# Uncertainty and Speed of Evolution in Presence of Quantum Correlation

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# *Outline*

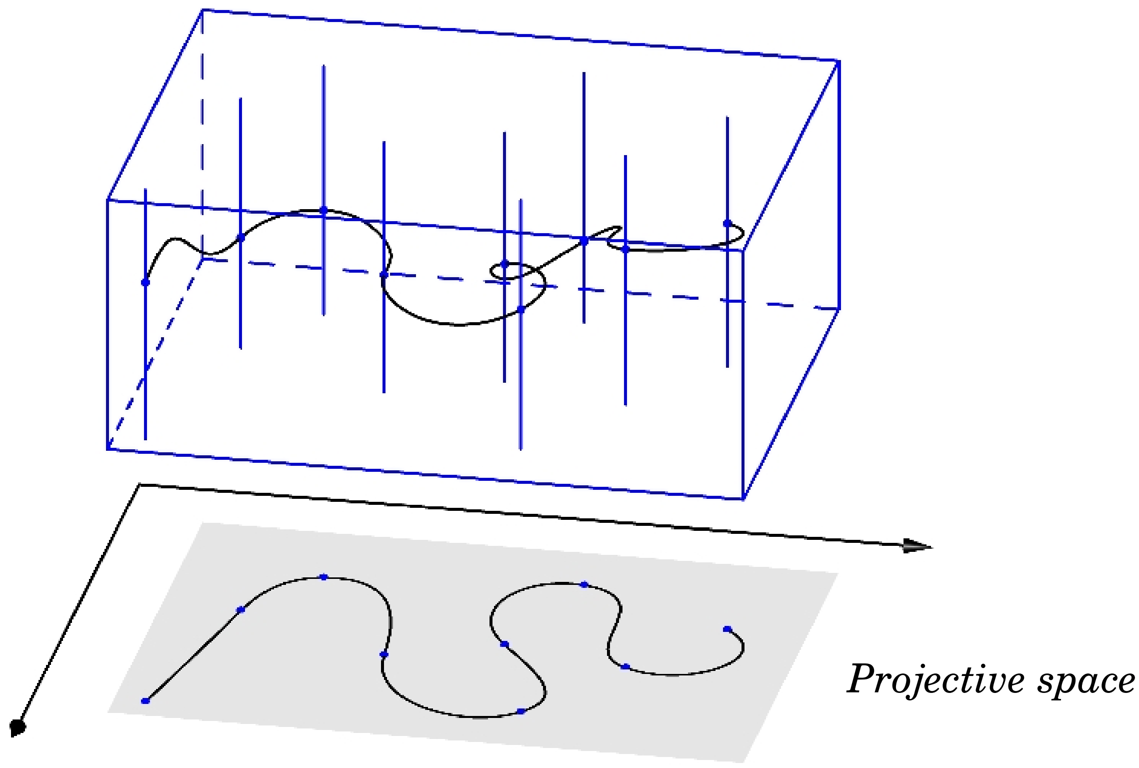
- *Introduction*
- *Quantum uncertainty and correlation*
  - *Geometric quantum uncertainty relation (GQUR)*
  - *Multipartite entanglement: geometric*
  - *GQUR and multiparty entanglement: applications*
  - *Mixed states*
- *Quantum speed and correlation*
  - *Local unitary evolution and quantum discord*
  - *Examples*
- *Conclusion*

# *Time-energy uncertainty relation*

# *Quantum geometry*

- *The projective Hilbert space*

$$\Pi : |\Psi\rangle \rightarrow |\Psi\rangle\langle\Psi|$$




*Complex projective Hilbert space may be given a natural metric, the Fubini-Study metric.*

# *Quantum geometry*

- *The projective Hilbert space*

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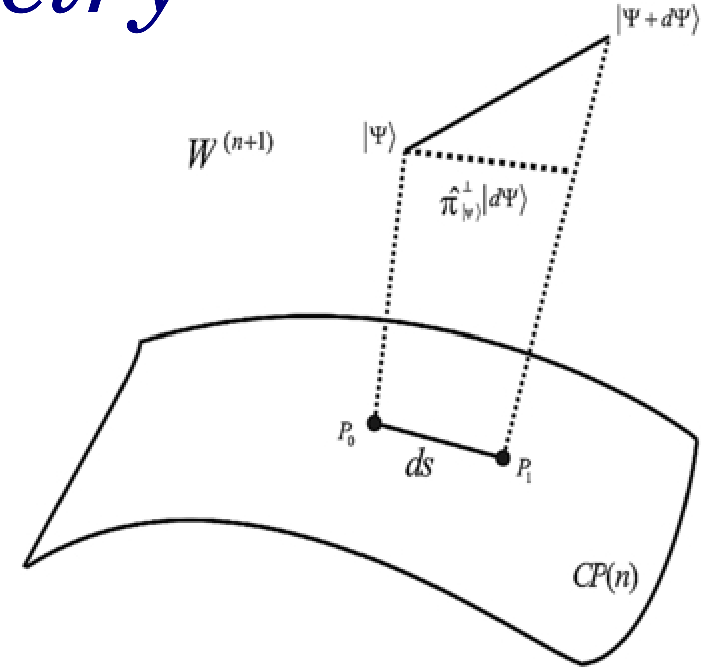


- *Fubini-Study (FS) metric:*

$$\begin{aligned} dS^2 &= 4\left(1 - |\langle\Psi(\bar{\lambda} + d\bar{\lambda})|\Psi(\bar{\lambda})\rangle|^2\right) \\ &= 4\left(\langle\partial_i\Psi|\partial_j\Psi\rangle - \langle\partial_i\Psi|\Psi\rangle\langle\Psi|\partial_j\Psi\rangle\right) d\lambda^i d\lambda^j \end{aligned}$$

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- *The distance, also called the Bargmann angle*

$$|\langle\Psi_1|\Psi_2\rangle|^2 = \cos^2\left(\frac{\mathcal{S}}{2}\right)$$

# *Geometry of quantum evolution*

# *Quantum dynamics*

- *Quantum state evolves following the Schrodinger equation*

$$i\hbar|\dot{\Psi}(t)\rangle = H(t)|\Psi(t)\rangle$$

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- *After an infinitesimal time translation*

$$|\Psi(t)\rangle \longrightarrow |\Psi(t+dt)\rangle$$

- *Infinitesimal distance using FS metric*

$$dS^2 = 4\left(1 - |\langle\Psi(t+dt)|\Psi(t)\rangle|^2\right)$$

*where*

$$|\Psi(t+dt)\rangle = |\Psi(t)\rangle + |\dot{\Psi}(t)\rangle dt + \frac{1}{2}|\ddot{\Psi}(t)\rangle dt^2 + \ldots$$

# *Quantum dynamics: distance...*

$$dS = \frac{2}{\hbar}\Delta H(t)dt$$

*where* $\Delta H(t)^2 = \langle\Psi(t)|H(t)^2|\Psi(t)\rangle - \langle\Psi(t)|H(t)|\Psi(t)\rangle^2$

