cos(2\omega\Delta t)$$



LGI with 4 measurements



$$C_{12} = \langle \sigma_z(0)\sigma_z(\Delta t)\rangle = \langle \sigma_z e^{-iH\Delta t}\sigma_z e^{iH\Delta t}\rangle = \cos(\omega\Delta t)$$

$$C_{23} = \langle \sigma_z(2\Delta t)\sigma_z(3\Delta t)\rangle \equiv \cos(\omega\Delta t)$$

$$C_{34} = \langle \sigma_z(3\Delta t)\sigma_z(4\Delta t)\rangle \equiv \cos(\omega\Delta t)$$

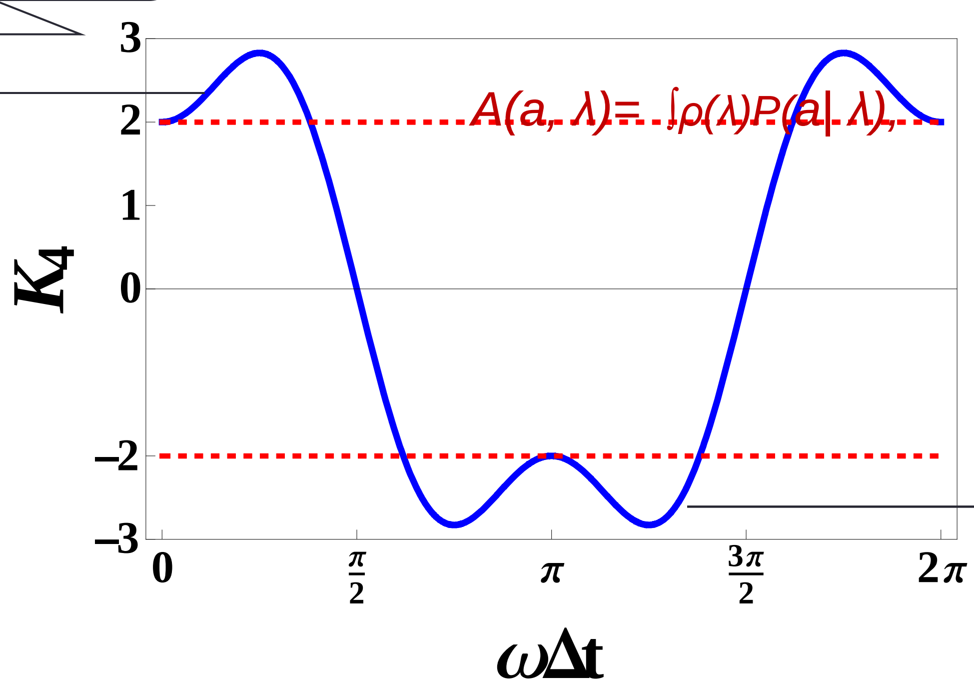
and

$$C_{14} = \langle \sigma_z(\Delta t)\sigma_z(4\Delta t)\rangle \equiv \cos(3\omega\Delta t)$$

Violation of four term LGI

$$K_4 = C_{12} + C_{23} + C_{34} - C_{14} = 3\cos(\omega\Delta t) - \cos(3\omega\Delta t)$$

LGI violation:
Quantum nature



Macrorealistic domain

LGI violation
Indicating Quantum behaviour

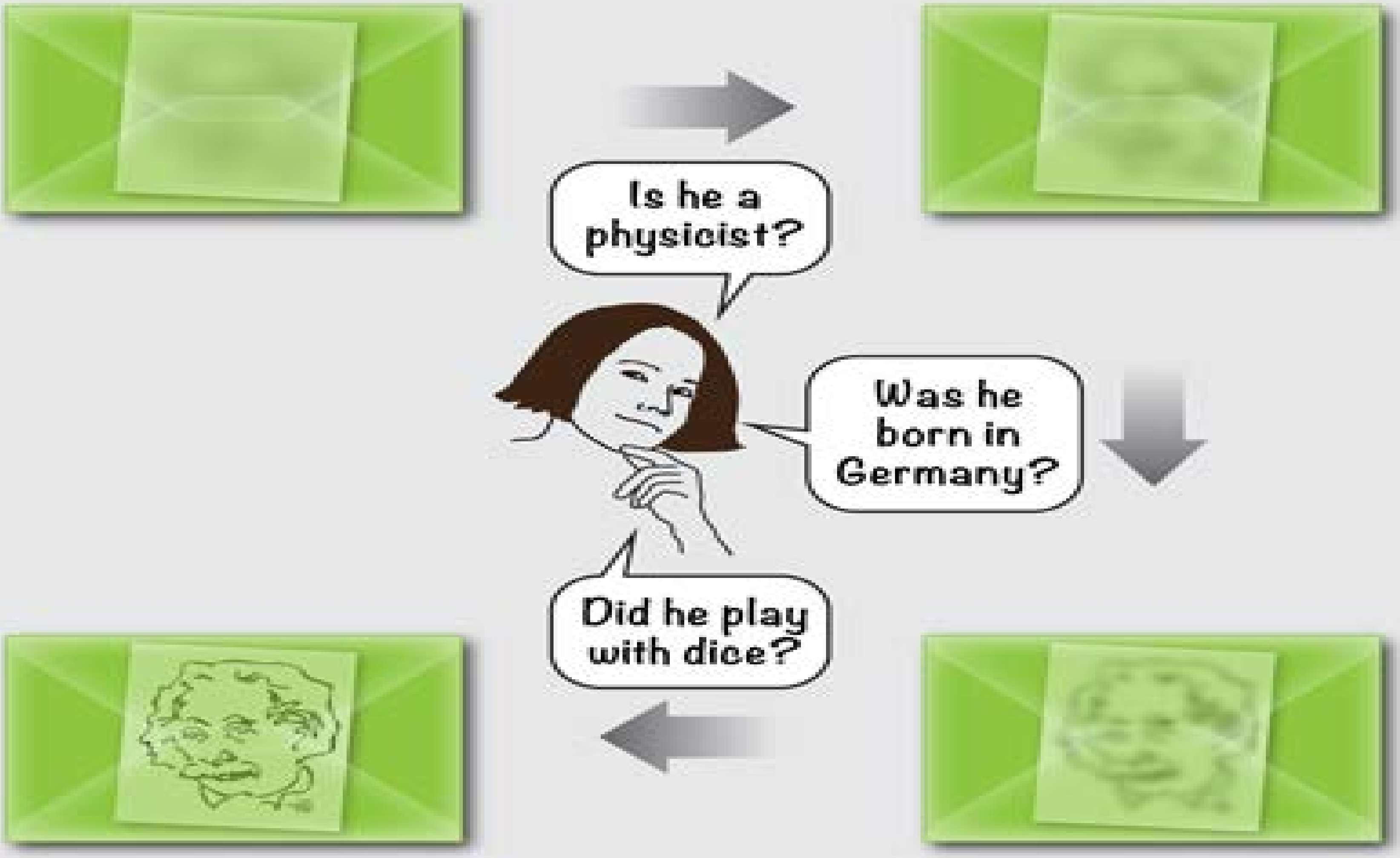
Bell and Leggett–Garg inequalities

statistical outcomes of spatial and temporal correlations

Violation points towards **non-existence of joint probabilities**

A. Fine, Phys. Rev. Lett. 48, 291 (1982); M. Markiewicz et.al., arXiv:1302.3502

Contextuality



Kochen-Specker Theorem (1967)

- Non-contextuality: All measurable properties of a physical system do not depend on the context in which they are measured.

- But a non-contextual assignment of values to the observables is not possible in quantum world

- Kochen-Specker studied the logical feature of the quantum theory in connection with the consistency of counterfactual propositions concerning the values of observables that are not co-measurable

J. S. Bell, Rev. Mod. Phys. 38, 447 (1966).
S. Kochen and E. P. Specker, J. Math. Mech. 17, 59 (1967).
N. D. Mermin, Rev. Mod. Phys. 65, 803 (1993).

Non-contextuality is a very plausible hypothesis based on our everyday experience. The colour of your car would be the same regardless if you looked at it together with Prof. Kochen or Prof. Specker. All classical theories of nature are compatible with Non-contexuality.

Kurzyński and Kaszlikowski, arXiv:1309.6777

Kochen-Specker inequality

- **Consider three boxes with gems such that when any two boxes are opened one of them contains a gem and the other doesn't.**
- **The situation could be expressed in terms of three dichotomic variables $X_i,\ \ i = 1, 2, 3$ with $X_i = 1\,(-1)$ corresponding to the case of gem present (absent) in the i^{th} box.**
- **Consequently, if we choose a pair of boxes uniformly at random, at most two of the three pairs could exhibit anticorrelation, so that the probability of obtaining anticorrelated outcomes is bounded from above by 2/3 i.e.,**

$$S_{NC} = \sum \frac{1}{3}\, p(X_i \neq X_{i\oplus 1}) \leq \frac{2}{3}$$

in any non-contextual model.

