Strong subadditivity in quantum information

Ujjwal Sen

Harish-Chandra Research Institute, Allahabad, INDIA

IPQI-2014

Outline

- What is strong subadditivity?
- SSA and the Holevo bound
- SSA and multiport quantum dense coding
- SSA and resonating valence bond states

Outline

- What is strong subadditivity?
- SSA and the Holevo bound
- SSA and multiport quantum dense coding
- SSA and resonating valence bond states

von Neumann entropy

- arb quantum state ρ
- von Neumann entropy of ρ, denoted $S(\rho)$, given by

$$S(\rho) = -\mathrm{tr}\,(\rho \log \rho)$$

Subadditivity

- $S(\rho\_AB) \leq S\_A + S\_B$

Strong subadditivity

- $S_{ABC} + S_A \leq S_{AB} + S_{AC}$

Strong subadditivity

- $S_{ABC} + S_A \leq S_{AB} + S_{AC}$

OR

- $S_B + S_C \leq S_{AB} + S_{AC}$

Strong subadditivity

- $S_{ABC} + S_A \leq S_{AB} + S_{AC}$

OR

- $S_B + S_C \leq S_{AB} + S_{AC}$

Lieb & Ruskai, J Math Phys 1973

Strong subadditivity

- $S_{ABC} + S_A \leq S_{AB} + S_{AC}$

OR

- $S_B + S_C \leq S_{AB} + S_{AC}$

- Both versions r equivalent & imply subadditivity.

Strong subadditivity

- $S_{ABC} + S_A \leq S_{AB} + S_{AC}$

OR

- $S_B + S_C \leq S_{AB}$

$r_{ABC} =$
$r_A \otimes r_{BC}$

- Both versions r equivalent & imply subadditivity.

Strong subadditivity

- S_ABC + S_A ≤ S_

For version 2 to subadditivity, first go to version 1.

OR

- S_B + S_C ≤ S_AB + S_AC

- Both versions r equivalent & imply subadditivity.

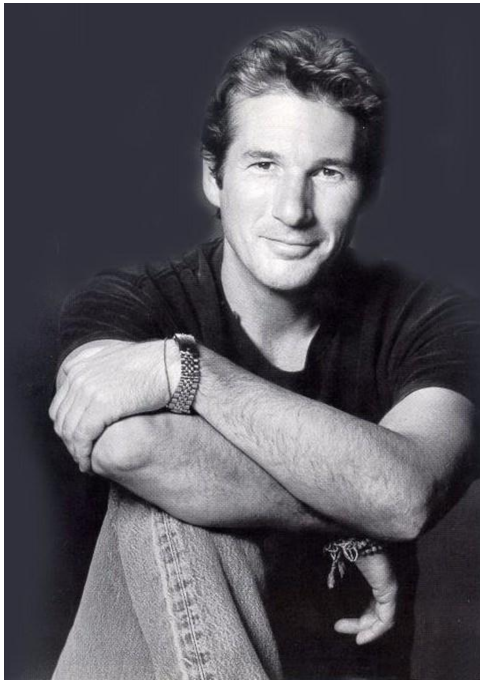
Outline

- What is strong subadditivity?
- SSA and the Holevo bound
- SSA and multiport quantum dense coding
- SSA and resonating valence bond states

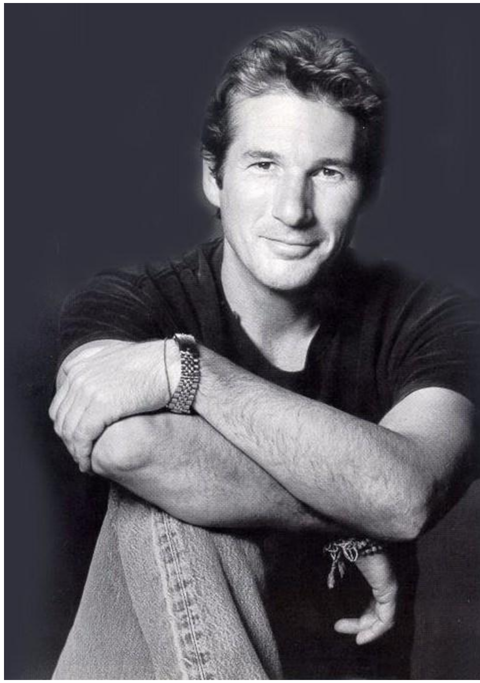
Encoding classical information in quantum states

Encoding classical information in quantum states

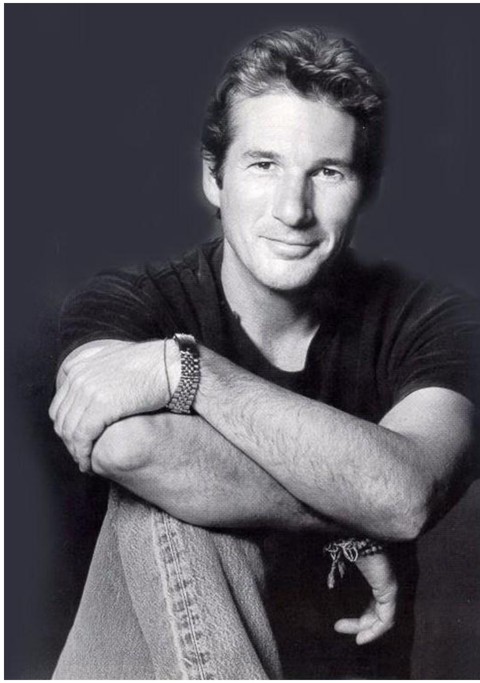
Encoding classical information in quantum states



Encoding classical information in quantum states

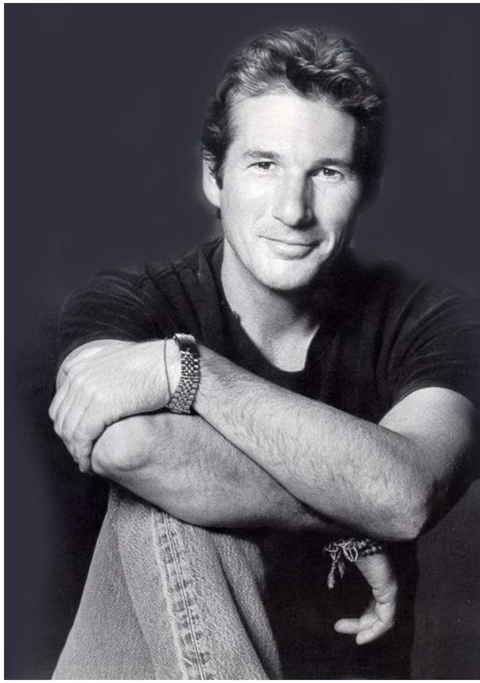


Encoding classical information in quantum states



- Alice (sender) encodes the classical variable x, that occurs with probability p_x, in the quantum state ρ_x. I.e., she produces the ensemble $\mathcal{E} = \{p_x, \rho_x\}$.

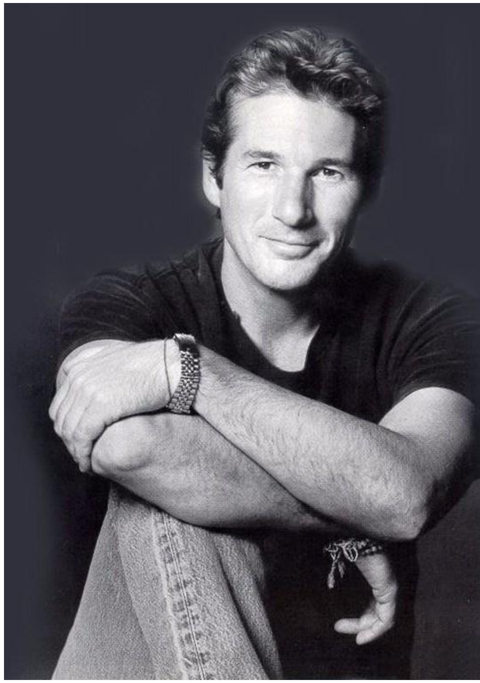
Encoding classical information in quantum states



- Alice (sender) encodes the classical variable x, that occurs with probability p_x, in the quantum state ρ_x. I.e., she produces the ensemble $\mathcal{E} = \{p_x, \rho_x\}$.

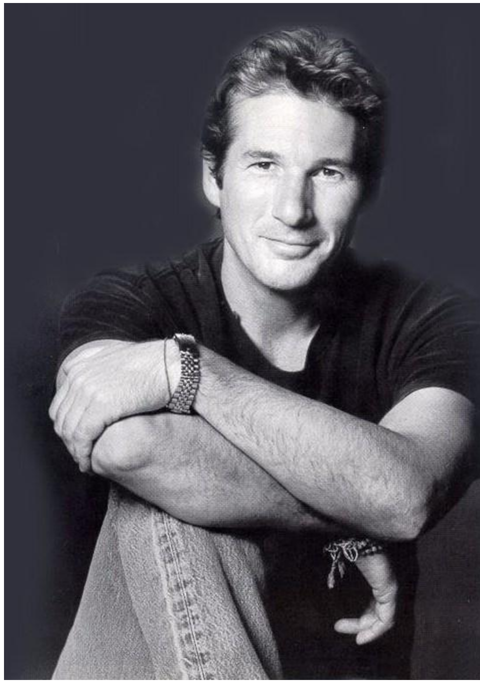
- She then sends it to Bob (receiver).

Encoding classical information in quantum states



- Task of Bob: Gather information about x.

Encoding classical information in quantum states



- Task of Bob: Gather information about x.
- Accessible information = Maximal classical information that can be extracted by Bob from $\mathcal{E} = \{p_x, \rho_x\}$.