*J. Anandan and Y. Aharonov (1990)*

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*Energy fluctuation of the state drives the quantum evolution !!*
*Distance traverse is directly proportional to the energy uncertainty present in the system.*

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*time-dep Hamiltonian*

$$S = \frac{2}{\hbar}\int_0^\tau \Delta H(t)dt$$

*time-indep Hamiltonian*

$$S = \frac{2}{\hbar}\tau\ \Delta H$$

*J. Anandan and Y. Aharonov (1990)*

# *Quantum dynamics: speed…*

$$dS = \frac{2}{\hbar}\Delta H(t)dt$$

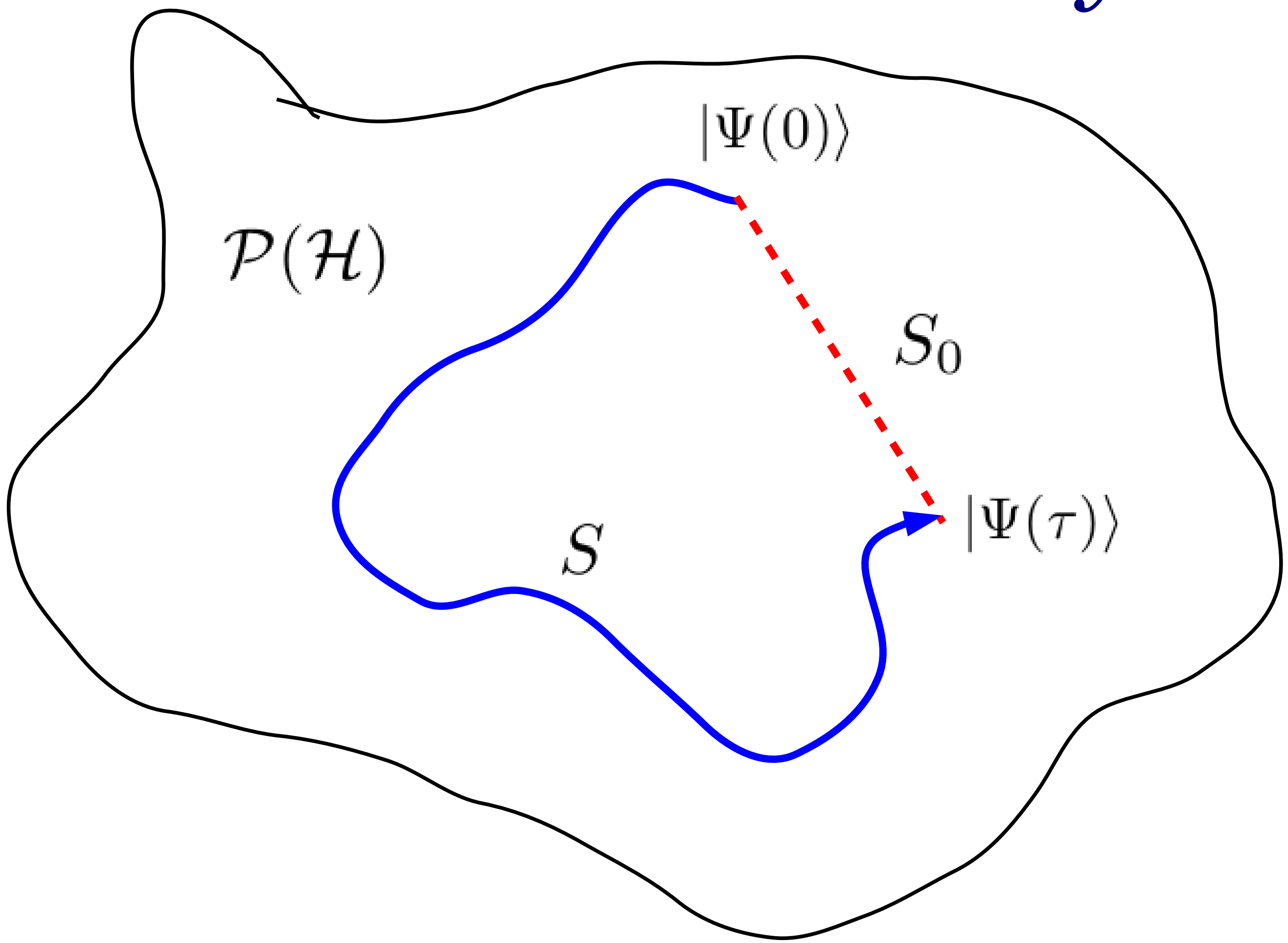
$$\Downarrow$$

$$\frac{dS}{dt} = \frac{2\Delta H(t)}{\hbar}$$

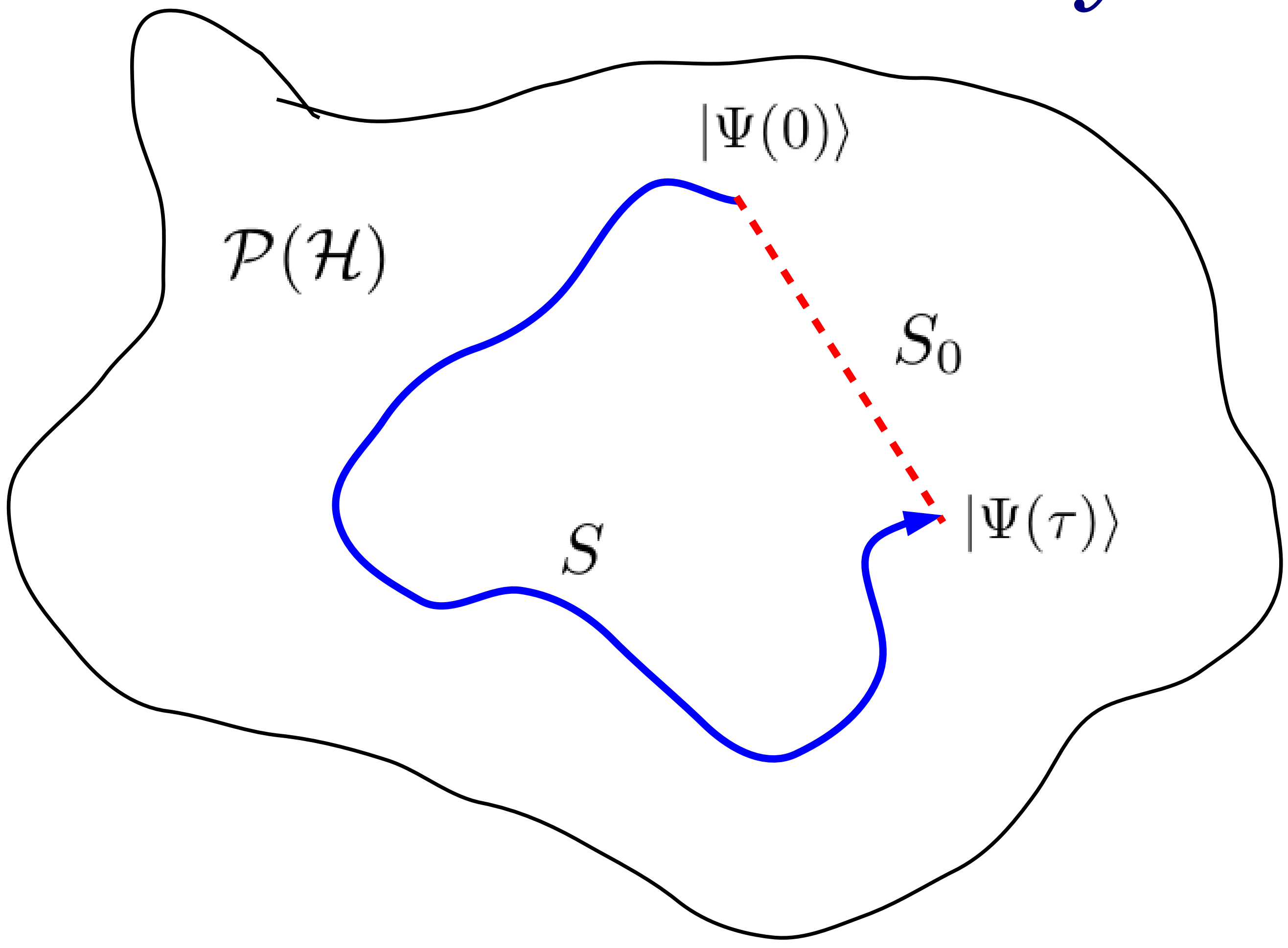
*Infinitely many Hamiltonians can be used to transport the same initial state to the same final state.*