- **Anticorrelation requires the following algebraic relations for relations for all the three pairs:**

$$
\begin{aligned}
X_1 X_2 &= -1 \\
X_2 X_3 &= -1 \\
X_1 X_3 &= -1.
\end{aligned}
$$

  However, these relations cannot be satisfied with non-contextual assignment of values because the product of the left-hand-sides is $X_1^2 X_2^2 X_3^2 = +1$, while the product of the right-hand-sides is -1. It is impossible to open different pairs of boxes and always find anti-correlation rather than correlation.
  N. D. Mermin, Rev. Mod. Phys., 65, 803 (1993)

- **Contextual assignment would require that if one of the boxes is full (empty), the other box would be empty (full). Assigning the pairwise probabilities as**

$$
P(X_i = 1, X_j = -1) = P(X_i = -1, X_j = 1) = \frac{1}{2} \quad i \neq j
$$

  one finds

$$
S = \sum \frac{1}{3} p(X_i \neq X_{i \oplus 1}) = 1
$$

If X_1 is co-measurable with X_2, X_2 with X_3, and X_3 with X_1, one may think that all the observables are jointly measurable (commuting observables in the case of quantum mechanics).
A non-contextuality inequality M. Araujo et. al., Phys. Rev. A **88**, 022118 (2013):

$$-\langle X_1 X_2\rangle + \langle X_2 X_3\rangle + \langle X_1 X_3\rangle \leq 1$$

is violated (maximally) by the generalized probabilities

$$\begin{aligned}
p(x_1=1,x_2=-1) &= \frac{1}{2} = p(x_1=-1,x_2=1),\ p(x_1=1,x_2=1)=0=p(x_1=-1,x_2=-1)\\
p(x_2=1,x_3=1) &= \frac{1}{2} = p(x_2=-1,x_3=-1),\ p(x_2=1,x_3=-1)=0=p(x_2=-1,x_3=1)\\
p(x_1=1,x_3=1) &= \frac{1}{2} = p(x_1=-1,x_3=-1),\ p(x_1=1,x_3=-1)=0=p(x_1=-1,x_3=1)
\end{aligned}$$

- **Pairwise compatible measurements are not jointly compatible.**

Non-existence of joint probabilities

- Suppose a given physical system has properties X_1, X_2, X_3 with outcomes x_1, x_2, x_3 and probability distributions $p(x_1)$, $p(x_2)$, $p(x_3)$. Suppose that the property X_1 can be co-measured with the property X_2 giving us a probability distribution $p(x_1, x_2)$ or it can be co-measured with the property X_3 giving a probability distribution $p(x_1, x_3)$. We say that X_1 can be measured in the context of X_2 or X_3. Non-contextuality states that there exists a joint probability distribution $p(x_1, x_2, x_3)$ such that $p(x_1, x_2)$ and $p(x_1, x_3)$ are recovered as marginals.

Kurzyński and Kaszlikowski, arXiv:1309.6777

The Kochen and Specker assertion that single quantum mechanical systems are contextual could be put to an experimentally testable format in the the paper by Klyachko-Can-Biniciouglu-Schumovsky (KCBS) (Phys. Rev. Lett. 101, 020403 (2008)). KCBS inequality – with a set of five observables in a three level system – was tested experimentally (Nature 474, 490 (2011)). (Note that it took 50 years to experimentally test Kochen-Specker theorem whereas Bell scenario was tested within 20 years of its formulation).

- **Entropic inequalities**
- **Moment matrix positivity**
- **Entropic uncertainty**

Entropic inequalities

The CHSH/LG/KS inequalities were originally formulated for dichotomic observables and they constrain linear combinations of correlation functions.

Braunstein & Caves recognized that classical Shannon entropies associated with correlation outcomes of any bipartite spatially separated parties obey certain constraints, violations of which would imply non-existence of a legitimate joint probability for all the measured quantities – which need not be dichotomic.

S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. 61, 662 (1988).

Entropic inequalities

If $P(q_k, q_{k+l})$ is the joint probability distribution associated with two observables Q_k and Q_{k+l}, the mean information associated with the measurements Q_k, Q_{k+l} is given by

Joint entropy: $H(Q_k, Q_{k+l}) = -\sum_{q_k, q_{k+l}} P(q_k, q_{k+l}) \log_2 P(q_k, q_{k+l})$

The information carried by Q_k, Q_{k+l} respectively is given by

$$\begin{aligned} H(Q_k) &= -\sum_{q_k} P(q_k) \log_2 P(q_k) \\ H(Q_{k+l}) &= -\sum_{q_{k+l}} P(q_{k+l}) \log_2 P(q_{k+l}) \end{aligned}$$

Notice that

$$\begin{aligned} P(q_k) &= \sum_{q_{k+l}} P(q_k, q_{k+l}) \\ P(q_{k+l}) &= \sum_{q_k} P(q_k, q_{k+l}) \end{aligned}$$

Entropic inequalities

If $P(q_k|q_{k+l})$ denotes the conditional probability of the observable Q_k assuming the value q_k when the observable Q_{k+l} has assumed a value q_{k+l}, then the conditional Shannon information is given by

$$H(Q_k|Q_{k+l}) = -\sum_{q_k, q_{k+l}} P(q_k, q_{k+l}) \log_2 P(q_k|q_{k+l})$$

Relation between conditional and joint probabilties:

Bayes' theorem $\Longleftrightarrow$ $$P(q_k|q_{k+l}) = \frac{P(q_k, q_{k+l})}{P(q_k)}$$

Thus $$H(Q_{k+l}|Q_k) = H(Q_k, Q_{k+l}) - H(Q_k).$$

Entropic approach

Two basic inequalities from information theory:

$$H(Q_{k+l}|Q_k) \leq H(Q_k) \leq H(Q_k, Q_{k+l})$$

- Left Hand Inequality: Removing a condition never decreases the information

- Right Hand Inequality: Two variables never carry less information than that carried by one of them.

Entropic approach

For three variables say, Q_1, Q_2, Q_3,

$$H(Q_3, Q_1) \quad \leq \quad H(Q_3, Q_2, Q_1)$$

As

$$\Downarrow$$

$$\begin{aligned} H(Q_3, Q_2, Q_1) &= H(Q_3|Q_2, Q_1) + H(Q_2, Q_1) \\ &= H(Q_3|Q_2, Q_1) + H(Q_2|Q_1) + H(Q_1) \end{aligned}$$

we have,

$$H(Q_3, Q_1) \quad \leq \quad H(Q_3|Q_2, Q_1) + H(Q_2|Q_1) + H(Q_1)$$

As $H(Q_3|Q_1) = H(Q_3, Q_1) - H(Q_1)$ follows from Bayes' theorem and as $H(Q_3|Q_2, Q_1) \leq H(Q_3|Q_2)$, we have,

$$H(Q_3|Q_1) \quad \leq \quad H(Q_3|Q_2) + H(Q_2|Q_1)$$

Entropic approach

For any observable Q at three different instants say, $Q_k,\ Q_{k+l}, Q_{k+m}$ with $t_{k+m} > t_{k+l} > t_k$, we similarly have

$$H(Q_{k+m}, Q_k) \quad \leq \quad H(Q_{k+m}, Q_{k+l}, Q_k) = H(Q_{k+m}|Q_{k+l}, Q_k) + H(Q_{k+l}|Q_k) + H(Q_k)$$

$$\Downarrow$$

$$H(Q_{k+m}, Q_k) \quad \leq \quad H(Q_{k+m}|Q_{k+l}, Q_k) + H(Q_{k+l}|Q_k) + H(Q_k)$$

$H(Q_{k+m}, Q_k) = H(Q_{k+m}|Q_k) + H(Q_k)$ (Bayes' theorem) implies

$$H(Q_{k+m}|Q_k) \quad \leq \quad H(Q_{k+m}|Q_{k+l}) + H(Q_{k+l}|Q_k).$$

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Information-Theoretic Bell Inequalities

Samuel L. Braunstein
Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125

and

Carlton M. Caves
Center for Laser Studies, University of Southern California, Los Angeles, California 90089
(Received 2 May 1988)

We formulate information-theoretic Bell inequalities, which apply to any pair of widely separated physical systems. If local realism holds, the two systems must carry information consistent with the inequalities. Two spin-s particles in a state of zero total spin violate these information Bell inequalities.

$$\boxed{H(A\,|\,B) \le H(A\,|\,B') + H(B'\,|\,A') + H(A'\,|\,B)}$$

$$|\phi\rangle = (2s+1)^{-1/2} \sum_{m=-s}^{s} (-1)^{s-m} |sm\rangle_{\mathcal{A},\mathbf{c}} \otimes |s\,-m\rangle_{\mathcal{B},\mathbf{c}}$$

Singlet state of two spin-s particles

Quantum mechanics predicts the probability

$$p(a=m_1, b=m_2) = |{}_{\mathcal{A},\mathbf{a}}\langle sm_1| \otimes {}_{\mathcal{B},\mathbf{b}}\langle sm_2|\,|\phi\rangle|^2 = (2s+1)^{-1} |D_{m_1,-m_2}(R_{\mathbf{n}}(\theta))|^2$$

that $\mathbf{S}_{\mathcal{A}} \cdot \mathbf{a}$ has value m_1 and $\mathbf{S}_{\mathcal{B}} \cdot \mathbf{b}$ has value m_2.

$$H^{\mathrm{QM}}(A \mid B) = H^{\mathrm{QM}}(B \mid A) \equiv H^{\mathrm{QM}}(\theta)$$

$$H^{\mathrm{QM}}(\theta) = -\frac{1}{2s+1} \sum_{m_1, m_2} \left| D_{m_1 - m_2}(R_{\mathbf{n}}(\theta)) \right|^2 \log \left| D_{m_1 - m_2}(R_{\mathbf{n}}(\theta)) \right|^2.$$

Coplanar geometry: a, b, a', b' are coplanar and successive vectors successive vectors are separated by angle $\theta/3$

Entropic Bell inequality is violated if the information difference

$$\mathcal{H}^{\mathrm{QM}}(\theta) \equiv 3H^{\mathrm{QM}}(\theta/3) - H^{\mathrm{QM}}(\theta)$$

is negative

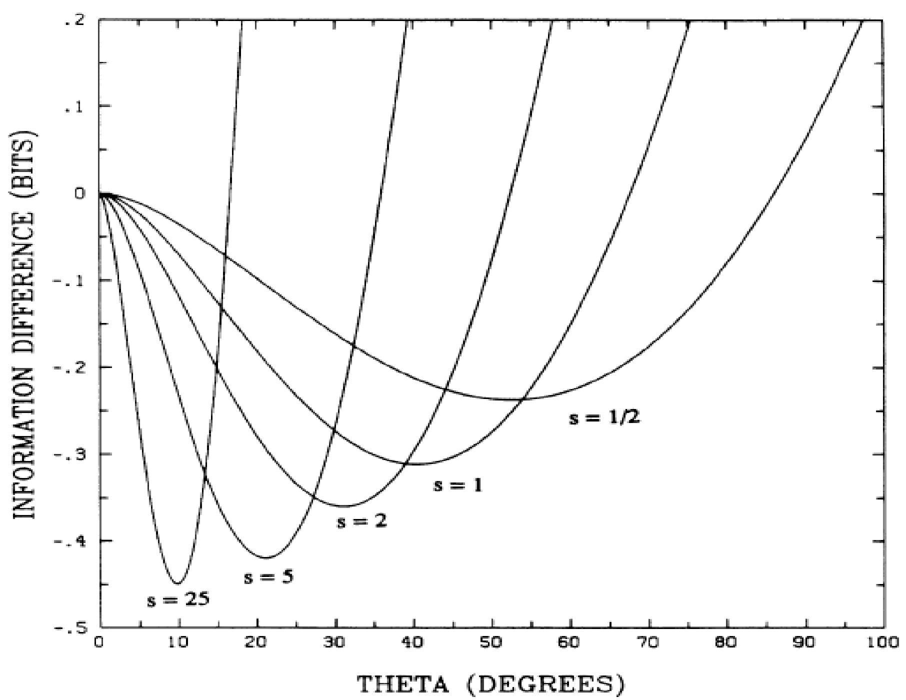

FIG. 1. Information difference $\mathcal{H}^{\mathrm{QM}}(\theta)$ in bits vs angle θ in degrees for $s=\frac{1}{2}$, 1, 2, 5, and 25. The maximum information deficit for $s=\frac{1}{2}$ is -0.2369 bits at 52.31°; for $s=25$, -0.4493 bits at 9.798°.