Holevo theorem (1973): Upper bound on accessible information

- Initial Ensemble: $\mathcal{E} = \{p_x, \rho_x\}$

- Accessible information $\leq \chi$ ← Holevo quantity

- $\chi = S(\rho) - \Sigma\, p_x\, S(\rho_x)$; $\rho = \Sigma\, p_x\, \rho_x$.

- Accessible information $\leq S(\rho) \leq \log_2 d$

Lower bound: Jozsa, Robb, & Wootters, Phys. Rev. A '94

Outline of the proof of Holevo bound

- Usually involved.

Outline of the proof of Holevo bound

- Usually involved.
- But, becomes uncluttered with use of SSA.

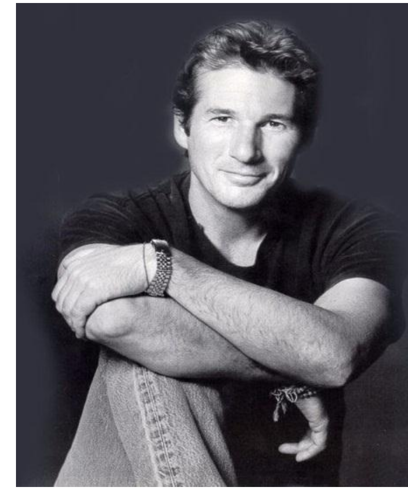
Schumacher, Westmoreland, Wootters, PRL 1996

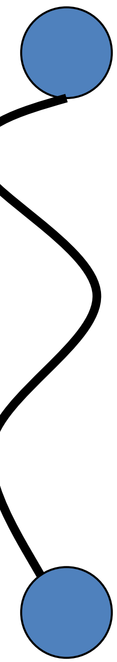
Outline of the proof of Holevo bound

- Usually involved.
- But, becomes uncluttered with use of SSA.
- Essential step is to prove that $\chi\_AB \geq \chi\_A$, and this is where SSA is used.

Schumacher, Westmoreland, Wootters, PRL 1996

LOCC counterpart of Holevo theorem: Upper bound on *locally* accessible info





$\mathcal{E} = \{p_x, \rho_x\}$

Entanglement correction to Holevo theorem

- Accessible information $\leq \log_2 d$

- Locally accessible information $\leq \log_2 d - E_{av}$

<u>Universally true</u>: Holds for arbitrary bipartite ensembles.

SSA plays an imp role, once again.

Badziag, Horodecki, Sen(De), & Sen, Phys. Rev. Lett. '03

Sen(De), Sen, & Lewenstein, Phys. Rev. A '06

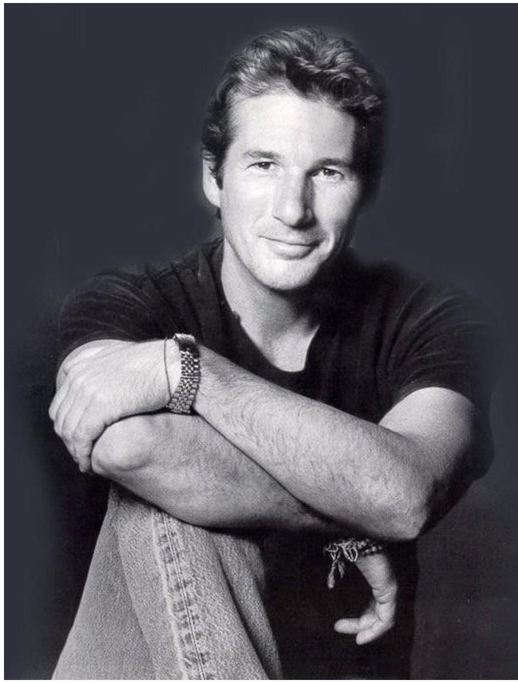
Outline

- What is strong subadditivity?
- SSA and the Holevo bound
- SSA and multiport quantum dense coding
- SSA and resonating valence bond states

Quantum dense coding

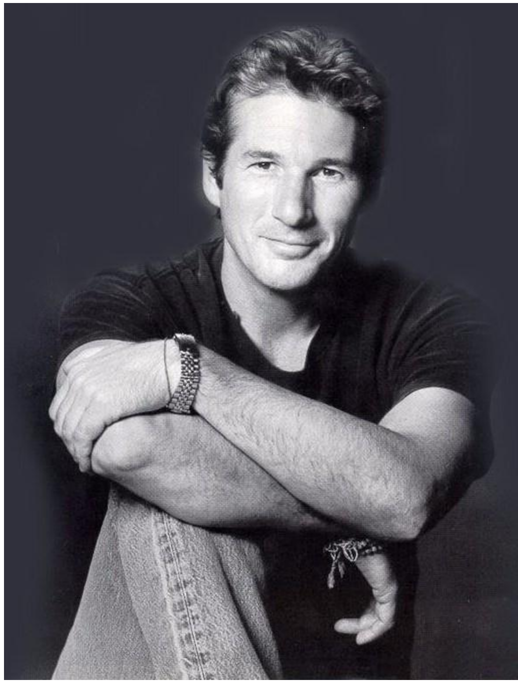
Alice

Sender

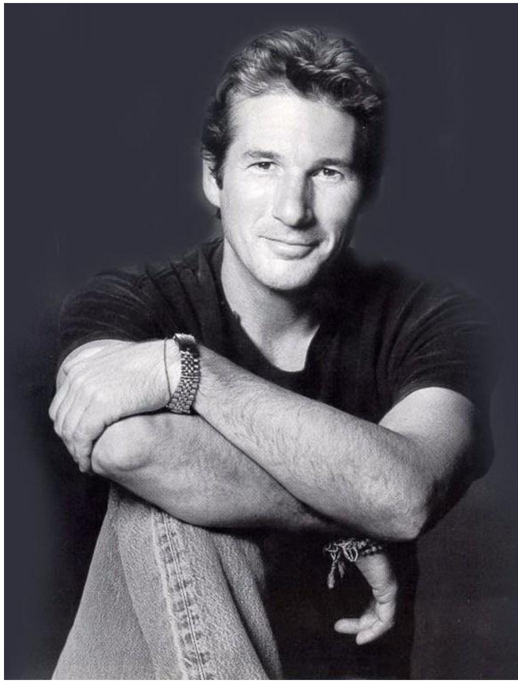


Bob

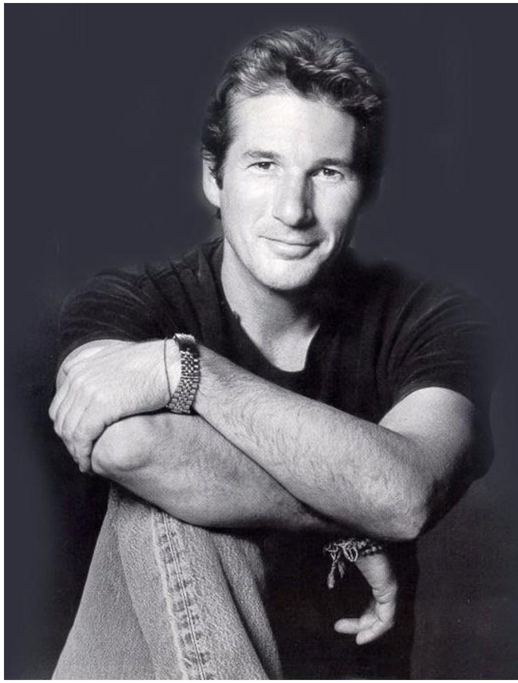
Receiver



Alice is in Bhubaneswar.
Bob is in Delhi.



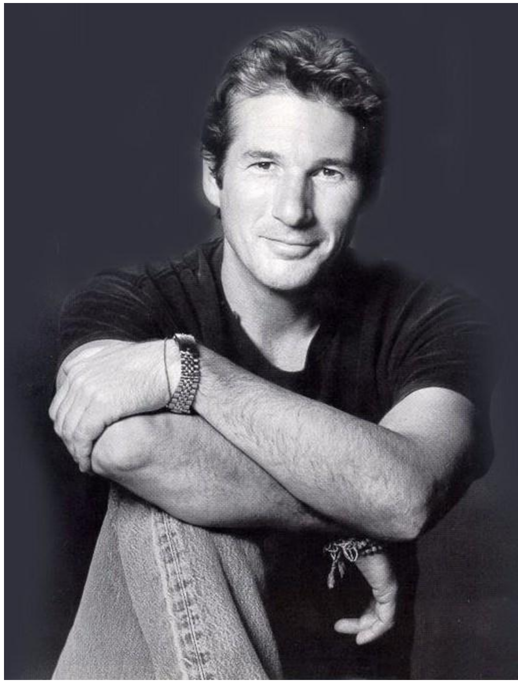
Alice wants to send info about weather in Bhubaneswar to Bob.



Alice wants to send info about weather in Bhubaneswar to Bob.