*J. Anandan and Y. Aharonov (1990)*

# *Geometry ...*

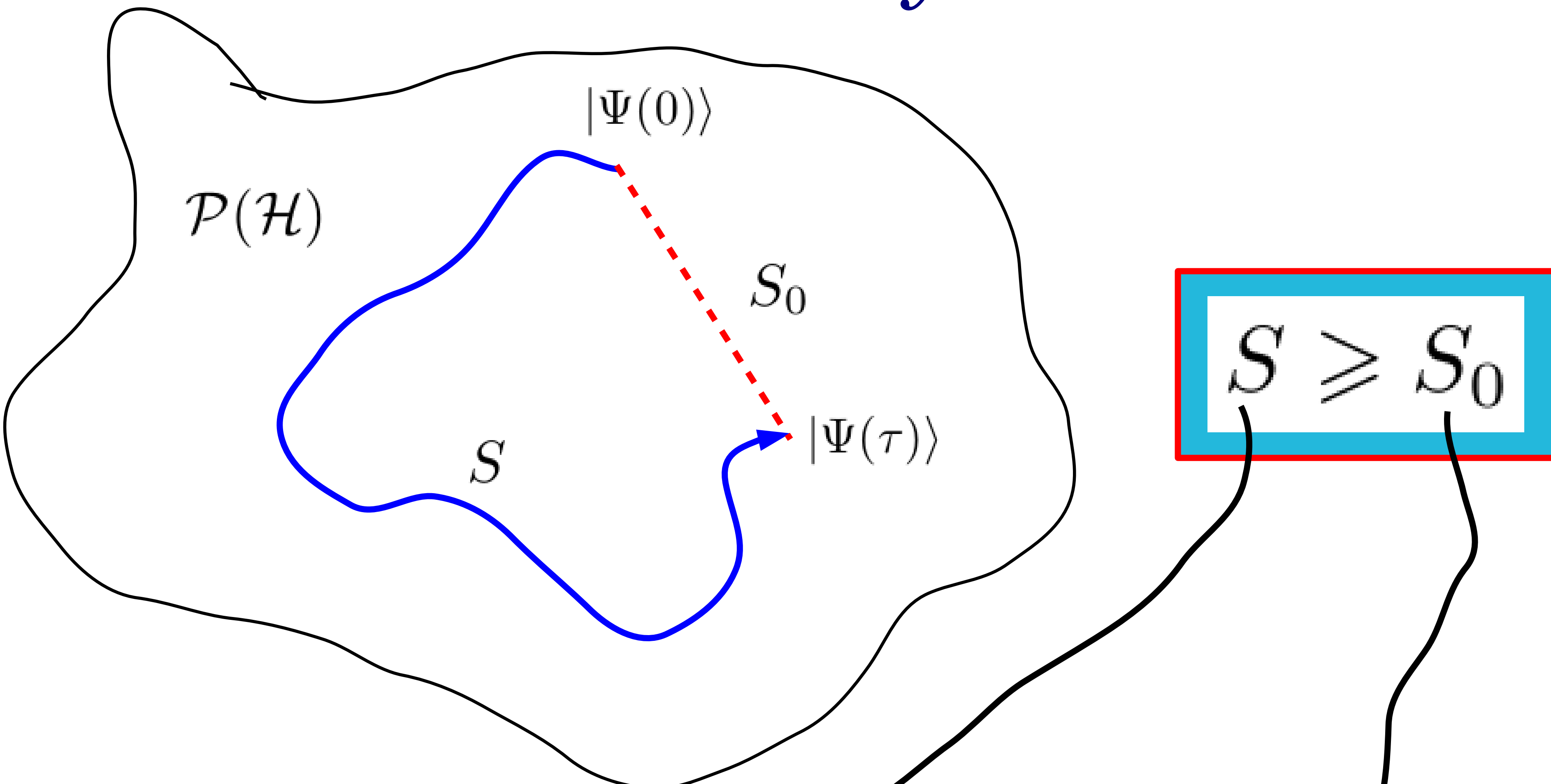


# *Geometry ...*




$$S \geqslant S_0$$

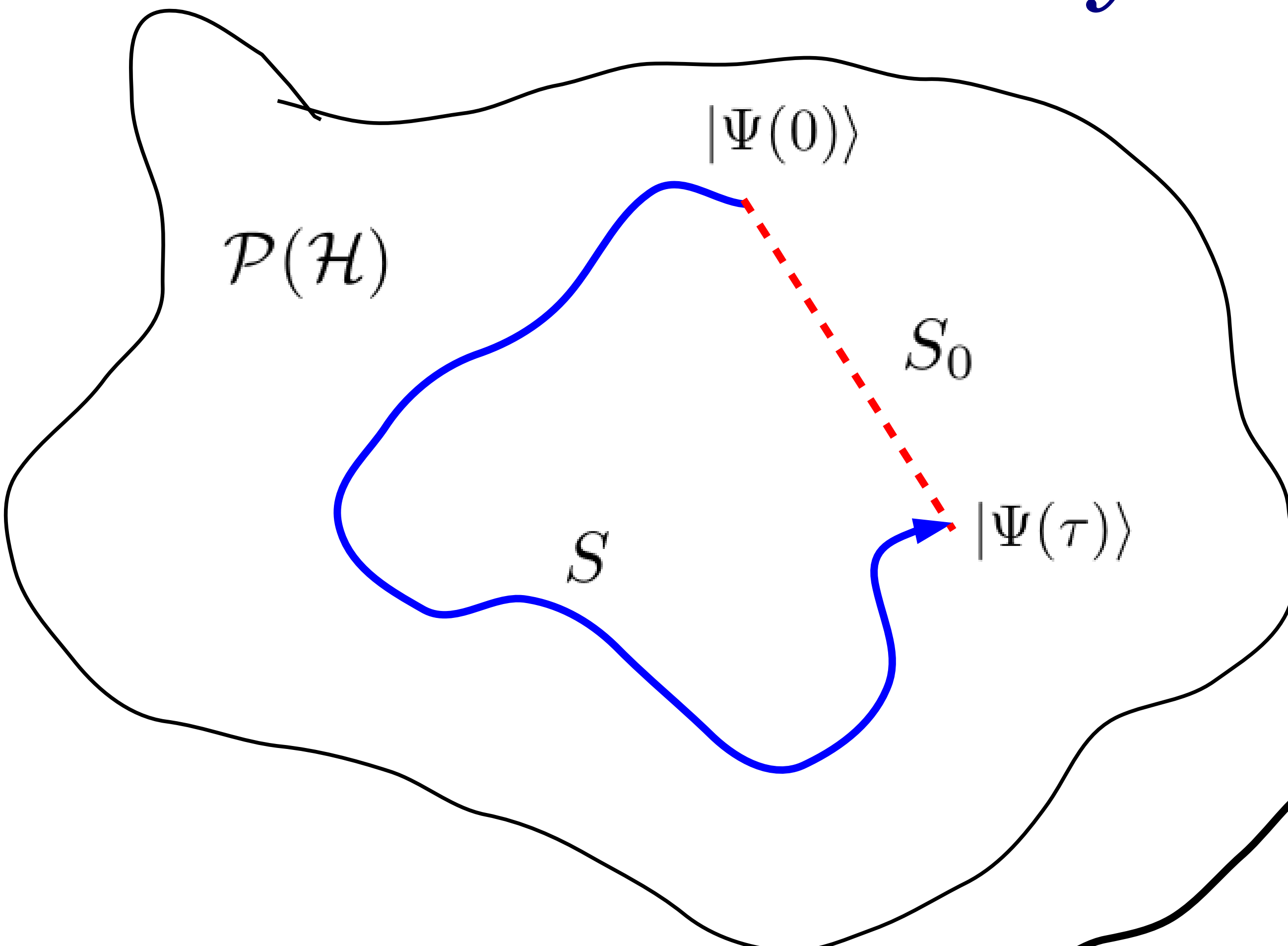
# *Geometry ...*




*Again...*

$$S = \frac{2}{\hbar}\int_0^{\tau} \Delta H(t)dt \quad \textit{and} \qquad S_0 = 2\,\cos\left(\langle\psi(0)|\psi(t)\rangle\right)$$

# *Geometry ...*




$$S \geqslant S_0$$

$$\int_0^\tau \Delta H(t)dt \geqslant \hbar \cos^{-1}(|\langle\Psi(0)|\Psi(\tau)\rangle|)$$