PRL 109, 020404 (2012) PHYSICAL REVIEW LETTERS week ending 13 JULY 2012

Entropic Test of Quantum Contextuality

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We study the contextuality of a three-level quantum system using classical conditional entropy of measurement outcomes. First, we analytically construct the minimal configuration of measurements required to reveal contextuality. Next, an entropic contextual inequality is formulated, analogous to the entropic Bell inequalities derived by Braunstein and Caves [Phys. Rev. Lett. **61**, 662 (1988)], that must be satisfied by all noncontextual theories. We find optimal measurements for violation of this inequality. The approach is easily extendable to higher dimensional quantum systems and more measurements. Our theoretical findings can be verified in the laboratory with current technology.

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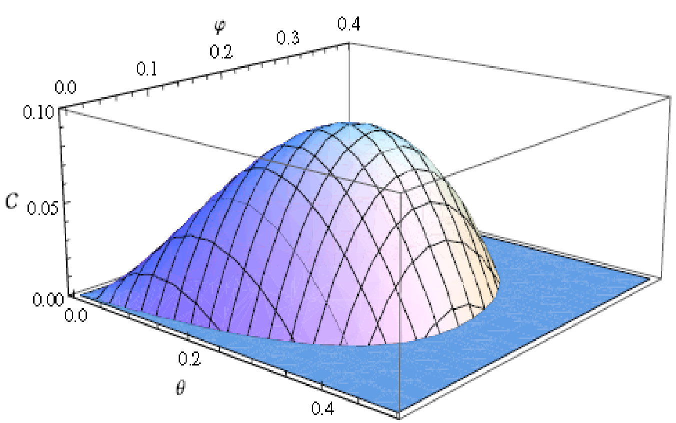
$$H(A_1|A_5) \leq H(A_1|A_2) + H(A_2|A_3) + H(A_3|A_4) + H(A_4|A_5).$$

$$C = H(A_1|A_5) - H(A_1|A_2) - H(A_2|A_3) - H(A_3|A_4) - H(A_4|A_5);$$

$$|\psi\rangle = (\sin\theta, \cos\theta, 0)^T, \qquad |A_1\rangle = \left(\frac{\sqrt{\cos 2\varphi}}{\sqrt{2}\cos\varphi}, \frac{\tan\varphi}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T,$$

$$|A_2\rangle = (0, \cos\varphi, -\sin\varphi)^T, \qquad |A_3\rangle = (1, 0, 0)^T,$$

$$|A_4\rangle = (0, \cos\varphi, \sin\varphi)^T, \qquad |A_5\rangle = \frac{|A_1\rangle \times |A_4\rangle}{\||A_1\rangle \times |A_4\rangle\|},$$



Our work (A. R. Usha Devi, H. S. Karthik, Sudha and A. K. Rajagopal, Phys. Rev. A **87**, 052103 (2013)) extends these information theoretic notions to develop Leggett-Garg entropic inequality to test macrorealism.

Entropic approach to Leggett-Garg Inequalities

- $Q(t_k)$ is a dynamical observable (not necessarily dichotomic!) at time t_k.
- Outcomes of measurements of the observable $Q(t_k) \rightarrow q_k$.
- Probability of observation of $q_k \rightarrow P(q_k)$.
- Macrorealism demands that the outcomes q_k of $Q(t_k)$ at all instants of time pre-exist independent of their measurement. Mathematically this implies the existence of a joint probability distribution $P(q_1, q_2, \ldots)$ characterizing the statistics of the outcomes
- The joint probability yields the marginals $P(q_k)$ of individual observations at time t_k.

Entropic Leggett-Garg Inequality

Entropic inequality for n consecutive measurements $Q_1, Q_2, \ldots, Q_n$ at time instants $t_1 < t_2 < \ldots < t_n$:

$$H(Q_n|Q_1) \leq H(Q_n|Q_{n-1}) + H(Q_{n-1}|Q_{n-2}) + \ldots + H(Q_2|Q_1).$$

Entropic LGI

Entropic Leggett-Garg Inequality implies that the macrorealistic information underlying the statistical outcomes of the observable at n different times must be consistent with the information associated with pairwise non-invasive measurements.

Quantum joint Probabilities

- Suppose $Q_k = S_z(t_k)$ takes the value m_k at time t_k and at a later instant of time t_{k+l} the measurement outcome is m_{k+l}.
- The quantum mechanical joint probability is given by

$$P(m_k, m_{k+l}) = p_{m_k}(t_k)\, q(m_{k+l}, t_{k+l}|m_k, t_k)$$

Here $p_{m_k}(t_k) = \mathrm{Tr}[\rho\, \Pi_{m_k}(t_k)]$ is the probability of obtaining the outcome m_k at time t_k. Also, as $\rho(t_k) = [\Pi_{m_k}(t_k)\rho\Pi_{m_k}(t_k)]/p_{m_k}(t_k)$,

$$\begin{aligned} q(m_{k+l}, t_{k+l}|m_k, t_k) &= \mathrm{Tr}[\rho(t_k)\, \Pi_{m_{k+l}}(t_{k+l})] \\ &= \mathrm{Tr}[\Pi_{m_k}(t_k)\rho\Pi_{m_k}(t_k)\, \Pi_{m_{k+l}}(t_{k+l})]/p_{m_k}(t_k) \end{aligned}$$

is the conditional probability of obtaining the outcome m_{k+l} for the spin component S_z at time t_{k+l}, if it had already taken the value m_k at an earlier time t_k.

Quantum joint Probabilities

Thus the quantum mechanical joint probability $P(m_k, m_{k+l})$ of obtaining the result m_k at time t_k and m_{k+l} at time t_{k+l} is given by

$$P(m_k, m_{k+l}) \quad = \quad \mathrm{Tr}[\Pi_{m_k}(t_k)\rho\Pi_{m_k}(t_k)\,\Pi_{m_{k+l}}(t_{k+l})]$$

Here,

$$\Pi_m(t) = U^\dagger(t)\,|s,m\rangle\langle s,m|\,U(t)$$

Projection Operator at time t

Entropic LGI for a quantum spin-s rotor

- Consider an initial state of the rotor in a maximally mixed state $\rho = \frac{1}{2s+1}\sum_{m=-s}^{s} |s,m\rangle\langle s,m| = \frac{I}{2s+1}$
  $\{|s,m\rangle\} \rightarrow$ simultaneous eigenstates of the squared spin operator $S^2 = S_x^2 + S_y^2 + S_z^2$ and the z-component of spin S_z
- Hamiltonian governing the evolution: $H = \omega\, S_y$
- Unitary evolution: $U(t) = e^{-i\omega t\, S_y/\hbar}$ (corresponds to a rotation about the y-axis by an angle $\omega\, t$).
- Dynamical observable $Q(t)$: We choose z-component of spin $Q(t) = S_z(t) = U^\dagger(t)\, S_z\, U(t)$ as the dynamical observable for our investigation of macro-realism.

Quantum joint Probabilities for spin-s Rotor

For the maximally mixed initial state the quantum mechanical joint probabilities are given by,

$$
\begin{aligned}
P(m_k, m_{k+l}) &= \frac{1}{2s+1}\,\mathrm{Tr}[\Pi_{m_k}(t_k)\,\Pi_{m_{k+l}}(t_{k+l})] \\
&= \frac{1}{2s+1}\,\mathrm{Tr}\left[U(t_k)|s,m_k\rangle\langle s,m_k|U^\dagger(t_k)U(t_{k+l})|s,m_{k+l}\rangle\langle s,m_{k+l}|U^\dagger(t_{k+l})\right] \\
&= \frac{1}{2s+1}\,\left[\langle s,m_k|U^\dagger(t_k)U(t_{k+l})|s,m_{k+l}\rangle\langle s,m_{k+l}|U^\dagger(t_k)U(t_{k+l})|s,m_k\rangle\right] \\
&= \frac{1}{2s+1}\,\left|\langle\, s,m_{k+l}|e^{-i\omega(t_{k+l}-t_k)\,S_y}\,|s,m_k\rangle\right|^2 \\
&= \frac{1}{2s+1}\,\left|\,d^s_{m_{k+l}\;m_k}(\theta_{kl})\right|^2
\end{aligned}
$$

Here $d^s_{m'm}(\theta_{kl}) = \langle s,m'|e^{-i\theta_{kl}\,S_y/\hbar}|s,m\rangle$ are the matrix elements of the $2s+1$ dimensional irreducible representation of rotation about y-axis by an angle $\theta_{kl} = \omega(t_{k+l} - t_k)$

Entropic LGI for equidistant time measurements

- For measurements at equidistant time intervals $\Delta t = t_{k+1} - t_k$, $k = 1, 2, \ldots n$ quantum mechanical information entropy depends only on the time separation (denoted by $\theta = (n-1)\omega\,\Delta t$):

$$
\begin{aligned}
H(Q_k|Q_{k+1}) &\equiv H\left[\frac{\theta}{n-1}\right] \\
&= \frac{-1}{2s+1}\sum_{m_k,m_{k+1}}\left|d^s_{m_{k+1},m_k}\left[\frac{\theta}{n-1}\right]\right|^2\log_2\left|d^s_{m_{k+1},m_k}\left[\frac{\theta}{n-1}\right]\right|^2.
\end{aligned}
$$

- Recall the entropic Leggett-Garg inequality given by

$$
H(Q_n|Q_1) \le H(Q_n|Q_{n-1}) + H(Q_{n-1}|Q_{n-2}) + \ldots + H(Q_2|Q_1)
$$

which implies $(n-1)H(Q_n|Q_{n-1}) - H(Q_n|Q_1) \ge 0$ for a spin-s rotor.