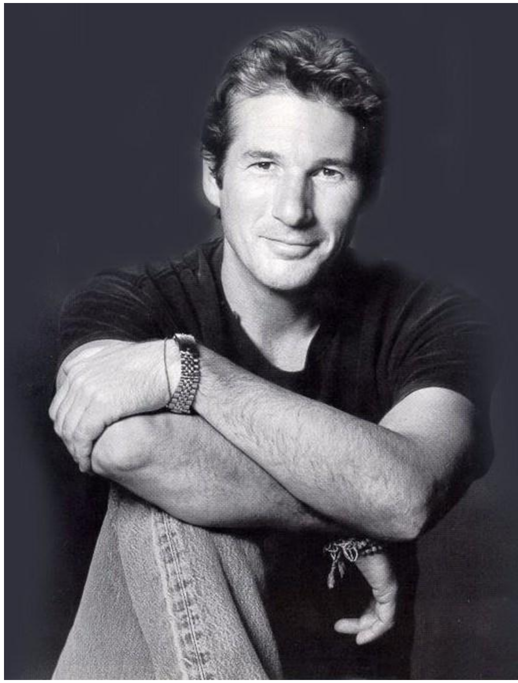
Sunny or not

Windy or not



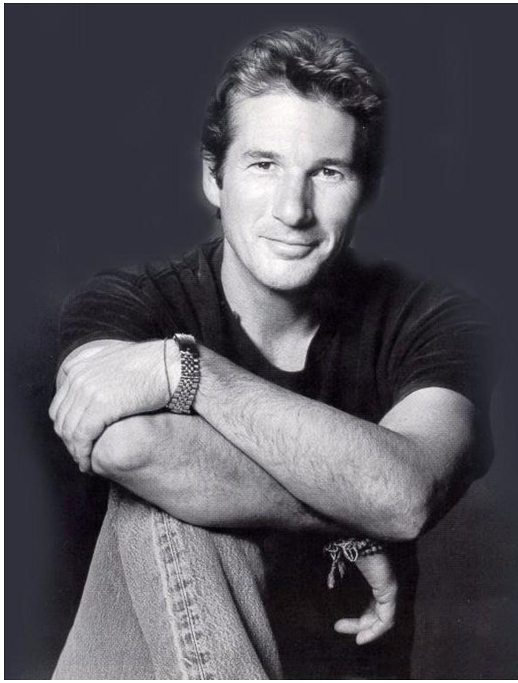
Alice wants to send info about weather in Bhubaneswar to Bob.

2 bits



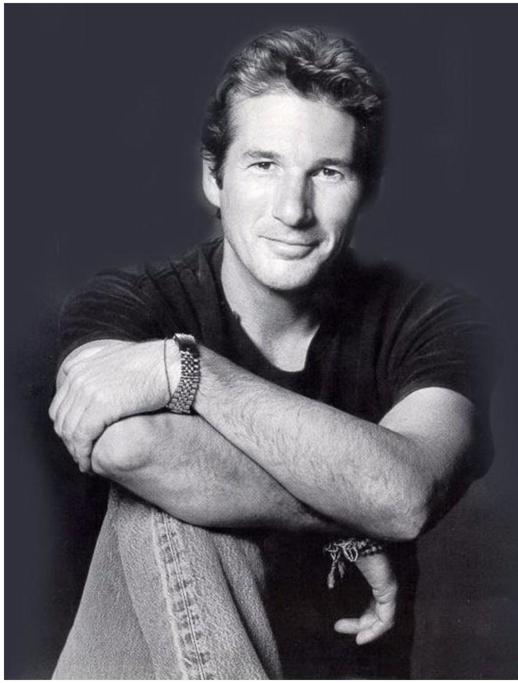
Alice wants to send info about weather in Bhubaneswar to Bob.

Can be sent by using …

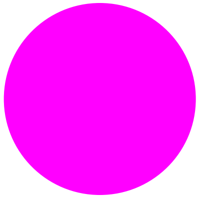


Alice wants to send info about weather in Bhubaneswar to Bob.

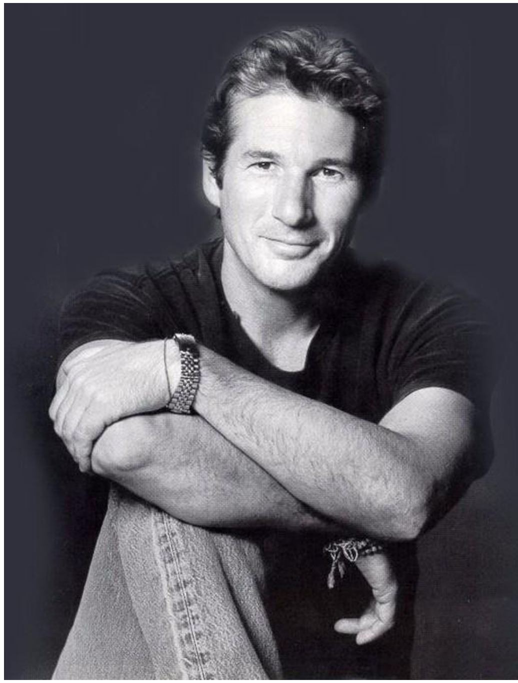
Four balls of different colors.



Alice wants to send info about weather in Bhubaneswar to Bob.

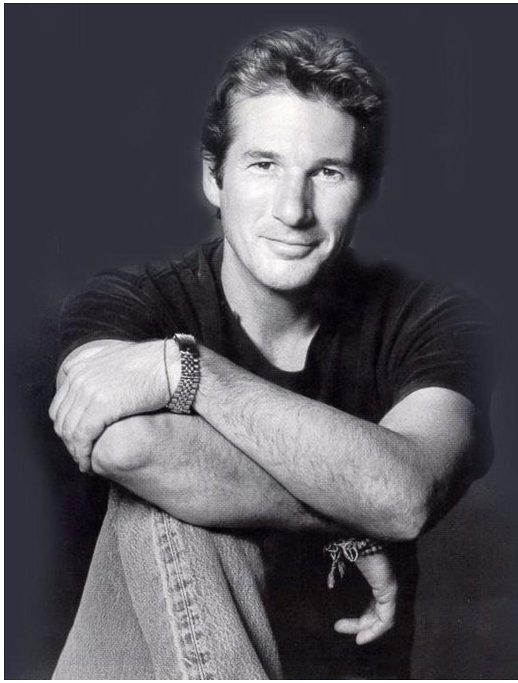


sunny and not windy



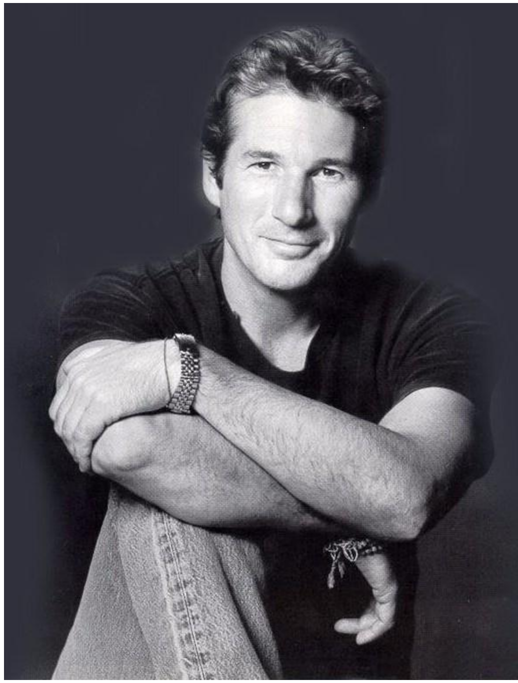
Alice wants to send info about weather in Bhubaneswar to Bob.

sunny and windy



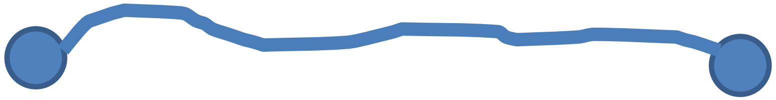
Alice wants to send info about weather in Bhubaneswar to Bob.

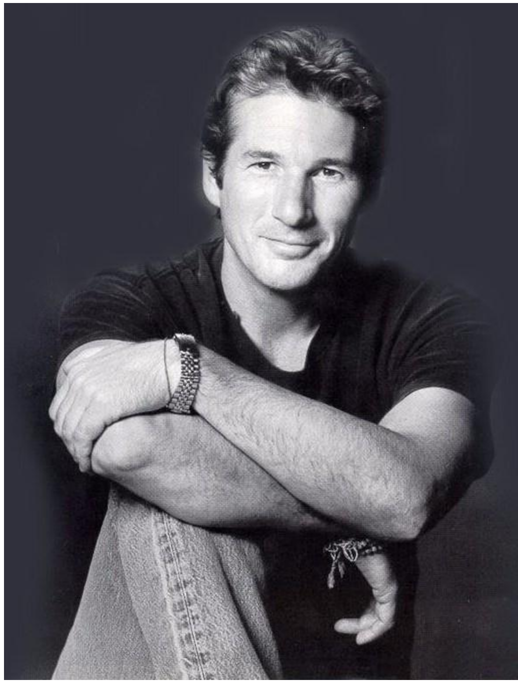
2 bits require 4 dim.



Alice wants to send info about weather in Bhubaneswar to Bob.

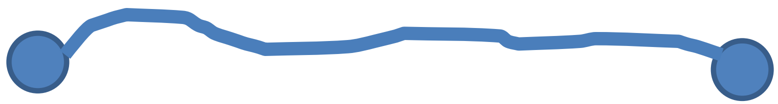
Using shared entanglement between Alice & Bob, …

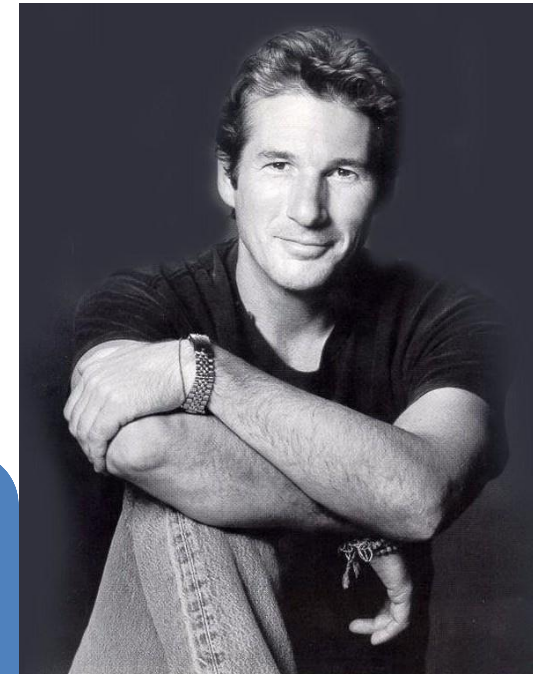




Alice wants to send info about weather in Bhubaneswar to Bob.