# *Geometric Quantum Uncertainty Relation*

$$\int_0^\tau \Delta H(t)dt \geqslant \hbar \cos^{-1}(|\langle\Psi(0)|\Psi(\tau)\rangle|)$$

- *For a time-independent H*

$$\tau \Delta H \geqslant \hbar \cos^{-1}(|\langle\Psi(0)|\Psi(\tau)\rangle|)$$

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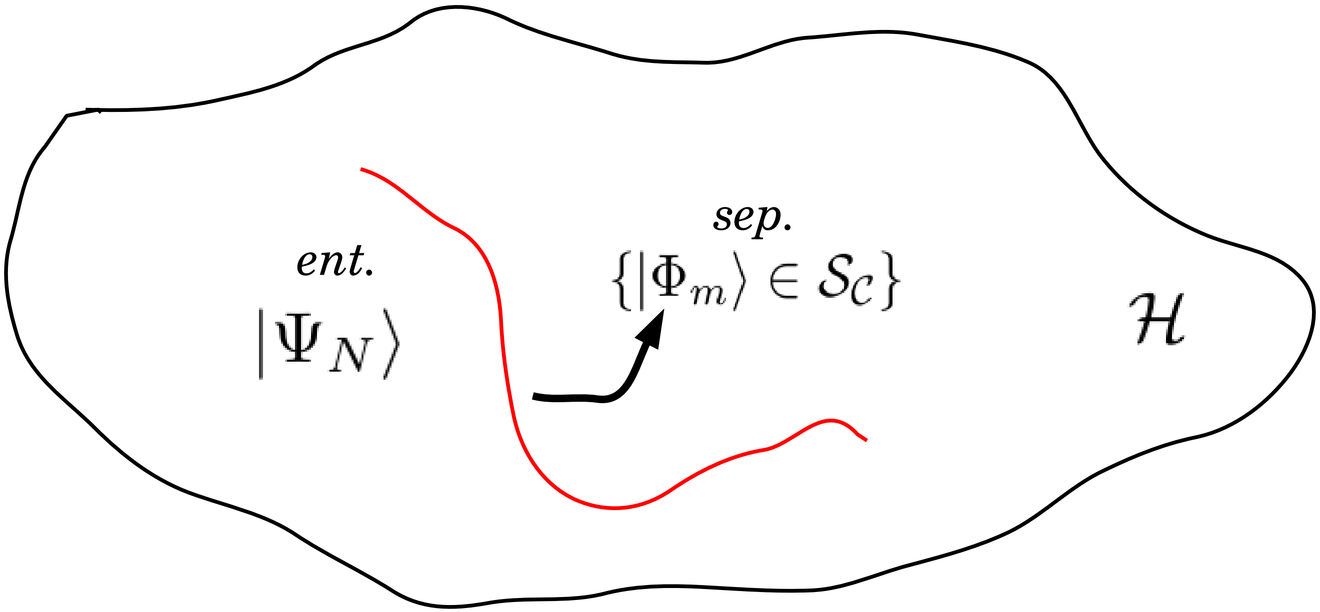
- *When the initial and the final states are orthogonal to each other, the above raltion is the celebrated A-A time-energy uncertainty relation!!*