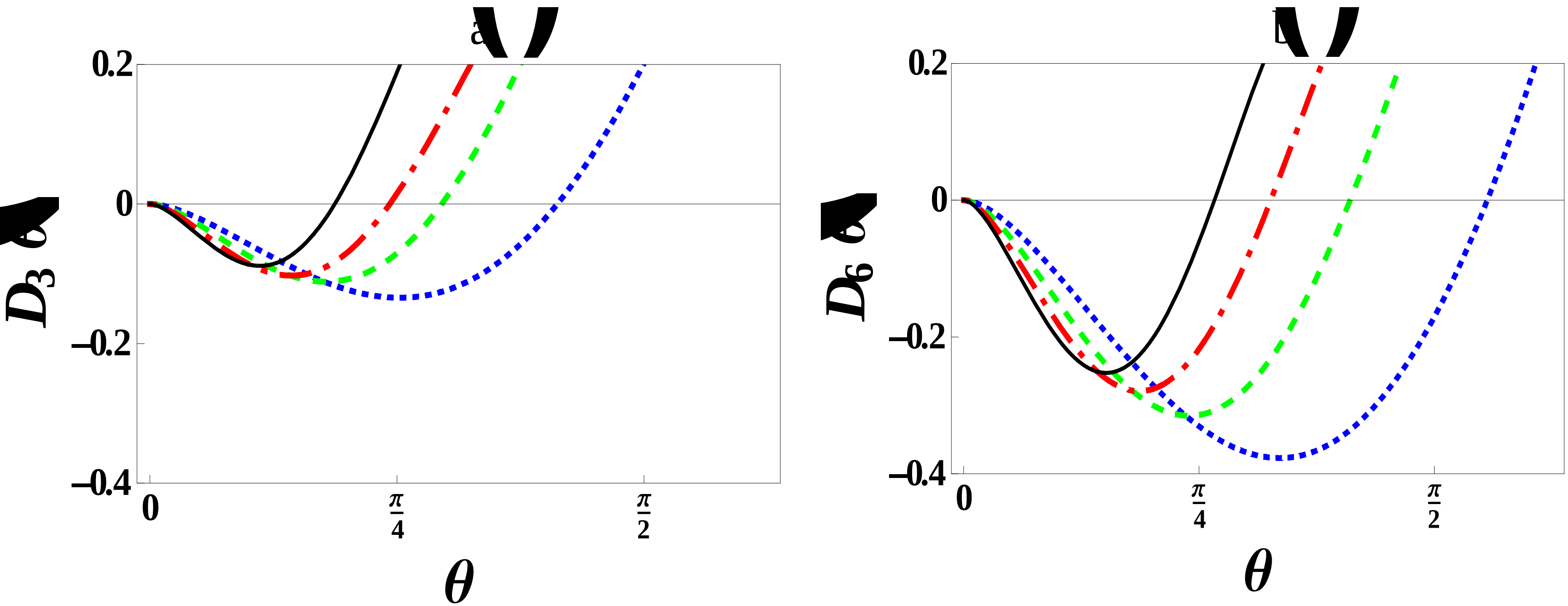
- The n-term entropic inequality for observations at equidistant time steps:

$$
\begin{aligned}
(n-1)H\left[\frac{\theta}{n-1}\right] - H(\theta) &= \frac{-1}{2s+1}\sum_{m_k,m_{k+1}}\left((n-1)\left|d^s_{m_{k+1},m_k}\left[\frac{\theta}{n-1}\right]\right|^2\log_2\left|d^s_{m_{k+1},m_k}\left[\frac{\theta}{n-1}\right]\right|^2\right. \\
&\qquad \left. -|d^s_{m_{k+1},m_k}(\theta)|^2\log_2|d^s_{m_{k+1},m_k}(\theta)|^2\right) \ge 0
\end{aligned}
$$

Violation of entropic LGI by a spin-s rotor

Information deficit D_n in units of $\log_2(2s+1)$ bits)

$$\mathcal{D}_n(\theta) = \frac{(n-1)\,H[\theta/(n-1)] - H(\theta)}{\log_2(2s+1)} \geq 0$$



Spin-1/2: Dotted Spin-1: Dashed Spin-3/2: Dot Dashed Spin 2: Solid

H. Katiyar, A. Shukla, K. R. K. Rao, and T. S. Mahesh, Phys. Rev. A 87, 052102 (2013).

Violation of Entropic Leggett-Garg Inequality in Nuclear Spin Ensembles

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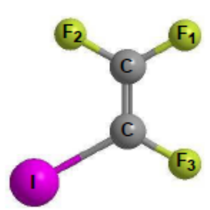
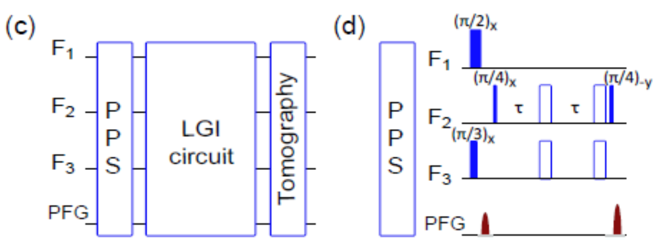
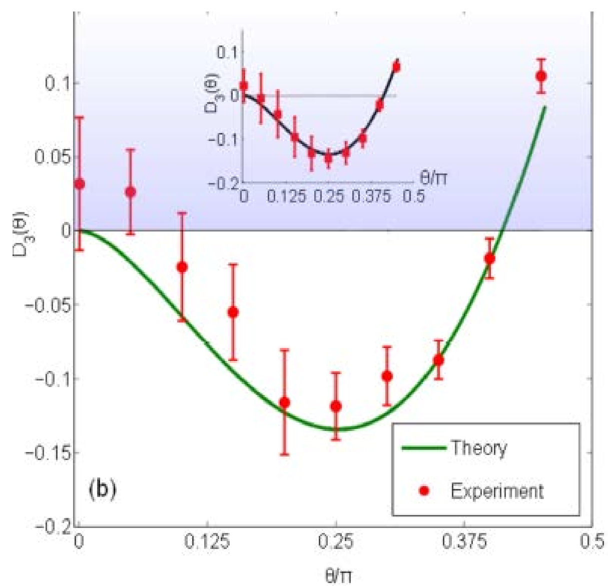
(a)



(b)

	F_1	F_2	F_3
F_1	0.0	69.9	-128.3
F_2		11860.8	47.4
F_3			-17379.1





^{19}F nuclear spins of trifluoroiodoethylene

Moment matrix positivity

Classical Moment Problem

$\Longrightarrow$ Addresses the issue of determining a probability distribution given a set of moments.

It brings forth the fact that

A given sequence of real numbers qualifies to be moment sequence of a legitimate probability distribution if and only if the associated moment matrix is positive.

Existence of joint probability distribution $\Longleftrightarrow$ *Moment matrix positive*

J.A Sholat and J.D. Tamarkin, *The problem of moments,* AMS (1943)

N.J. Akhiezer, *The Classical Moment Problem*, Hofuer Publishing Co., (1965)

- When does a sequence of real numbers qualify to be a moment sequence and thereby correspond to a valid joint probability distribution?

- The answer is, when the corresponding moment matrix is positive definite. The nature of physically valid joint probability distribution can be brought out with the help of positive moment matrix.

Positivity of moment matrix and the nature of grand joint probabilities

We consider three dichotomic random variables $X_1,\ X_2,\ X_3$. A sequence of eight moments $\{1, \langle X_1\rangle, \langle X_2\rangle, \langle X_3\rangle, \langle X_1X_2\rangle, \langle X_2X_3\rangle, \langle X_1X_3\rangle,\ \langle X_1X_2X_3\rangle\}$ faithfully encodes the details of the joint probability distribution $P(x_1, x_2, x_3),\ x_i = \pm 1$. This encryption of trivariate probabilities in these eight moments is reflected in the positivity of the 8×8 moment matrix

$$M = \langle \xi\, \xi^T\rangle,\ \text{where } \xi^T = (1,\ X_1,\ X_2,\ X_3,\ X_1X_2,\ X_2X_3,\ X_1X_3,\ X_1X_2X_3).$$

In other words, given a set of real numbers (which is supposed to be the moment sequence), positivity of the moment matrix ensures that there exists a valid joint probability distribution. .