Using shared entanglement between Alice & Bob, …

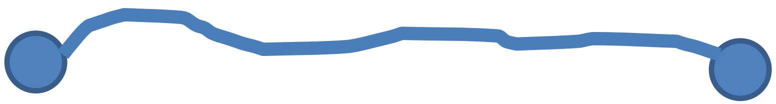




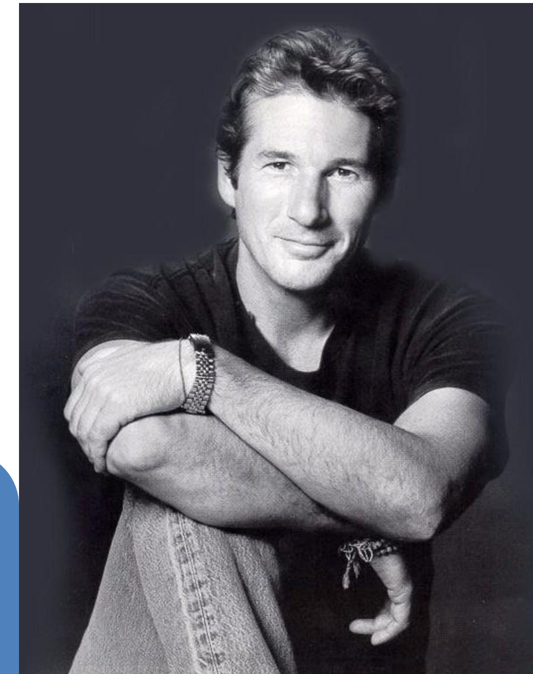
$$|\uparrow\rangle\otimes|\downarrow\rangle-|\downarrow\rangle\otimes|\uparrow\rangle$$

Alice wants to send info about weather in Bhubaneswar to Bob.

Using shared entanglement between Alice & Bob, …



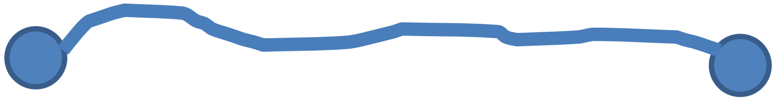
$$|\uparrow\rangle\otimes|\downarrow\rangle-|\downarrow\rangle\otimes|\uparrow\rangle$$



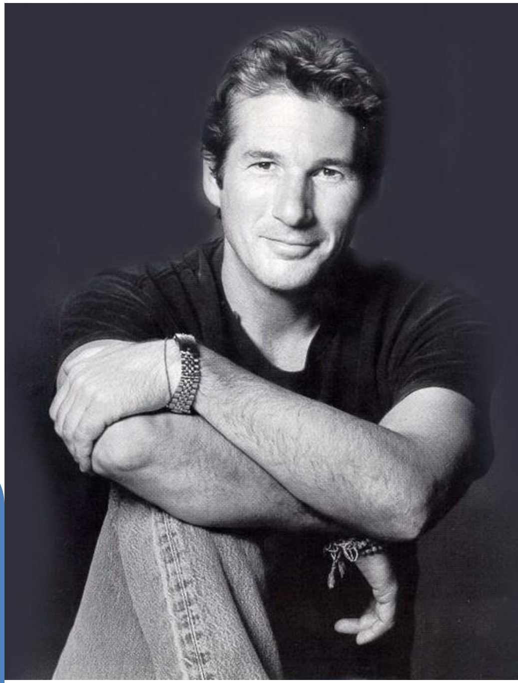
Alice wants to send info about weather in Bhubaneswar to Bob.

2 bits require 2 dim.

Quantum dense coding



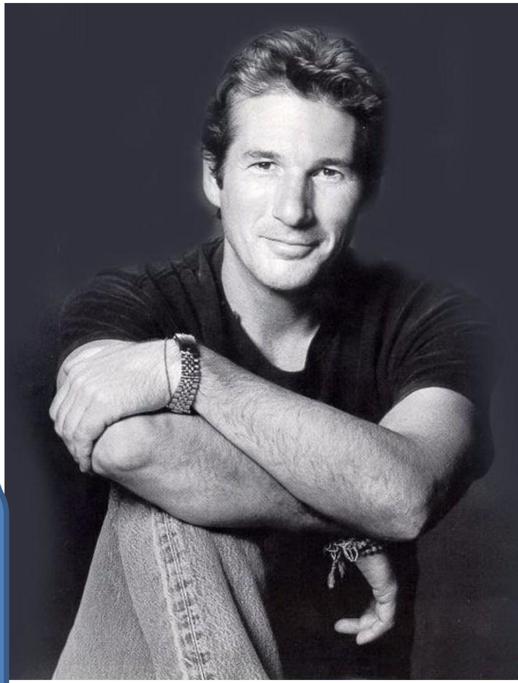
$$|\uparrow\rangle\otimes|\downarrow\rangle-|\downarrow\rangle\otimes|\uparrow\rangle$$

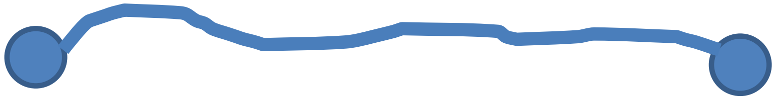


Alice wants to send info about weather in Bhubaneswar to Bob.

2 bits require 2 dim.

Quantum dense coding



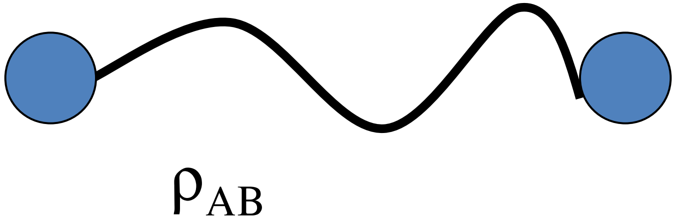


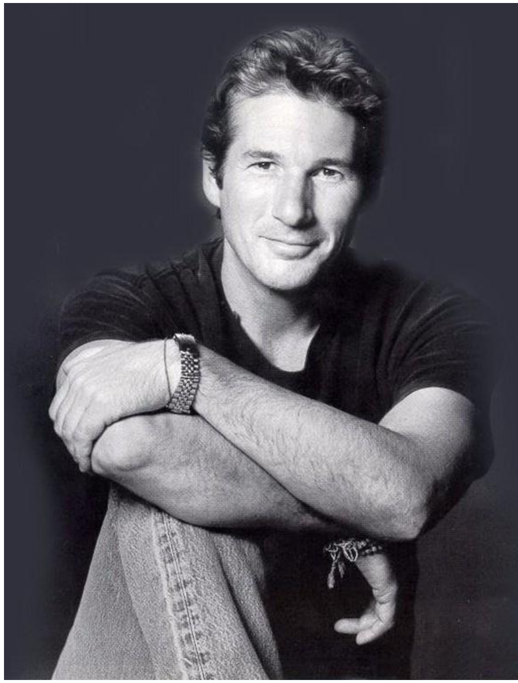
Bennett & Wiesner
1992

Alice wants to send info about weather in Bhubaneswar to Bob.

2 bits require 2 dim.

Capacity of quantum dense coding



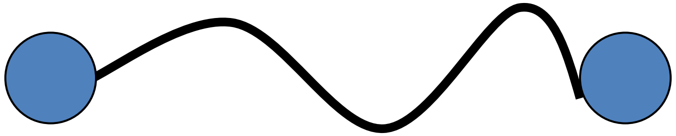


- Capacity of quantum dense coding = amount of classical info that can be sent via a given shared state.

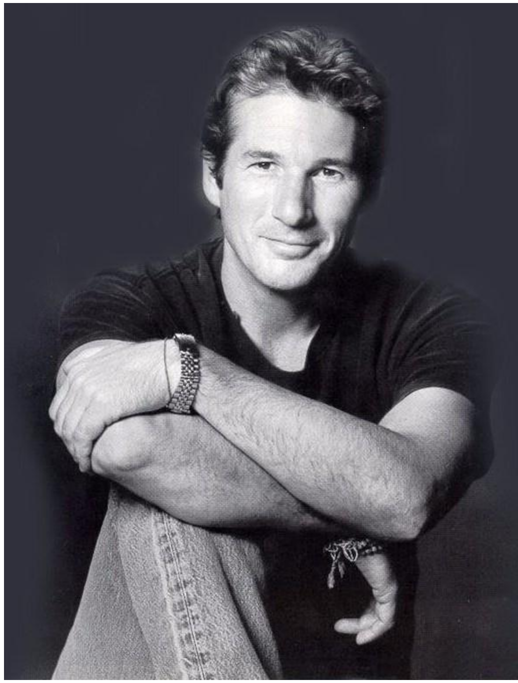
- The Holevo theorem can be used to find this capacity.