$$\tau\Delta H \geqslant \frac{h}{4}$$

*J. Anandan and Y. Aharonov (1990)*

# *Multipartite entanglement*

# *Multiparty entanglement: geometric*




- *The Geometric measure of entanglement (GME)*

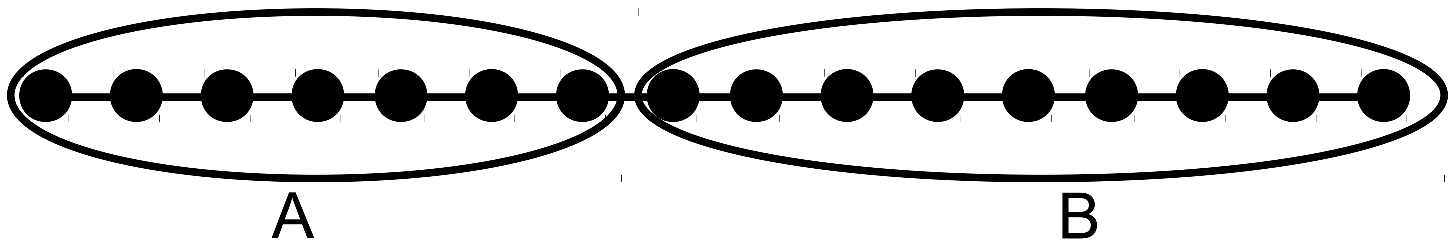
$$\mathcal{E}_{\mathcal{C}}(|\Psi_N\rangle) = \min_{\{|\Phi_m\rangle \in S_{\mathcal{C}}\}} \left(1 - |\langle \Phi_m | \Psi_N \rangle|^2\right)$$

*The distance is the measure of entanglement where the minimization is carried out over all pure states that are m-separable states.*

*Wei, Goldbart (2003)*

# *Genuine multiparty entanglement (GGM)*

- *A multiparty pure quantum state is said to be genuinely multiparty entangled if it is entangled across every bipartition of its constituent parties.*




*A. Sen(De) and U. Sen (2010)*

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- *GGM: generalized geometric measure*

$$\mathcal{E}_{\mathcal{G}}(|\Psi_N\rangle) = \min_{\{|\Phi_m\rangle \in \mathcal{S}_{\mathcal{B}}\}} \left(1 - |\langle \Phi_m|\Psi_N\rangle|^2\right)$$

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- *For pure states, it can be calculated, analytically, for arbitrary number of parties and dimensions.*

*A. Sen(De) and U. Sen (2010)*

# *Geometric entanglement & GQUR*

*From GQUR* ⇨

$$\int_0^\tau \Delta H(t)dt \geqslant \hbar \cos^{-1}(|\langle\Psi(0)|\Psi(\tau)\rangle|)$$

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⇐ *From GME*

- *By definition* $|\langle\Psi(0)|\Psi(\tau)\rangle|^2 \leqslant 1 - \mathcal{E}_\mathcal{C}\left(|\Psi(\tau)\rangle\right)$

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*From GQUR* ⇒

$$\int_0^{\tau} \Delta H(t)dt \geqslant \hbar \cos^{-1}(|\langle\Psi(0)|\Psi(\tau)\rangle|)$$

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⇐ *From GME*

- *By definition* $|\langle\Psi(0)|\Psi(\tau)\rangle|^2 \leqslant 1 - \mathcal{E}_{\mathcal{C}}(|\Psi(\tau)\rangle)$

- *We may write* $\cos^{-1}(|\langle\Psi(0)|\Psi(\tau)\rangle|) \geqslant \mathcal{G}(\mathcal{E}_{\mathcal{C}})$

*where another entanglement measure*

$$\mathcal{G}(\mathcal{E}_{\mathcal{C}}) = \cos^{-1}\sqrt{1 - \mathcal{E}_{\mathcal{C}}}$$

# Geometric entanglement & GQUR

*From GQUR* ⇒

$$\int_0^{\tau} \Delta H(t)dt \geqslant \hbar \cos^{-1}(|\langle\Psi(0)|\Psi(\tau)\rangle|)$$

$$\mathcal{E}_{\mathcal{C}}(|\Psi_N\rangle) = \min_{\{|\Phi_m\rangle \in S_{\mathcal{C}}\}} \left(1 - |\langle\Phi_m|\Psi_N\rangle|^2\right)$$

⇐ *From GME*

- *Now we have...*

$$\int_0^{\tau} \Delta H dt \geqslant \hbar\, \mathcal{G}\,(\mathcal{E}_{\mathcal{C}})$$

# *Geometric entanglement & GQUR*

- *In terms of the time averaged energy fluctuation*

$$\boxed{\overline{\Delta H}\tau \geqslant \hbar\, \mathcal{G}\left(\mathcal{E}_{\mathcal{C}}\right)}$$

*where* $\overline{\Delta H} = \frac{1}{\tau}\int_0^\tau \Delta H(t)dt$

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*where* $$\overline{\Delta H} = \frac{1}{\tau}\int_0^{\tau} \Delta H(t)dt$$

*For an arbitrary quantum evolution, the time interval multiplied by the time-averaged energy fluctuation is bounded below by the geometric measure of multipartite entanglement, provided the initial state is unentangled.*

- *Application:*
  *Ising Hamiltonian*

# *Evolution with Ising Hamiltonian*

$$|\Psi(0)\rangle = |\varphi\rangle^{\otimes 8}$$

*where*

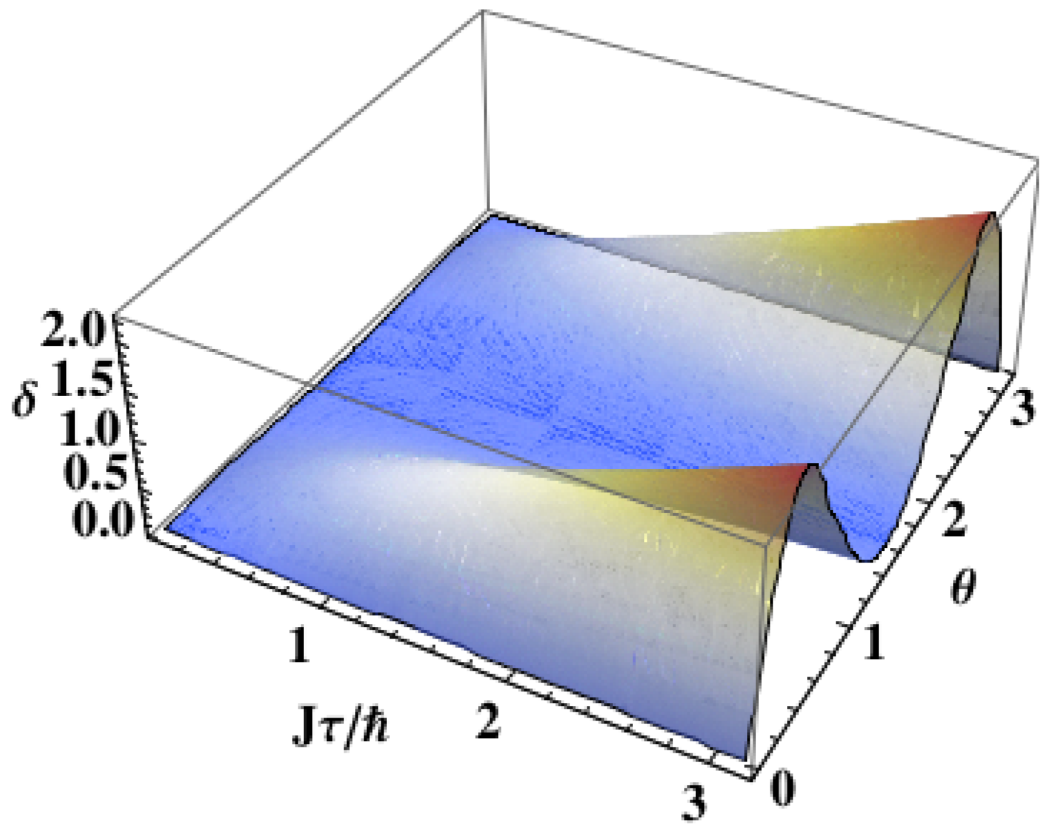
$$|\varphi\rangle = \cos\theta|0\rangle + \exp(-i\phi)\sin\theta|1\rangle,\ \theta \in [0,\pi],\ \phi \in [0,2\pi)$$