Denoting $\langle X_1 X_2 \rangle = a$, $\langle X_2 X_3 \rangle = b$, $\langle X_1 X_3 \rangle = c$ and considering the 4×4 principal minor of the moment matrix constructed from $M = \langle \xi \xi^T \rangle$:

$$M = \begin{pmatrix} 1 & a & b & c \\ a & 1 & c & b \\ b & c & 1 & a \\ c & b & c & 1 \end{pmatrix}.$$

Here, $\xi^T = \{1, X_1X_2, X_2X_3, X_1X_3\}$

Eigenvalues of M:

$$\begin{aligned} \lambda_1 &= 1+a-b-c, \qquad \lambda_2 = 1-a+b-c, \\ \lambda_3 &= 1-a-b+c, \qquad \lambda_4 = 1+a+b+c \end{aligned}$$

Moment matrix associated with temporal correlations

Consider the dynamical evolution of a qubit governed by the Hamiltonian $H = \frac{1}{2}\hbar\omega\sigma_x$. We consider measurement of three observables

$$X_i = \sigma_z(t_i), \quad t_1 = 0,\ t_2 = \Delta t,\ t_3 = 2\,\Delta t.$$

The dynamical observable σ_z at different times is given explicitly by,

$$\sigma_z(t_i) = e^{iHt_i}\sigma_z e^{-iHt_i} = \sigma_z\,\cos(\omega t_i) + \sigma_y\,\sin(\omega t_i).$$

When the qubit is initially prepared initially in a maximally mixed state $\rho_{\rm in} = I/2$, sequential measurements of X_1, X_2, X_3 leads to

$$\begin{aligned} \langle X_1\rangle &= \langle\sigma_z\rangle = 0;\ \ \langle X_2\rangle = \langle\sigma_z(\Delta t)\rangle = 0; \\ \langle X_3\rangle &= \langle\sigma_z\rangle(2\Delta t) = 0;\ \ \langle X_1\,X_2\,X_3\rangle = 0. \end{aligned}$$

$$\begin{aligned}
\langle X_1\, X_2\rangle &= \langle\{\sigma_z, \sigma_z(\Delta t)\}\rangle = \cos(\omega\Delta t) \\
\langle X_2\, X_3\rangle &= \langle\{\sigma_z(\Delta t), \sigma_z(2\Delta t)\}\rangle = \cos(\omega\Delta t) \\
\langle X_1\, X_3\rangle &= \langle\{\sigma_z, \sigma_z(2\Delta t)\}\rangle = \cos(2\omega\Delta t)
\end{aligned}$$

On associating the parameters a, b, c of the moment matrix as

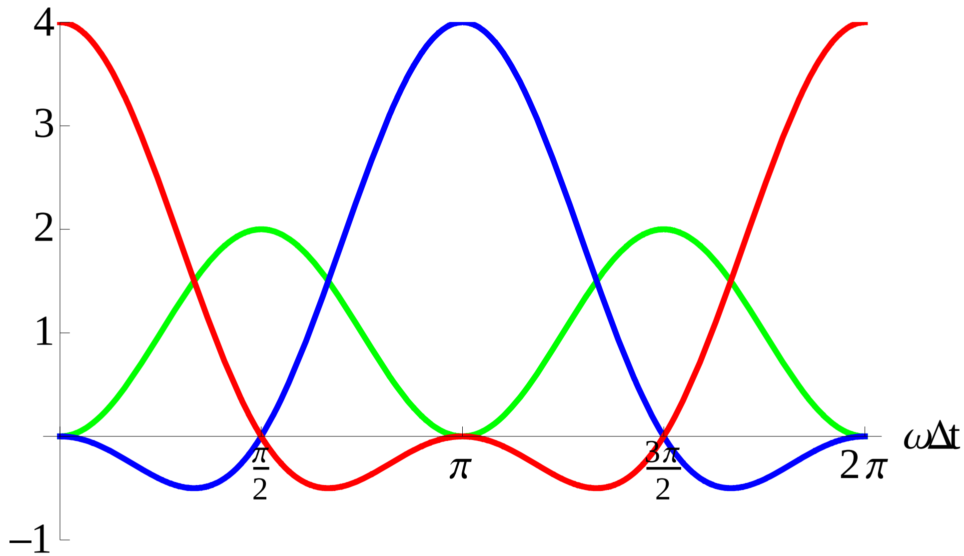
$$a = \cos(\omega\Delta t), \quad b = \cos(\omega\Delta t), c = \cos(2\omega\Delta t),$$

positivity of the eigenvalues of the moment matrix results in the conditions:

$$1 - \cos(2\omega\Delta t) \geq 0,$$

$$1 - 2\cos(\omega\Delta t) + \cos(2\omega\Delta t) \geq 0$$

$$1 + 2\cos(\omega\Delta t) + \cos(2\omega\Delta t) \geq 0.$$



The graph above illustrates that the moment matrix corresponding to sequential measurements on a quantum system at three different times is *negative* for all values of Δt.

Moment matrix associated with spatial correlations

We now consider a spatially separated two qubit system in a Bell state

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left[|0_A, 1_B\rangle - |1_A, 0_B\rangle\right].$$

We consider measurements of three observables

$$X_1 = \vec{\sigma}\cdot\hat{a}\otimes I, \quad X_2 = I\otimes\vec{\sigma}\cdot\hat{b}, \quad X_3 = \vec{\sigma}\cdot\hat{a}'\otimes I.$$

We obtain,

$$\begin{aligned}
\langle X_1\rangle = \langle X_2\rangle &= \langle X_3\rangle = 0, \quad \langle X_1\, X_2\, X_3\rangle = 0 \\
\langle X_1\, X_2\rangle &= -\hat{a}\cdot\hat{b} = -\cos\theta_{ab} \\
\langle X_2\, X_3\rangle &= -\hat{a}'\cdot\hat{b} = -\cos\theta_{a'b} \\
\langle X_1\, X_3\rangle &= \hat{a}\cdot\hat{a}' = \cos\theta_{aa'}.
\end{aligned}$$

Moment matrix associated with spatial correlations

Choosing coplanar geometry for $\hat{a}, \hat{b}, \hat{a}'$ such that $\theta_{ab} = \pi - \phi$, $\theta_{a'b} = \pi - \phi$ and $\theta_{aa'} = 2\pi - 2\phi$, we obtain,

$$a = \cos\phi, \quad b = \cos\phi, \quad c = \cos 2\phi,$$

which results in analogous conclusion as in the case of temporal correlations i.e.,

Moment matrix turns out to be negative for any arbitrary value of ϕ.

Specker's probabilities for anticorrelated variables:

$$
\begin{aligned}
P(X_i = 1, X_j = -1) &= P(X_i = -1, X_j = 1) = \frac{1}{2} \quad i \neq j \\
\langle X_1 X_2 \rangle &= -1 = \langle X_2 X_3 \rangle = \langle X_1 X_3 \rangle
\end{aligned}
$$

$$
M = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}
$$

Eigenvalues: $-2,\ 2,\ 2,\ 2 \Rightarrow M < 0$

Connection between positivity of moment matrix with the positive partial transpose of a 2-qubit density matrix

We consider a 2-qubit density matrix

$$\rho_{AB} = \frac{1}{4}\left[I \otimes I + (\vec{\sigma}\cdot\vec{r}) \times I + I \otimes (\vec{\sigma}\cdot\vec{s}) + \sum_{i,j=x,y,z} (\sigma_i \otimes \sigma_j)\, t_{ij}\right]$$

where $r_i = \mathrm{Tr}[\rho_{AB}(\sigma_i \otimes I)]$, $s_i = \mathrm{Tr}[\rho_{AB}\,(I \otimes \sigma_i)]$ and $t_{ij} = \mathrm{Tr}[\rho_{AB}(\sigma_i \otimes \sigma_j)]$ denote 15 parameters characterizing the 2-qubit density matrix.

When $r_i = s_i = 0$ and $t_{ij} = t_i\delta_{ij}$, we find that the eigen values of the density matrix are given by,

$$1 - t_1 + t_2 + t_3, \quad 1 + t_1 - t_2 + t_3,$$

$$1 + t_1 + t_2 - t_3, \quad 1 - t_1 - t_2 - t_3.$$

In view of the fact that $-1 \leq t_i \leq 1$ and under partial transpose, $t_i \to -t_i$, we have the eigenvalues of the partially transposed density matrix to be

$$1 + t_1 - t_2 - t_3, \quad 1 - t_1 + t_2 - t_3,$$

$$1 - t_1 - t_2 + t_3, \quad 1 + t_1 + t_2 + t_3.$$

It is readily seen that the eigen values of the moment matrix and that of the partially transposed density matrix match identically if we make an association

$$a \to t_1, \quad b \to t_2, \quad c \to t_3.$$

$$\Downarrow$$

Positivity of the moment matrix is *e*quivalent to the positivity of the partially transposed density matrix.

- Positivity of the partially transposed density matrix implies that the two qubit density matrix is separable i.e.,

$$\rho_{AB} = \sum_{\lambda} p_\lambda \, (\rho_{A\lambda} \otimes \rho_{B\lambda}).$$

- Positivity of the moment matrix thus implies that the two variable correlations can be expressed as

$$\begin{aligned} a &= \sum_{\lambda} p_\lambda \, \mathrm{Tr}[\rho_{A\lambda}\sigma_x] \, \mathrm{Tr}[\rho_{B\lambda}\sigma_x] \\ &= \sum_{\lambda} p_\lambda \, \mathrm{Tr}\left[\rho_{A\lambda}\{\sum_{m_1=\pm 1} m_1 \, \Pi^{(x)}_{m_1}\}\right] \, \mathrm{Tr}\left[\rho_{B\lambda}\{\sum_{m_2=\pm 1} m_2 \, \Pi^{(x)}_{m_2}\}\right] \\ &= \sum_{m_1, m_2=\pm 1} P^{(x)}(m_1, m_2) \, m_1 \, m_2 \end{aligned}$$

Here $\Pi^{(x)}_{m_1} = \dfrac{1}{2}\,[I + m_1\,\sigma_x], \quad \Pi^{(x)}_{m_2} = \dfrac{1}{2}\,[I + m_2\,\sigma_x]$

and $P^{(x)}(m_1, m_2) = \sum_\lambda p_\lambda\, P^{(x)}_\lambda(m_1) Q^{(x)}_\lambda(m_2)$ with

$$P^{(x)}_\lambda(m_1) = \mathrm{Tr}[\rho_{A\lambda}\sigma_x], \qquad Q^{(x)}_\lambda(m_2) = \mathrm{Tr}[\rho_{B\lambda}\sigma_x]$$

- **We find a connection between positivity of the moment matrix and that of a *partially transposed two qubit density matrix*.**
- **Positivity of partial transpose criterion comes to help now – and it ascertains that *admissiblity of a joint probability distribution with the given sequence (moment matrix positivity) is ensured if and only if the associated two qubit density matrix is separable*.**
- **This in turn leads to our identification that the given set of *moments* should necessarily allow a convex product decomposition of the joint probabilities, so as to be declared as a physically valid sequence of moments.**

Entropic uncertainty relations

U*ncertainty* relation for any two non-commuting observables **A** and **B** i.e.,

$$(\Delta \boldsymbol{X})_{\rho}\,(\Delta \boldsymbol{Z})_{\rho} \geq |\langle[\boldsymbol{X}, \boldsymbol{Z}]\rangle|/2$$

W. Heisenberg, Z. Phys. 43, 172 (1927); E. H. Kennard, Zeitschr. Phys. **44** 326 (1927); H. P. Robertson, Phys. Rev. 34, 163 (1929)

- Uncertainty relation constraining the product of standard deviations suffers from the drawback that the right hand side depends on the quantum state. In the specific example of a state ρ prepared in an eigenstate of X, the standard deviation $(\Delta X)_\rho$ as well as the commutator $|\langle [X, Z]\rangle_\rho|$ vanish and in turn, the uncertainty relation doesn't reveal any constraint on the *spread* $(\Delta Z)_\rho$ of the observable Z.