Hiroshima, J. Phys. A ’01; Ziman & Buzek, PRA ‘03

Capacity of quantum dense coding



ρ_{AB}

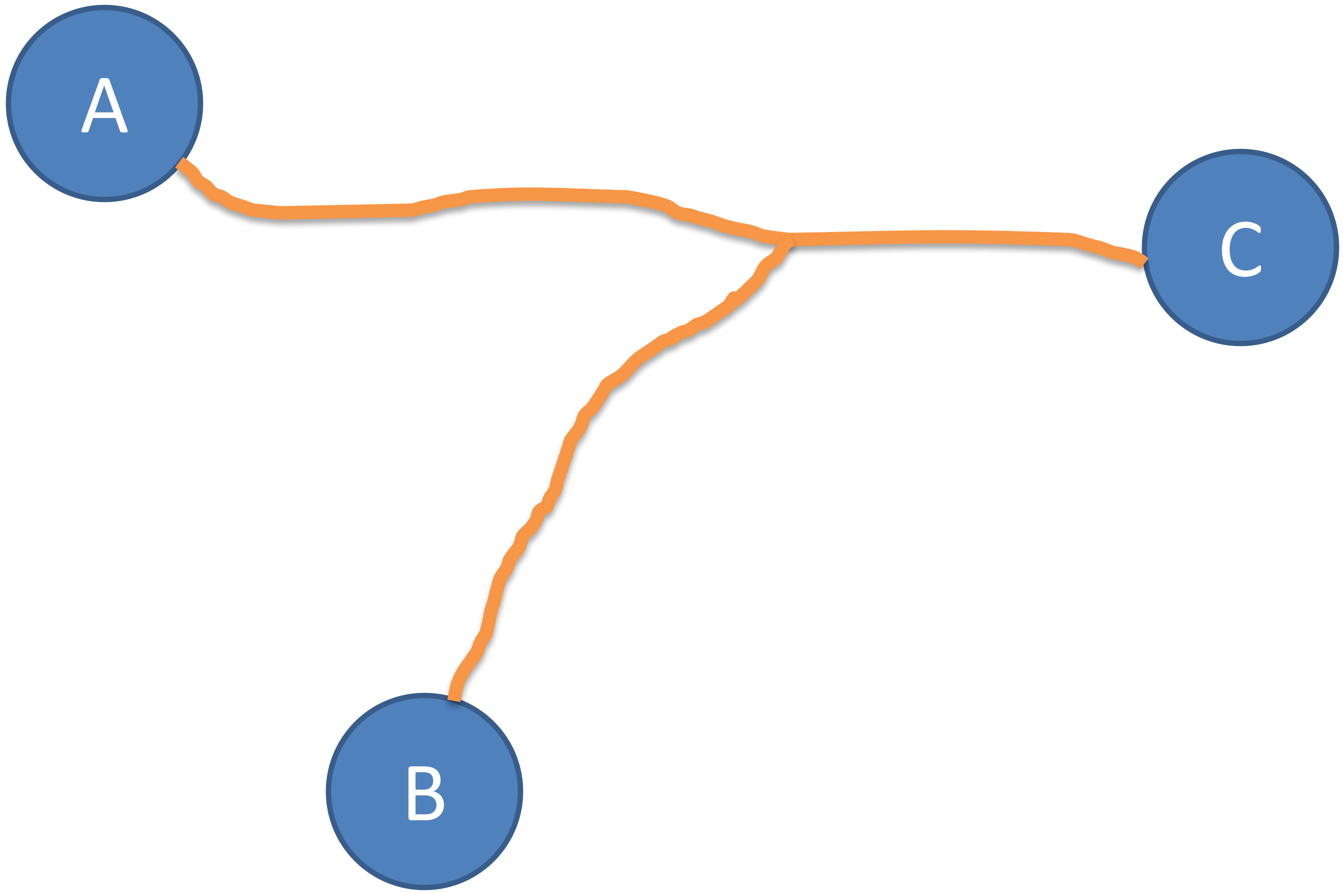


- Capacity of quantum dense coding =

$$\log d + S_B - S_{AB}$$

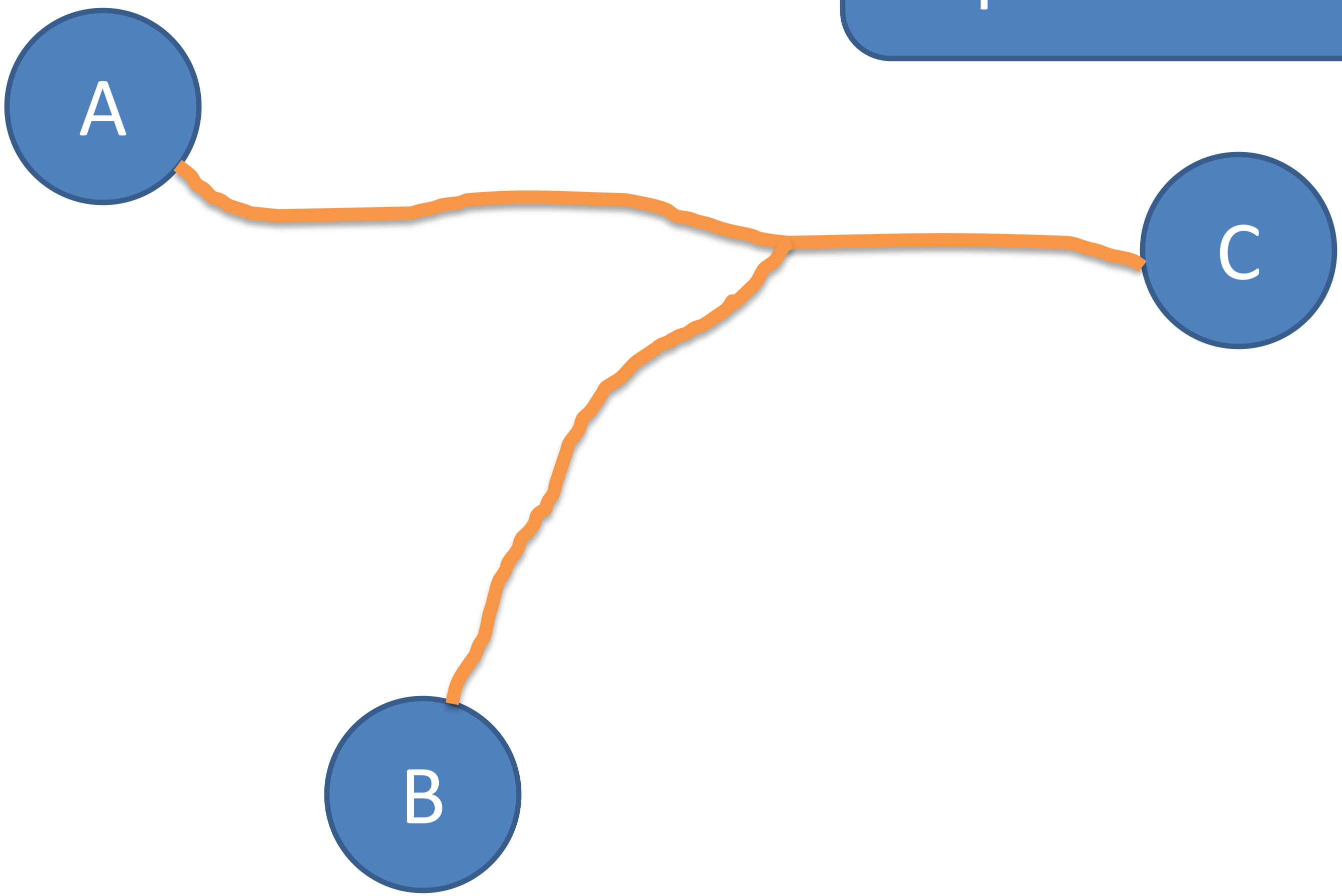
Hiroshima, J. Phys. A '01; Ziman & Buzek, PRA '03