*Driving with Ising Haniltonian*

$$H_I = \frac{J}{4}\left(\sum_{i=1}^{N-1}(I-\sigma_i^z)(I-\sigma_{i+1}^z)\right)$$

# *Evolution with Ising Hamiltonian*

$$\delta = \frac{\tau \Delta H}{\hbar} - \mathcal{G}(\mathcal{E}_G)$$



# *Mixed states*

# *For mixed states*

- *Hilbert-Schmidt distance*

$$dS_{HS}^2 = \mathrm{Tr}\left[\rho(t+dt) - \rho(t)\right]^2$$
$$= \mathrm{Tr}\left(\dot{\rho}\right)^2 dt^2$$

- *Speed of evolution*

$$\frac{dS_{HS}^2}{dt^2} = \frac{4}{\hbar^2}\mathrm{Tr}\left[(\rho^2 H^2) - (\rho H)^2\right]$$

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- *Fubini-Study metric*

$$dS^2_{FS} = 4\left(1 - \frac{\mathrm{Tr}\left[\rho(t+dt)\rho(t)\right]}{\mathrm{Tr}\left[\rho(t)^2\right]}\right)$$
$$= 2\, dS^2_{HS}$$

$\Longrightarrow$ $$\frac{dS_{FS}}{dt} = \frac{2}{\hbar}\Delta H_Q(t)$$

*Anandan (1991)*

# *Uncertainty relation: mixed state*

*Bargmann angle: for unitarily connected quantum states*

$$\frac{\mathrm{Tr}\left[\rho_1 \rho_2\right]}{\mathrm{Tr}\left[\rho_1^2\right]} = \cos^2\left(\frac{S_0}{2}\right)$$

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*GQUR:*

$$\frac{1}{\hbar}\int_0^{\tau} \Delta H_Q(t)dt \geqslant S_0 = 2\cos^{-1}\sqrt{\frac{\mathrm{Tr}\left[\rho(0)\rho(\tau)\right]}{\mathrm{Tr}\left[\rho(0)^2\right]}}$$

*For a time-independent Hamiltonian*

$$\tau\ \Delta H_Q \geqslant \hbar\ S_0$$

*Anandan (1991)*

# *GQUR and entanglement*

- *Geometric entanglemt measure*

$$\mathcal{E}_{\mathcal{C}}^{FS}(\rho_{A_1 \ldots A_N}) = \min_{\rho^S_{A_1 \ldots A_N} \in \mathcal{S}_C} \left( 1 - \frac{\mathrm{Tr}\left[\rho_{A_1 \ldots A_N} \rho^S_{A_1 \ldots A_N}\right]}{\mathrm{Tr}\left[\rho^2_{A_1 \ldots A_N}\right]} \right)$$

- *Similarly as in the case of pure states*

$$\mathcal{G}\left(\mathcal{E}_{\mathcal{C}}^{FS}\right) = \cos^{-1}\sqrt{1 - \mathcal{E}_{\mathcal{C}}^{FS}}$$

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- *Similarly as in the case of pure states*

$$\mathcal{G}\left(\mathcal{E}_{\mathcal{C}}^{FS}\right) = \cos^{-1}\sqrt{1 - \mathcal{E}_{\mathcal{C}}^{FS}}$$

$$\int_0^\tau \Delta H_Q(t)dt \geqslant \hbar \, \mathcal{G}\left(\mathcal{E}_{\mathcal{C}}^{FS}\right)$$

# *Quantum speed and correlation*

# *Quantum speed: geometric approach*

*Let us consider an evolution* $\rho(t) \rightarrow \rho(t+dt)$

*Distance traverse in the projective Hilbert space, following HS distance:*

$$dS^2 = \mathrm{Tr}\left[\rho(t+dt) - \rho(t)\right]^2$$
$$= \mathrm{Tr}\left(\dot{\rho}\right)^2 dt^2$$

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*Following* $i\hbar\dot{\rho} = [H(t), \rho]$

*Speed of quantum evolution:*

$$\frac{dS^2}{dt^2} = v^2 = \frac{2}{\hbar^2}\mathrm{Tr}\{[\rho, H(t)][\rho, H(t)]^\dagger\}$$
$$= \frac{4}{\hbar^2}\mathrm{Tr}\left[(\rho^2 H(t)^2) - (\rho H(t))^2\right]$$
$$= \frac{4}{\hbar^2}\Delta H(t)^2$$

# *Speed: local unitary evolution*

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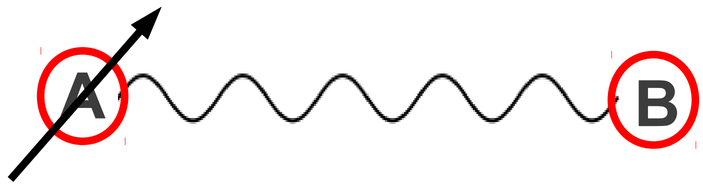

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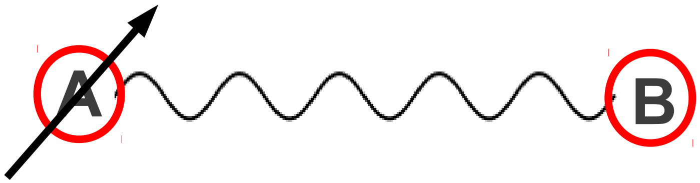


*1-qubit Hamiltonian:* $H = h(h_0 \mathbb{I}_2 + \overline{r}.\overline{\sigma})$

*Local Hamiltonian:* $H_a = H \otimes \mathbb{I}_2$

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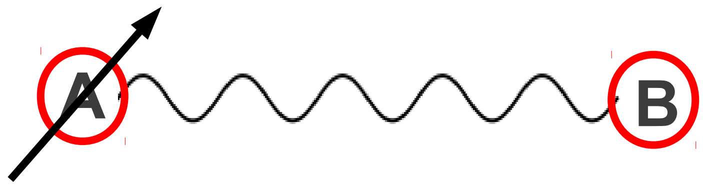