- It has been identified subsequently that Shannon entropies of the probabilities of measurement outcomes of the observables X, Z given by, $H_\rho(X) = -\sum_x P(x)\log_2 P(x)$, $H_\rho(Z) = -\sum_z P(z)\log_2 P(z)$ offer a more general framework to quantify the *intrinsic ignorance* associated with incompatible measurements.

x, z are the measurement outcomes of the observable X, Z and $P(x) = \langle x|\rho|x\rangle$, $P(z) = \langle z|\rho|z\rangle$ denote the probability of outcomes x, z; $\{|x\rangle\}$ $(\{|z\rangle\})$ is the set of eigenvectors of X (Z).

- Trade-off between the entropies of a pair of discrete non-commuting observables X and Z was formulated by Deutsch (Phys. Rev. Lett. 50, 631 (1983)) and was subsequently improved.

- The conjecture put forth by Kraus (Phys. Rev. D 35, 3070 (1987)) was proved by Maassen and Uffink (Phys. Rev. Lett. 60, 1103-1106 (1988)):

$$H_\rho(X) + H_\rho(Z) \geq -2\log_2 C(X,Z)$$

where $C(X,Z) = \max_{x,z}| < x|z > |$.

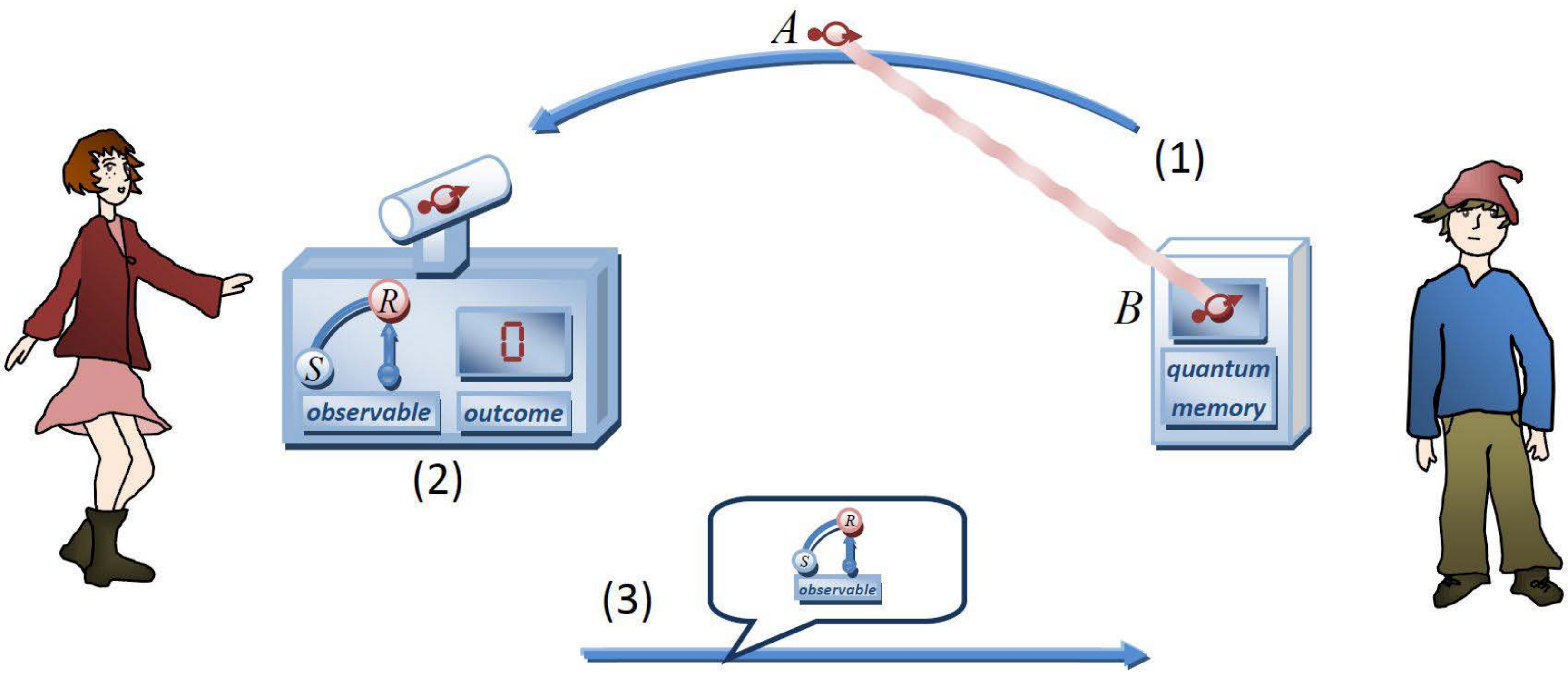
- The lower bound limiting the sum of entropies is independent of the state ρ.

- The term $C(X,Z)$ can assume a maximum value $\frac{1}{\sqrt{d}}$ resulting in the maximum entropic bound of $\log_2 d$, where d denotes the dimension of the system.

Extension of entropic uncertainty relation assisted by the presence of a quantum memory (Berta et al., Nature Physics 6, 659(2010) refined the lower bound. Here an observer Bob, whose task is to minimize the uncertainty of Alice's measurement of the observables $X,\ Z$, is allowed to share an entangled quantum state ρ_{AB} with that in Alices possession.

A Quantum Game

. Berta, M. Christandl, R. Colbeck, J. M. Renes, and R. Renner, Nature Physics **6**, 659(2010)



entangled with his quantum memory.

(2) Alice measures either R or S and notes her outcome.

(3) Alice announces her measurement choice to Bob.

Berta et. al EUR

- The uncertainty principle, when Bob possesses a quantum memory, is given by

BERTA et.al EUR:
$S(X|B) + S(Z|B) \geq -2\log_2 C(X,Z) + S(A|B)$

where $S(X|B)$ & $S(Z|B)$ are the conditional von Neumann entropies of the post measured states and $S(A|B)$ is the conditional von Neumann entropy of the state $\boldsymbol{\rho_{AB}}$.

- ***S(A|B) can assume negative values when the state ρ_{AB} is entangled***

Berta et. al EUR

- When Alice's system is in a maximally entangled state with Bob's quantum memory, **$S(A|B) = -\log_2 d$** and as **$-2\log_2 C(X,Z) \le \log_2 d$** one can achieve a trivial lower bound of **zero**. Thus, with the help of a quantum memory maximally entangled with Alice's state, Bob can beat the uncertainty bound and can predict the outcomes of incompatible observables X, Z precisely.

Two Experiments

Singlet state

- Alice and Bob share a Singlet state(maximally entangled)
- Measuring the spins at both ends, ask what's $P(m_a,m_b)$?
- $P(m_a,m_b) =$

$[1 + m_a m_b \mathrm{Cos}(\theta_{ab})]/4$

where θ_{ab} is the angle between

the spin directions **a** and **b**

Rotor in a maximally mixed state

- Consider a spin-1/2 system in a random mixture state i.e, $\boldsymbol{\rho}=\mathbf{I}/2$ (I denotes 2×2 identity matrix) evolving under a hamiltonian
- Make measurements at time t_1 and t_2. Ask what's $P(m_1,m_2)$?
- $P(m_1,m_2) = [1 + m_1 m_2 \mathrm{Cos}(\theta_{12})]/4$

where θ_{12} is the temporal difference $(t_2 - t_1)$

A. R. Usha Devi, H. S. Karthik, Sudha, and A. K. Rajagopal, Phys. Rev. A 87, 052103 (2013)

We ask…….

- ***Analogous to spatial correlations, do temporal correlations arising in sequential measurement of observables, play a distinct role in reducing the uncertainty of incompatible observables?***

QUESTION:
Is $H(X|X_o) + H(Z|Z_o) \leq -2\log_2 C(X,Z)$ always?

where X_o and Z_o are observables measured earlier to that of X and Z respectively.

Temporal correlations arising in sequential measurement of observables too play a distinct role in reducing the uncertainty of incompatible observables

***Theorem**: If temporal correlations of the outcomes of X_o, X and those of Z_o, Z obtained from sequential measurement runs on the quantum state are classical (the correlation probabilities are of the convex product form), the sum of conditional entropies obey the inequality*

$$H(X|X_o) + H(Z|Z_o) \geq -2\log_2 C(X,Z)$$

Karthik et al., arXiv eprint:1310.5079

Temporal correlation between the sequential outcomes x_0 and x of the observables X_0, X is iff the joint probabilities $P(x_0, x)$ can be expressed as a convex combination of products of probabilities,

$$\begin{aligned} P(x_0, x) &= \sum_\lambda p_\lambda P_\lambda(x_0) Q_\lambda(x), \\ \sum_{x_0} P_\lambda(x_0) &= 1, \ \sum_x Q_\lambda(x) = 1 \\ \sum_\lambda p_\lambda &= 1, \ \ 0 \leq p_\lambda \leq 1. \end{aligned}$$

- **Conditional information for the measurement outcomes of the observable X, given that in a prior measurement X_0 has taken the value x_0:**

$$H_\rho(X|X_0 = x_0) = -\sum_x P(x|x_0)\,\log_2 P(x|x_0)$$

- **The conditional probability $P(x|x_0) = P(x_0, x)/P(x_0)$ corresponding to *classical* temporal correlations is given by,**

$$\begin{aligned} P(x|x_0) &= \frac{\sum_\lambda p_\lambda\, P_\lambda(x_0)\, Q_\lambda(x)}{\sum_{\lambda'} p_{\lambda'}\, P_{\lambda'}(x_0)} \\ &= \sum_\lambda p_{\lambda,x_0}\, Q_\lambda(x) \end{aligned}$$

where

$$p_{\lambda,x_0} = \frac{p_\lambda\, P_\lambda(x_0)}{\sum_{\lambda'} p_{\lambda'}\, P_{\lambda'}(x_0)}$$

(Note that $\sum_\lambda p_{\lambda,x_0} = 1$, and $0 \le p_{\lambda,x_0} \le 1$.)