Distributed Q Dense Coding



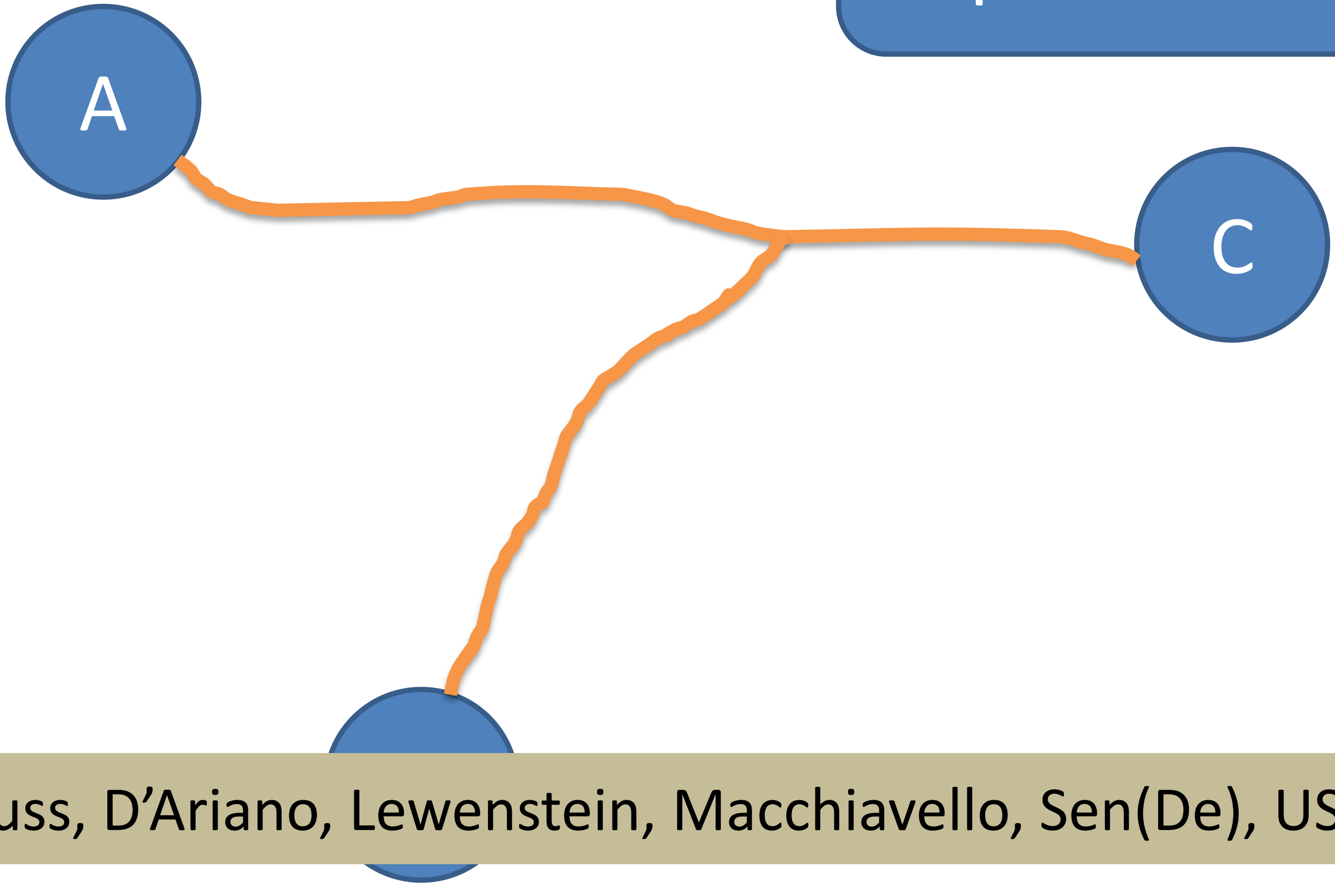
Distributed Q Dense Coding

Towards a quantum internet



Distributed Q Dense Coding

Towards a quantum internet



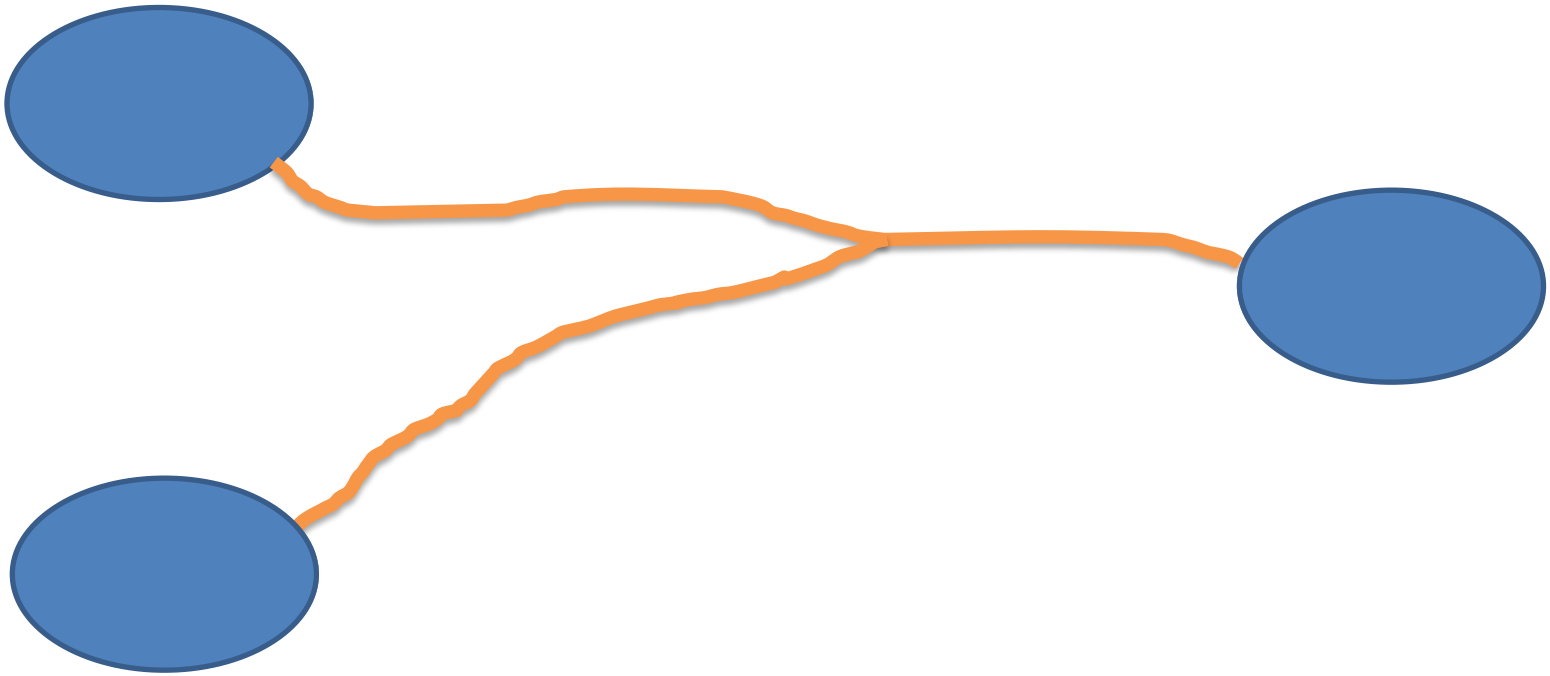
Bruss, D’Ariano, Lewenstein, Macchiavello, Sen(De), US, PRL 2004

Exclusion pple for Q Dense Coding

Exclusion pple for Q Dense Coding

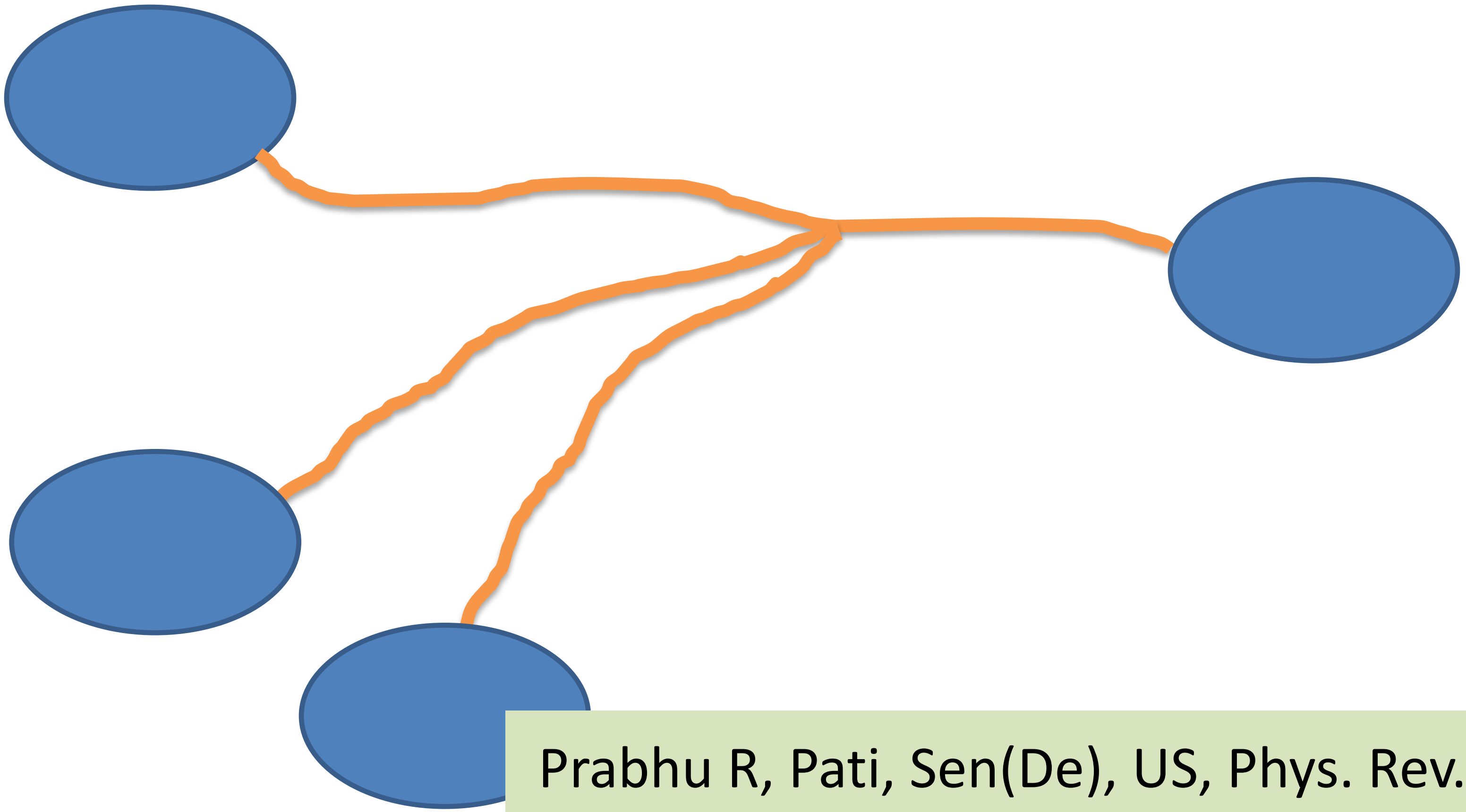
Prabhu R, Pati, Sen(De), US, Phys. Rev. A 2013