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*Local Hamiltonian:* $H_a = H \otimes \mathbb{I}_2$

*Speed of evolution*

$$\frac{dS^2}{dt^2} = v^2 = \frac{4}{\hbar^2}\Delta H_a^2(\rho)$$

*Local Anandan fluctuation:*

$$\begin{aligned}\Delta H_a^2(\rho) &= \mathrm{Tr}\left[(\rho^2 H_a^2) - (\rho H_a)^2\right] \\ &= h^2 \mathrm{Tr}\left[(\rho^{ab})^2 - \overline{r}.(\overline{\sigma} \otimes \mathbb{I}_2)\rho^{ab}\overline{r}.(\overline{\sigma} \otimes \mathbb{I}_2)\rho^{ab}\right]\end{aligned}$$

# *Quantum correlation: geometric discord*

*For 2-qubit state: geometric measure of quantum discord*

$$D_a^2(\rho) = \min_{\{\Pi_i^a\}} \parallel \rho - \sum_{i=0}^{1} (\Pi_i^a \otimes \mathbb{I}_2)\rho(\Pi_i^a \otimes \mathbb{I}_2) \parallel^2$$

*With* $\Pi_0^a \; = \; 1/2(\mathbb{I}_2 + \overline{r}.\overline{\sigma})$

$$\Pi_1^a \; = \; 1/2(\mathbb{I}_2 - \overline{r}.\overline{\sigma})$$

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$$\Pi_1^a = 1/2(\mathbb{I}_2 - \overline{r}.\overline{\sigma})$$

$$\Longrightarrow \quad D_a^2(\rho) = \min_{\{\overline{r}\}} \frac{1}{2} \mathrm{Tr} \left( \rho^2 - \overline{r}.(\overline{\sigma} \otimes \mathbb{I}_2)\rho\overline{r}.(\overline{\sigma} \otimes \mathbb{I}_2)\rho \right)$$

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$$2D_a^2(\rho) = \mathrm{Tr}\rho^2 - m_a^{max}$$

*where* $m_a^{max}$ *is the largest eigen value of the Matrix* $M_a$ *with the elements* $M_a^{ij} = Tr[\rho(\sigma^i \otimes \mathbb{I}_2)\rho(\sigma^j \otimes \mathbb{I}_2)]$

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*With*

$$\Pi_0^a = 1/2(\mathbb{I}_2 + \bar{r}.\bar{\sigma})$$

$$\Pi_1^a = 1/2(\mathbb{I}_2 - \bar{r}.\bar{\sigma})$$

$$\Longrightarrow D_a^2(\rho) = \min_{\{\bar{r}\}} \frac{1}{2} \mathrm{Tr}\left(\rho^2 - \bar{r}.(\bar{\sigma} \otimes \mathbb{I}_2)\rho\bar{r}.(\bar{\sigma} \otimes \mathbb{I}_2)\rho\right)$$

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*Local Anandan fluctuation:* $\Delta H_a^2(\rho) \geqslant 2h^2 D_a^2(\rho)$

# *Bounds on local speed*

*Speed of evolution:* $$\frac{dS^2}{dt^2} = v_a^2 = \frac{4}{\hbar^2}\Delta H_a^2(\rho)$$

*Lower bound:* $$v_a^2 \geqslant \frac{8h^2}{\hbar^2} D_a^2(\rho) = (v_a^{min})^2$$

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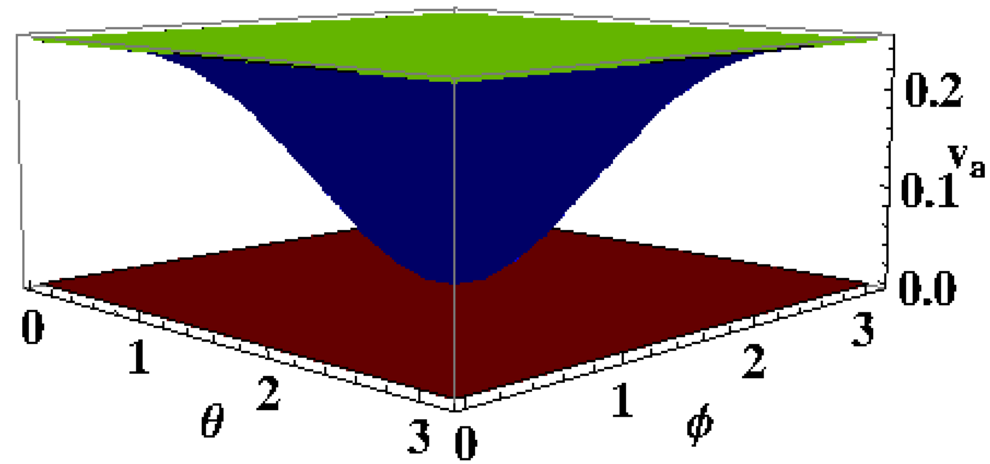
*Upper bound:* $$v_a^2 \leqslant \frac{4h^2}{\hbar^2}\left(\mathrm{Tr}\rho^2 - m_a^{min}\right) = (v_a^{max})^2$$

*where* $m_a^{min}$ *is the largest eigen value of the Matrix* $M_a$
*with the elements* $M_a^{ij} = Tr[\rho(\sigma^i \otimes \mathbb{I}_2)\rho(\sigma^j \otimes \mathbb{I}_2)]$

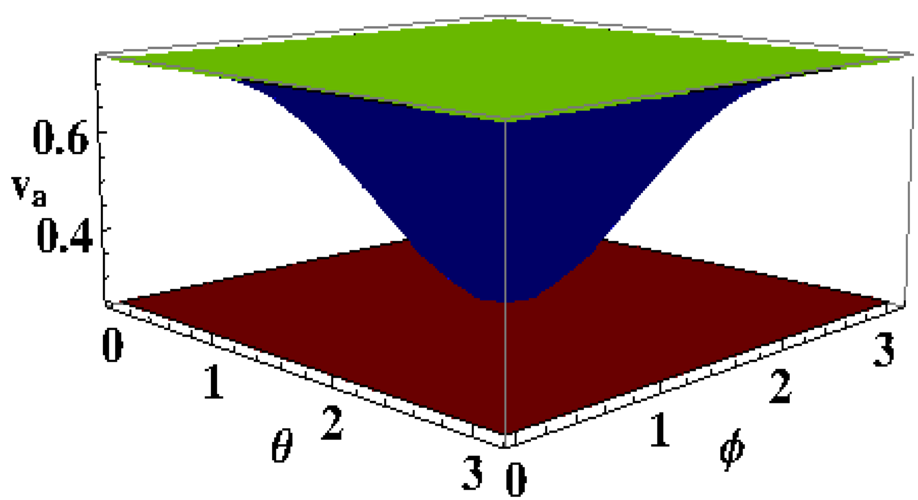
# *Bounds on local speed*

$h = 1$, p=1/2, $\bar{r} = \{\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi\}$