- **Conside the relative entropy $D(\mathcal{P}_{x_0}||\mathcal{Q}_{x_0})$ of the probabilitiy distributions $\mathcal{P}_{x_0}(\lambda, x) = p_{\lambda,x_0}\, Q_\lambda(x)$ and $\mathcal{Q}_{x_0}(\lambda, x) = p_{\lambda,x_0}\, P(x|x_0)$. Positivity of the relative entropy implies**

$$
\begin{aligned}
D(\mathcal{P}_{x_0}||\mathcal{Q}_{x_0}) &= \sum_\lambda \sum_x p_{\lambda,x_0}\, Q_\lambda(x) \log_2 \left[\frac{Q_\lambda(x)}{P(x|x_0)}\right] \geq 0 \\
&\Rightarrow H_\rho(X|x_0) \geq \sum_\lambda p_{\lambda,x_0}\; H_\rho^{(\lambda)}(X)
\end{aligned}
$$

where $H_\rho^{(\lambda)}(X) = -\sum_x Q_\lambda(x) \log_2 Q_\lambda(x)$.

Thus, the average conditional information

$$H_\rho(X|X_0) = -\sum_{x_0} p(x_0)\, H_\rho(X|x_0); \;\; p(x_0) \; = \; \sum_{x} P(x, x_0) = \sum_{\lambda} p_\lambda \, P_\lambda(x_0)$$

should obey the constraint

$$\begin{aligned} H_\rho(X|X_0) \; & \geq \; \sum_{x_0} p(x_0) \sum_{\lambda} p_{\lambda, x_0} \, H_\rho^{(\lambda)}(X) \\ & \geq \; \sum_{\lambda} \, p_\lambda \, H_\rho^{(\lambda)}(X), \end{aligned}$$

Similarly, we obtain

$$H_\rho(Z|Z_0) \geq \sum_\lambda p_\lambda H_\rho^{(\lambda)}(Z).$$

Thus, the sum of conditional entropies are constrained by

$$\begin{aligned} H_\rho(X|X_0) + H_\rho(Z|Z_0) &\geq \sum_\lambda p_\lambda [H_\rho^{(\lambda)}(X) + H_\rho^{(\lambda)}(Z)] \\ &\geq \sum_\lambda p_\lambda \left[-2\log_2 c(X,Z)\right] \\ &= -2\log_2 c(X,Z). \end{aligned}$$

using

$$H_\rho^{(\lambda)}(X) + H_\rho^{(\lambda)}(Z) \geq -2\log_2 c(X,Z).$$

Example

- ***Temporal correlations assisting in reducing the entropic spread of non-commuting observables***
- Consider:

A spin s particle precessing about y axis:

Hamiltonian: $H = \hbar\omega S_y$

Initial State : highly mixed state:

$\rho_{in} = (I_{2s+1})/(2s+1)$

- Measurement of non-commuting observables $X = S_x$ and $Z = S_z$ results in the probabilities of outcomes

$-s \leq m_x, m_z \leq s$

Example

- Under the Hamiltonian dynamics, the evolution of z component of spin is given by

$S_z(t) = U^\dagger(t)S_z(0)U(t) = S_z\,\mathrm{Cos}(\omega t) + S_x\,\mathrm{Sin}(\omega t)$;

$U(t)=\exp(-i\omega t S_y)$.

- ***First run*** :

  Measure $S_z(t)$ at time t_{x0} and $t_x = \pi/2\omega$

  Call $S_z(t_{x0}) = X_0 = S_z\,\mathrm{Cos}(\omega t_{x0}) + S_x\,\mathrm{Sin}(\omega t_{x0})$ and

  $S_z(t_x) = X = S_x$

- Define $\theta = \boldsymbol{\omega t_{x0}} - \pi/2$ -----> dimensionless time separation.

Example

- The sequential measurements enables one to record the temporal correlation probabilities $P(m_{x0}, m_x; \theta)$ of the outcomes $-s \le m_{x0}, m_x \le s$ of the observables
  $X_0 = S_z(t_{x0})$ and $X = S_x$.
- ***Second run:***
  Measure $S_z(t)$ at time t_{z0} and $t_z = \pi/\omega$
  Call $S_z(t_{z0}) = Z_0 = S_z \, Cos(\omega t_{z0}) + S_x \, Sin(\omega t_{z0})$ and
  $S_z(t_z) = Z = S_z$
- Define $\phi = \boldsymbol{\omega t_{z0}}$ **-** π-----> dimensionless time sep.
- Similarly obtain $P(m_{z0}, m_z; \phi)$

Example

$$
\begin{aligned}
P(m_{x_0}, m_x; \theta) &= P(m_{x_0}; t_{x0})\, P(m_x; t_x | m_{x_0}; t_{x0}) \\
&= \mathrm{Tr}[\rho\, \Pi_{m_{x_0}}(t_{x0})]\, \frac{\mathrm{Tr}[\Pi_{m_{x_0}}(t_{x0}) \rho\, \Pi_{m_{x_0}}(t_{x0})\, \Pi_{m_x}(t_x)]}{P(m_{x_0}; t_{x0})} \\
&= \frac{1}{2s+1}\, \mathrm{Tr}[\Pi_{m_{x_0}}(t_{x0})\, \Pi_{m_x}(t_x)] \\
&= \frac{1}{2s+1}\, |\langle s, m_{x_0} | e^{-i\omega(t_{x0} - t_x)\, S_y} | s, m_x \rangle|^2 \\
&= \frac{1}{2s+1}\, |\, d^s_{m_x\, m_{x_0}}(\theta)|^2
\end{aligned}
$$

Example

- The conditional entropies of measurement (which depend only on the time separations θ, ϕ) $H_\rho(X|X_0) = \mathcal{H}(\theta)$ and $H_\rho(Z|Z_0) = \mathcal{H}(\phi)$

$$\mathcal{H}(\theta) = -\frac{1}{2s+1} \sum_{m_{x_0}, m_x} |d^s_{m_{x_0}, m_x}(\theta)|^2 \log_2 |d^s_{m_{x_0}, m_x}(\theta)|^2$$

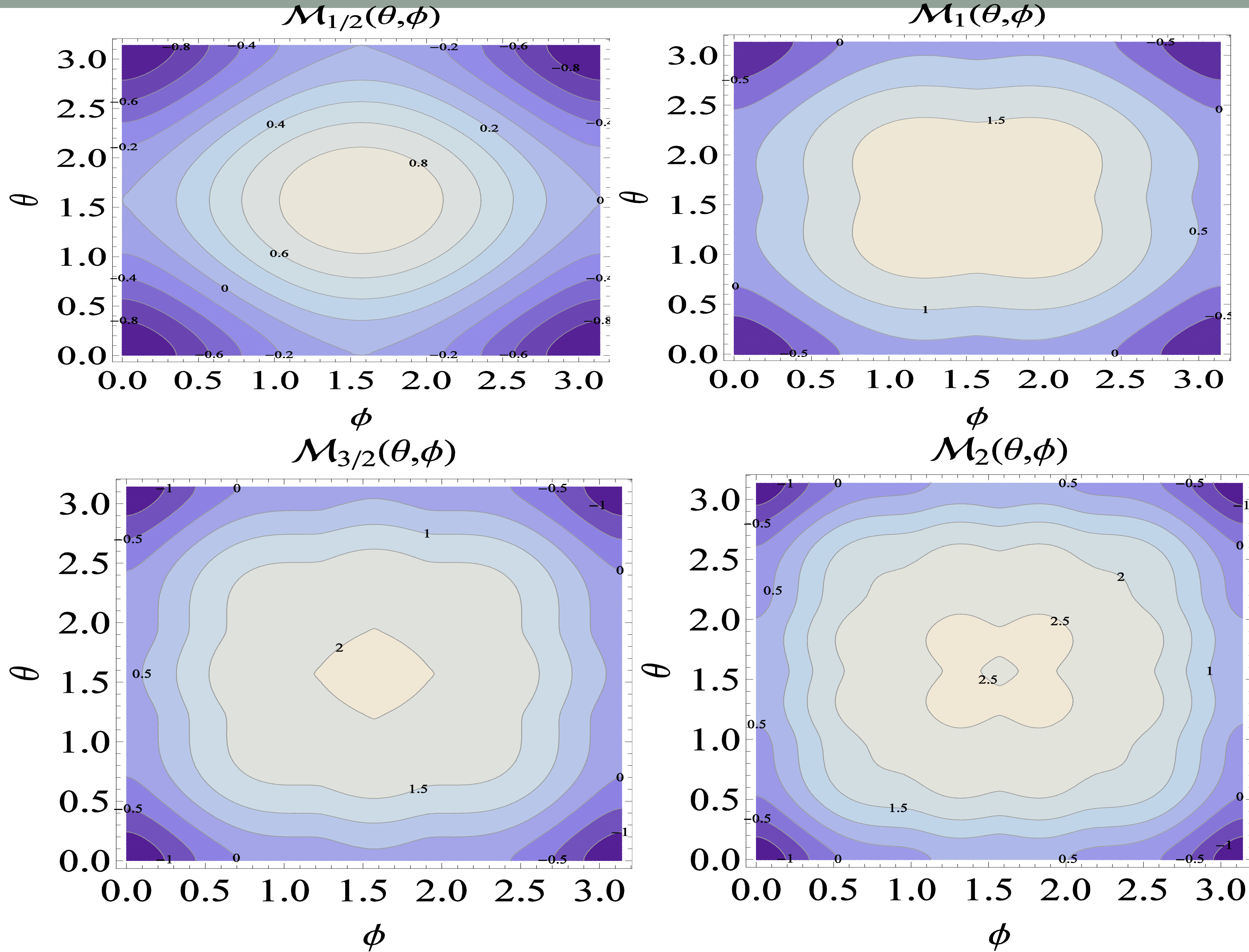
$$\mathcal{H}(\phi) = -\frac{1}{2s+1} \sum_{m_{z_0}, m_z} |d^s_{m_{z_0}, m_z}(\phi)|^2 \log_2 |d^s_{m_{z_0}, m_z}(\phi)|^2.$$