Exclusion pple for Q Dense Coding



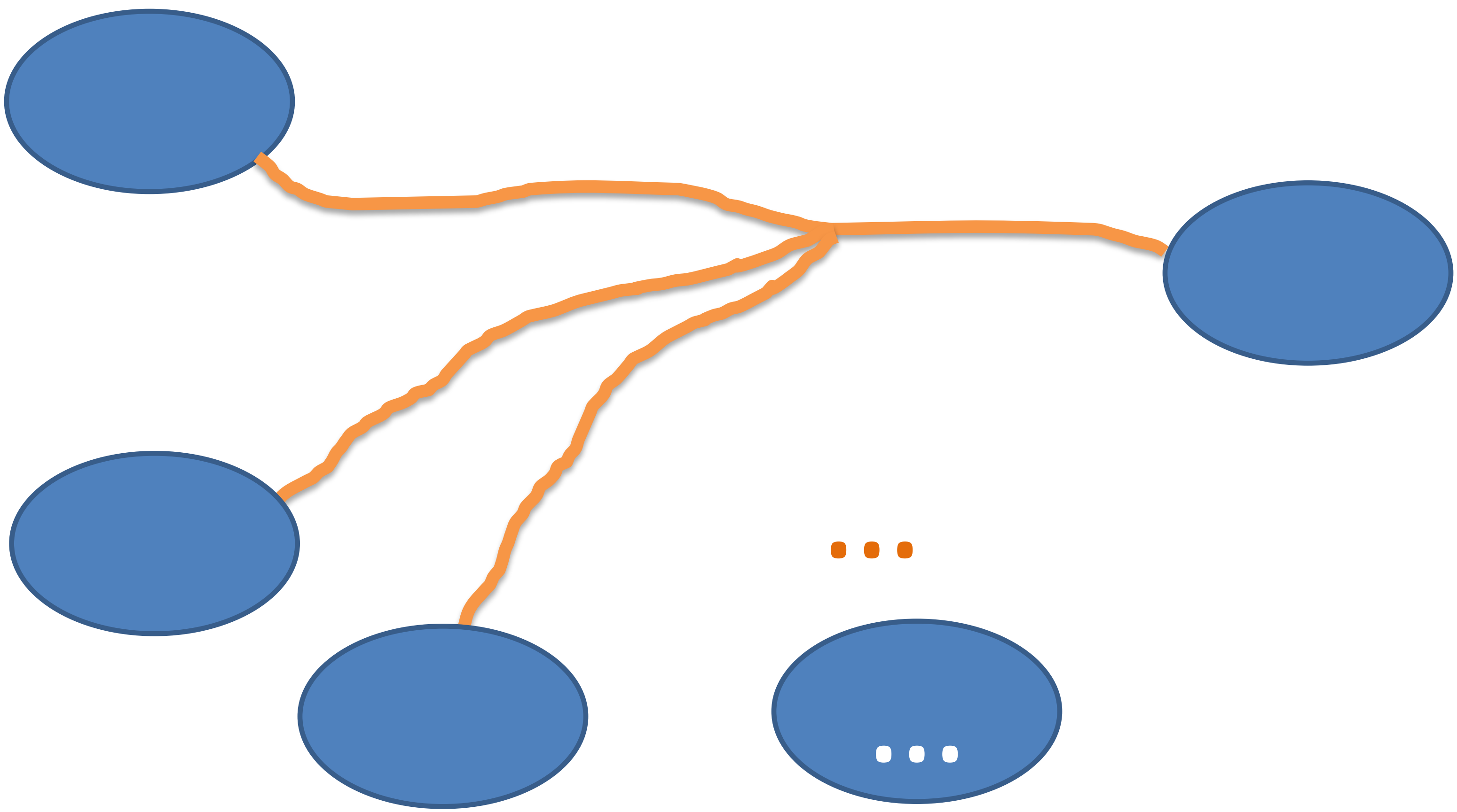
Prabhu R, Pati, Sen(De), US, Phys. Rev. A 2013