$\rho_{cq} = (1-p)|00\rangle\langle 00| + p/2|1+\rangle\langle 1+|$




$\rho_{me} = p|00\rangle\langle 00| + 1/2(1-p)(|00\rangle - |11\rangle)(\langle 00| - \langle 11|)$



# *Bounds on speed: two sided local unitary*

*Both the qubits follow local unitary evolution*

$$H_a = h(h_0\mathbb{I}_4 + \overline{r}.(\overline{\sigma} \otimes \mathbb{I}_2)); \;\; H_b = h(h_0\mathbb{I}_4 + \overline{r}.(\mathbb{I}_2 \otimes \overline{\sigma}))$$

$$\Delta H_t^2(\rho) = \frac{1}{2}\mathrm{Tr}\left([\rho, H_a][\rho, H_a]^\dagger + [\rho, H_b][\rho, H_b]^\dagger\right) + \mathrm{Tr}\left([\rho, H_a][\rho, H_b]^\dagger\right)$$

*where the third term takes the form,* $2h^2\, Tr[\rho^2 \sum_{i,j} r_i r_j(\sigma_i \otimes \sigma_j) - \rho \sum_i r_i(\sigma_i \otimes \mathbb{I}_2)\rho \sum_j r_j(\mathbb{I}_2 \otimes \sigma_j)]$ *and vanishes for non quantum-quantum states.*

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$$(\Delta H_a(\rho) - \Delta H_b(\rho))^2 \leqslant \Delta H_t^2(\rho) \leqslant (\Delta H_a(\rho) + \Delta H_b(\rho))^2$$

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$\Rightarrow$

$$\frac{dS^2}{dt^2} = v_t^2 = \frac{4}{\hbar^2}\Delta H_t^2(\rho)$$

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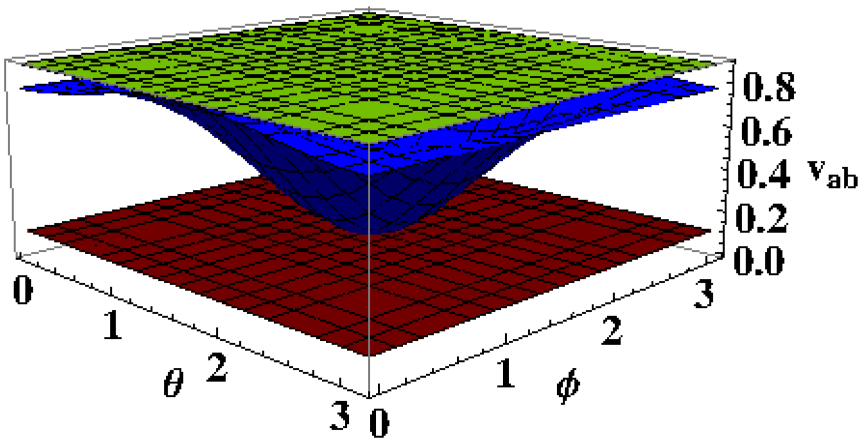


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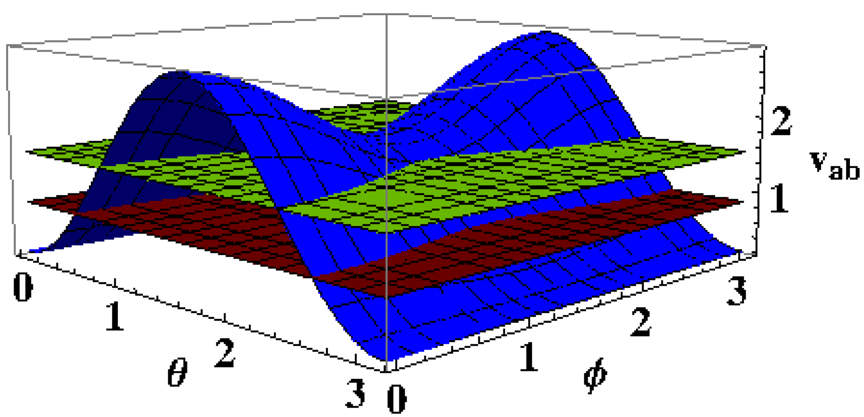

# *Conclusion*

- *Quantum entanglement that plays an important role in setting the limits for the quantum uncertainties.*

- *The geometric time-energy uncertainty relation is shown to be bounded below by the multipartite entanglement for pure and mixed quantum states. Example considered is evolution with Ising Hamiltonian.*

- *The speed of local quantum unitary evolution is lower bounded by the geometric measure of quantum discord.*

- *For two sided local (but same) unitary, the speed of quantum evolution can be increased more than the case of the non-quantum-quantum correlated systems.*

*Thank you*

# *Grover quantum search*

L. K. Grover (1997)

# *Grover quantum search*

$$|\psi_0\rangle = 1/\sqrt{n}\sum_{i=0}^{n-1}|i\rangle$$

Grover operator

$$G = -I_0 H^{\otimes n} I_m H^{\otimes n}$$

$$I_0 = \mathbb{I} - 2|\psi_0\rangle\langle\psi_0|, \; I_m = \mathbb{I} - 2|m\rangle\langle m|$$

where $|m\rangle$ being the target state
$H$ is the Hadamard transformation

*L. K. Grover (1997)*

# *Role of entanglement*

After $k$ iterations of the Grover operator the combined n-qubit state evolves to the state

$$|\psi_k\rangle = \frac{\cos\theta_k}{\sqrt{n-1}} \sum_{i \neq m} |i\rangle + \sin\theta_k |m\rangle$$

where $\theta_k = (2k+1)\theta_0$ and $\theta_0 = \sin^{-1}(1/\sqrt{n})$

*S. L. Braunstein and A. K. Pati (2001).*

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where $\theta_k = (2k+1)\theta_0$ and $\theta_0 = \sin^{-1}(1/\sqrt{n})$

The state can be decomposed in the Schmidt basis

$$|\psi_k\rangle = \sqrt{\lambda_1(k)}|g'\rangle|e\rangle - \sqrt{\lambda_2(k)}|e'\rangle|g\rangle$$

where $\{|g\rangle, |e\rangle\}$ describes an orthonormal basis for i-th qubit and $\{|g'\rangle, |e'\rangle\}$ describes an orthonormal basis for other (n-1) qubits.

*S. L. Braunstein and A. K. Pati (2001).*

# *Role of entanglement*

Where $\lambda_1(k)\lambda_2(k) = \frac{n(n-2)}{2(n-1)^2}\sin^2(\theta_k - \theta_0)\cos^2(\theta_k)$

The Grover search is complete when $\theta_k \to \pi/2$

The bound:

$$\delta = \frac{1}{\hbar}\int_{\theta_0}^{\theta_k} \Delta H_Q(\theta_k)d\theta_k - \mathcal{G}\left(\mathcal{E}_{\mathcal{C}}(|\psi_{\theta_k}\rangle)\right)$$

# *Grover quantum serach...*

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