Example

- We define a quantity $\mathbf{M}_s(\theta,\phi)$ as the difference between the sum of conditional entropies and the Massen-Uffink uncertainty bound $-2 \log_2 c(X,Z)$

$$\mathbf{M}_s(\theta,\phi) = H_\rho(X|X_0) + H_\rho(Z|Z_0) + 2 \log_2 c(X,Z)$$
$$= \mathscr{H}(\theta) + \mathscr{H}(\phi) + 2 \log_2 c(X,Z)$$

in order to demonstrate improved precision in the measurement of the spin components X and Z



$\mathcal{M}_{1/2}(\theta,\phi)$
$\mathcal{M}_{1}(\theta,\phi)$
$\mathcal{M}_{3/2}(\theta,\phi)$
$\mathcal{M}_{2}(\theta,\phi)$
θ
ϕ
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Contextuality and entropic uncertainty

Given three observables X_1, X_2, X_3 where in co-measurability of X_1, X_2 and X_1, X_3 is ensured i.e., $[X_1, X_2] = [X_1, X_3] = 0$ but $[X_2, X_3] \neq 0$, we explore the trade-off between the Shannon entropies of the non-commuting observables X_2 and X_3, both of which are conditioned with the measurement outcomes of the observable X_1

QUESTION: Is $H(X_2|X_1) + H(X_3|X_1) \leq -2\log_2 C(X_2,X_3)$ always?

***Theorem**: If the outcomes of X_1 do not depend on the context of measuring it with X_2 or X_3, there follows a "Contextual" entropic steering inequality*

$$H(X_2|X_1) + H(X_3|X_1) \geq -2\log_2 C(X_2,X_3)$$

This identification(theorem) reveals the crucial significance of *Quantum Contextuality* to achieve sharpened predictions of incompatible observables which indeed is counter intuitive!!

EXAMPLE: Contextuality of X_1 *assisting in reducing the entropic spread of non-commuting observables* X_2 and X_3 *.*

Consider three of the KCBS dichotomic observables $X_i = 2|v_i\rangle\langle v_i| - I$ with outcomes ± 1; $|v_1\rangle = (0,0,1)$; $|v_2\rangle = (\sin\theta, \cos\theta, 0)$; $|v_3\rangle = (1,0,0)$ in the quantum state $|\psi\rangle = (1/\sqrt{(1+\sin^2\alpha)})(\sin\alpha, \cos\alpha, \sin\alpha)$

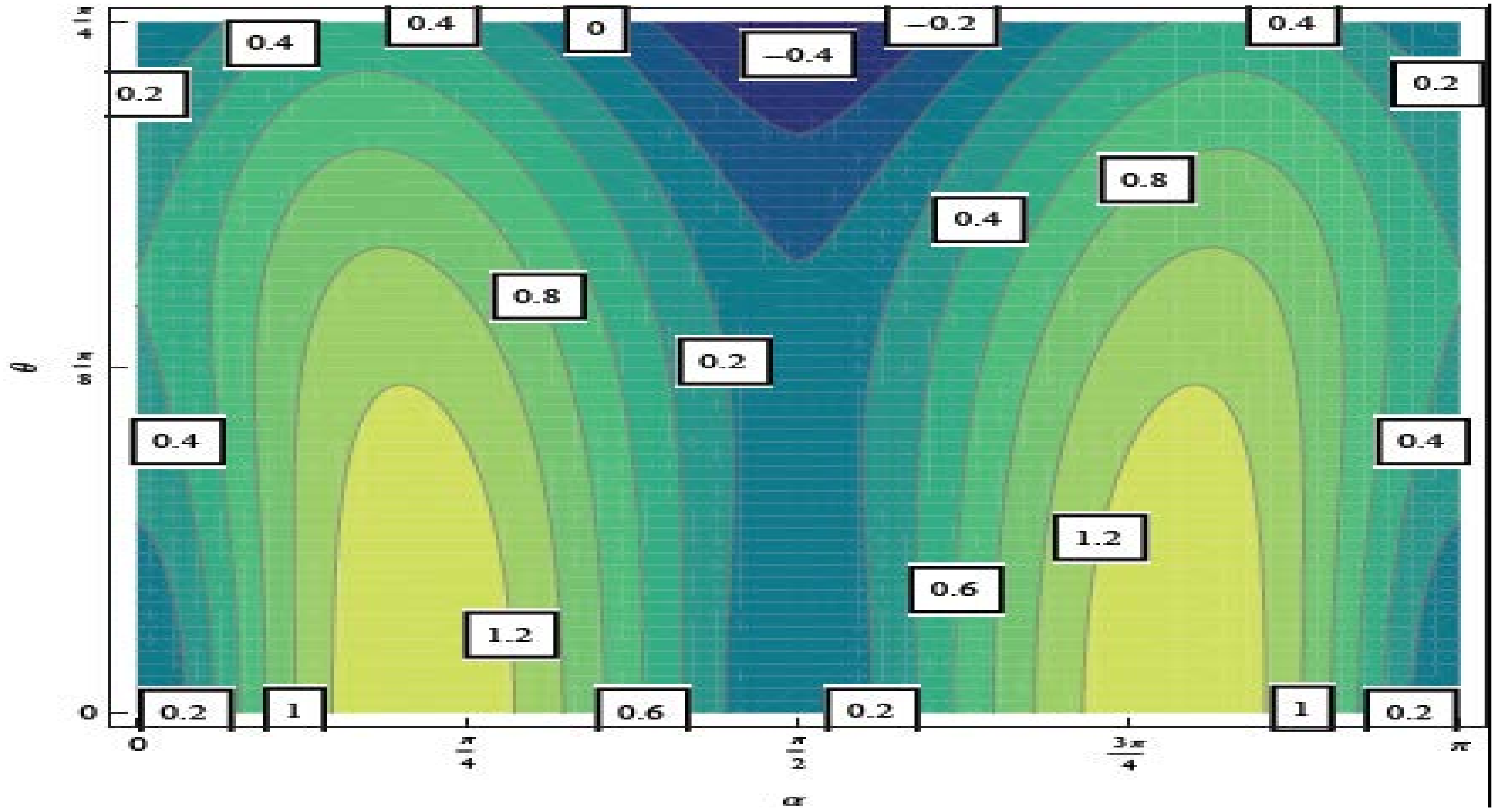
(PRL 101, 020403 (2008))

Probabilities			
	$P(X_1, X_2)$		$P(X_1, X_3)$
P(1,1)	0	P(1,1)	0
P(1,-1)	$(1/(1+\sin^2\alpha))\sin^2\alpha$	P(1,-1)	$(1/(1+\sin^2\alpha))\sin^2\alpha$
P(-1,1)	$(1/\sqrt{(1+\sin^2\alpha)})(\cos^2\alpha\cos^2\theta + \sin^2\alpha\sin^2\theta)$	P(-1,1)	$(1/(1+\sin^2\alpha))\sin^2\alpha$
P(-1,-1)	$(1/\sqrt{(1+\sin^2\alpha)})(\sin^2\alpha\cos^2\theta + \cos^2\alpha\sin^2\theta)$	P(-1,-1)	$(1/(1+\sin^2\alpha))\cos^2\alpha$

We define a quantity M(θ, α) as the difference between the sum of conditional entropies and the Massen-Uffink uncertainty bound $-2\log_2 c(X_2,X_3)$:

$$M(\theta,\alpha) = H(X_2|X_1) + H(X_3|X_1) + 2\log_2 c(X_2,X_3)$$

to demonstrate improved precision in the measurement of the non-commuting observables X_2, X_3.



The bound limiting the trade-off is smaller than that given by the Massen-Uffink uncertainty relation. This clearly brings out an instance to reveal that contextuality of the observable X_1 assists in enhancing the precision of measuring non-commuting observables X_2 and X_3. This is essentially because of the *the non-existence of the joint probability distribution* for all the three observables – unlike in the non-contextual theory.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"WHAT'S COME OVER HEISENBERG? HE SEEMS TO BE CERTAIN ABOUT EVERYTHING THESE DAYS."

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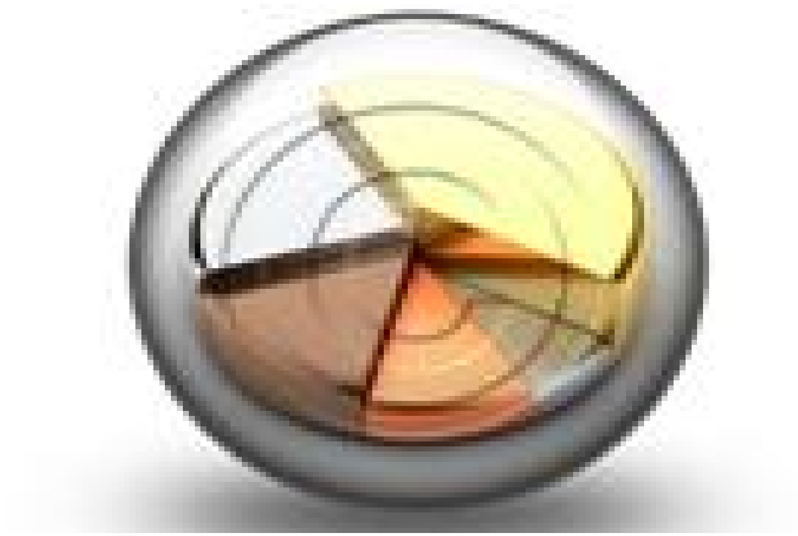

Thank you

International Tech Park, Bengaluru
"Silicon Valley of India"

Lalbagh garden, Bengaluru

Gopuram sculpture

Bull Temple

THE ART OF LIVING

Art of Living Spiritual Foundation