Exclusion pple for Q Dense Coding

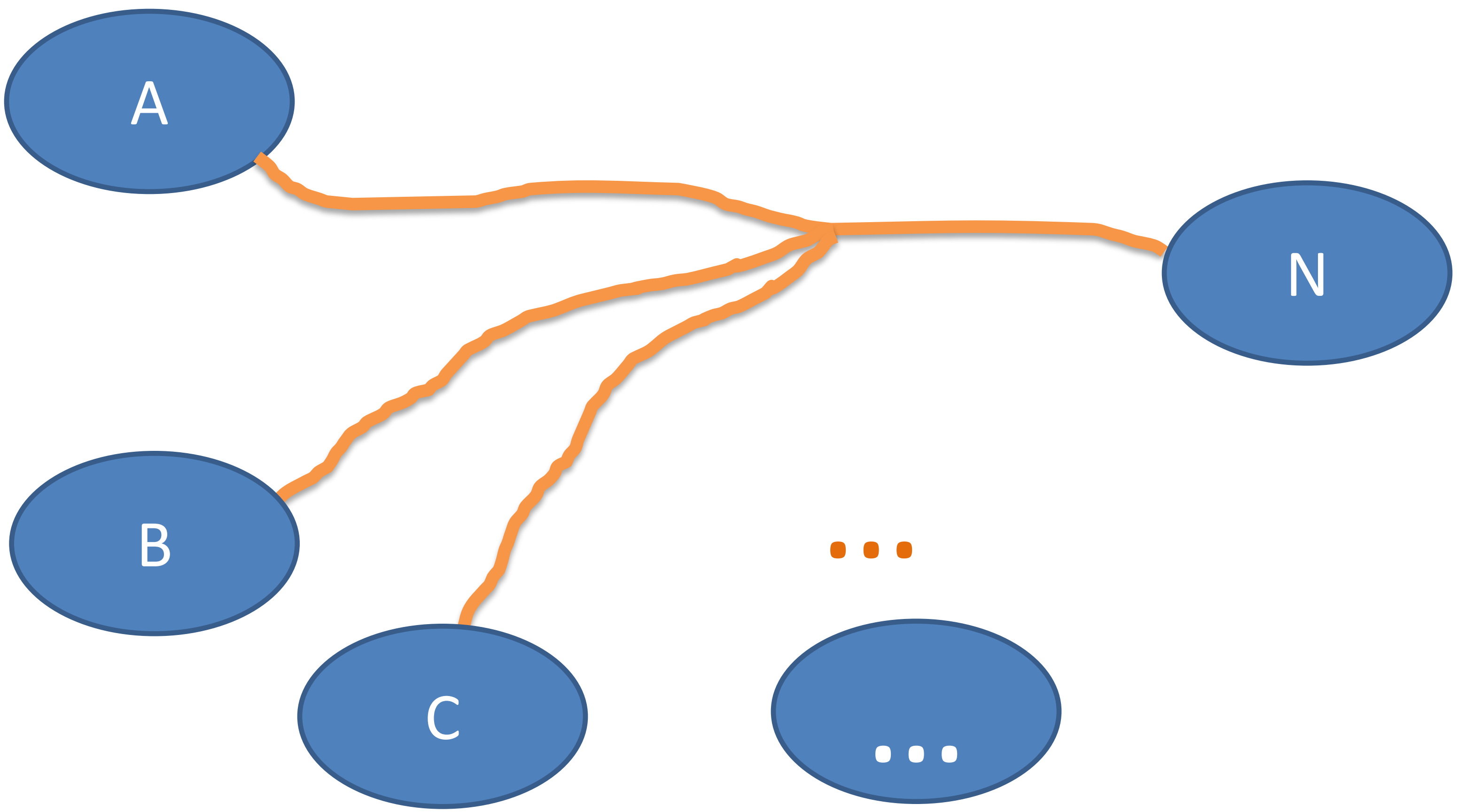


Prabhu R, Pati, Sen(De), US, Phys. Rev. A 2013

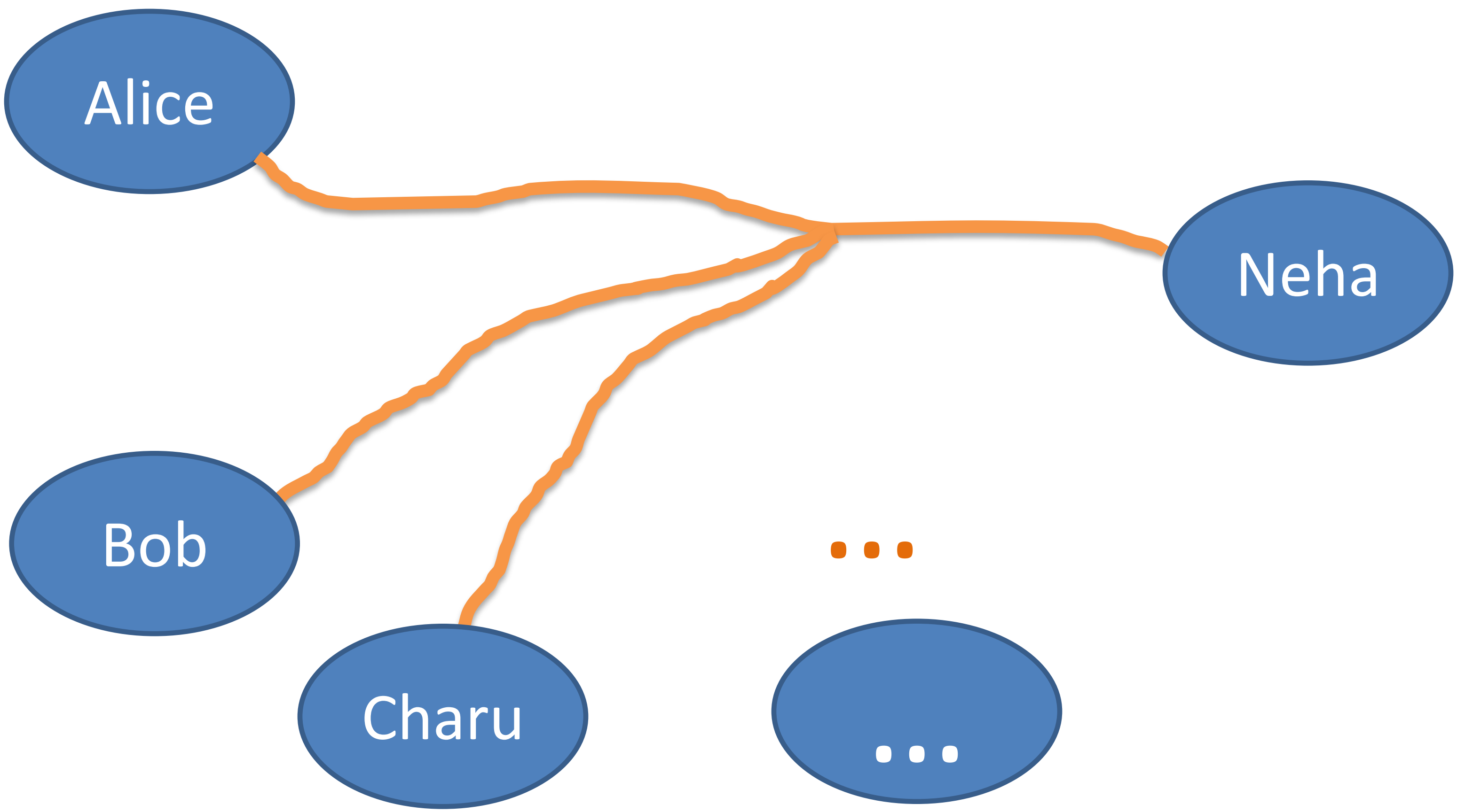
Exclusion pple for Q Dense Coding



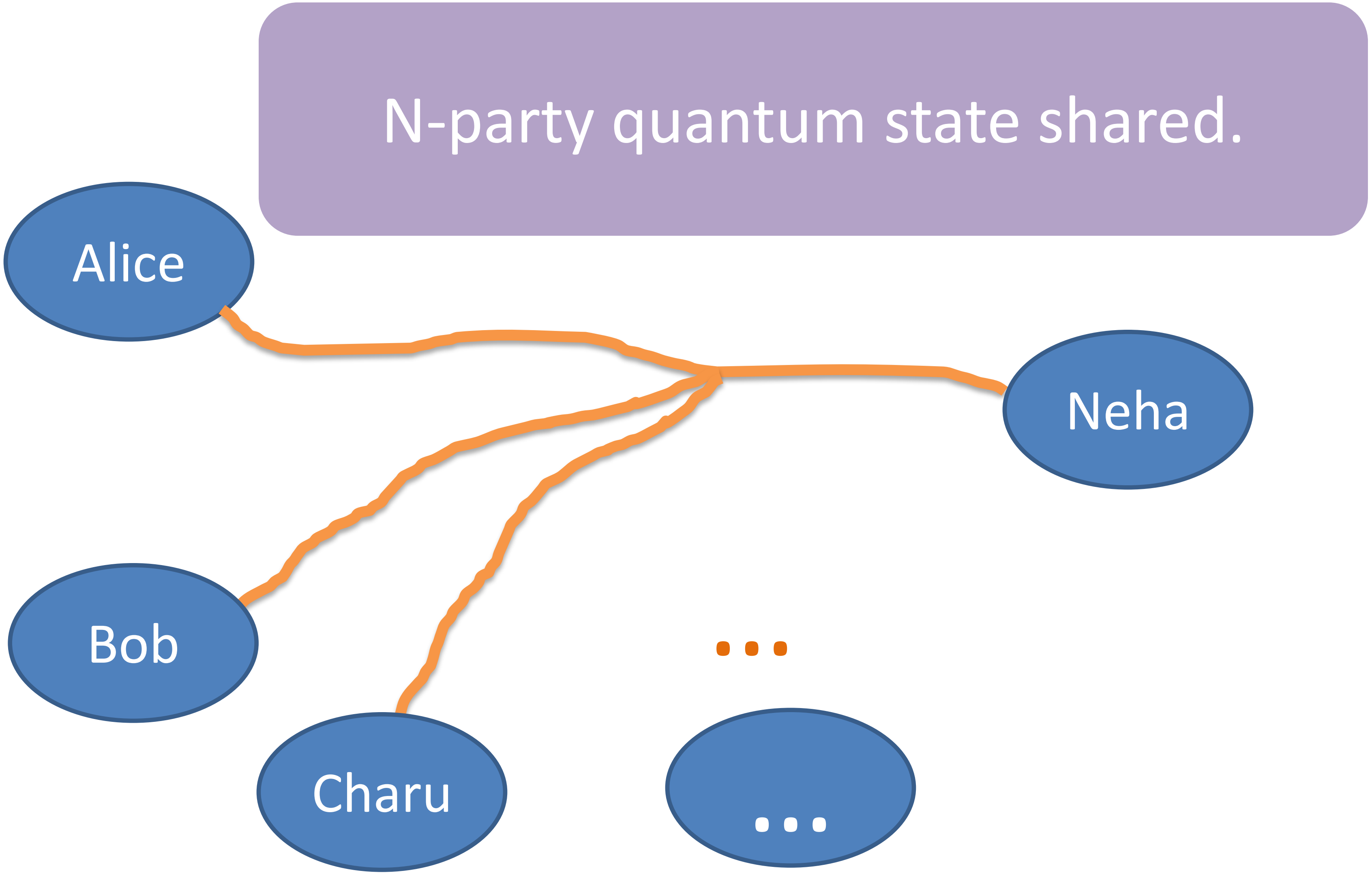
Exclusion pple for Q Dense Coding



Exclusion pple for Q Dense Coding

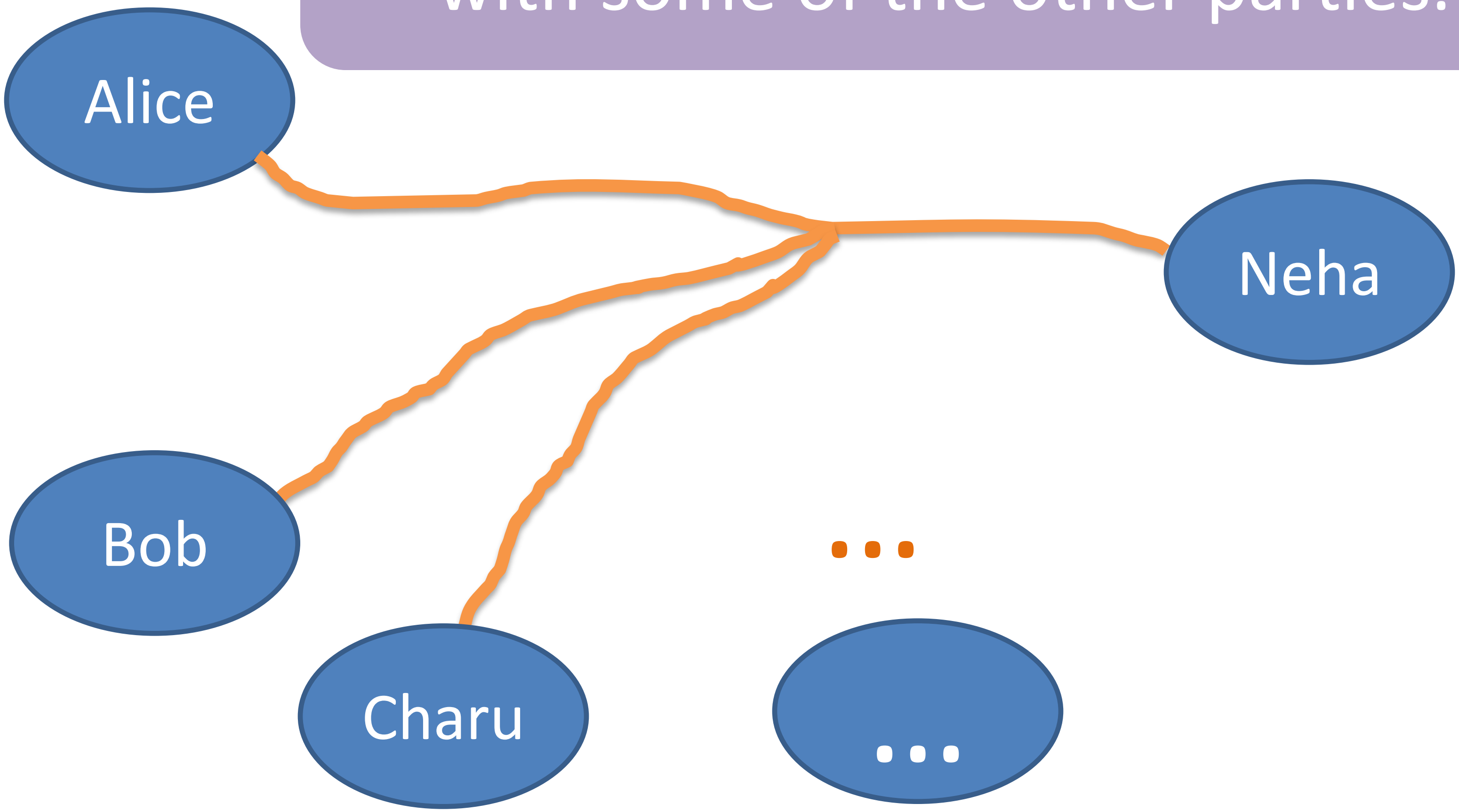


Exclusion pple for Q Dense Coding



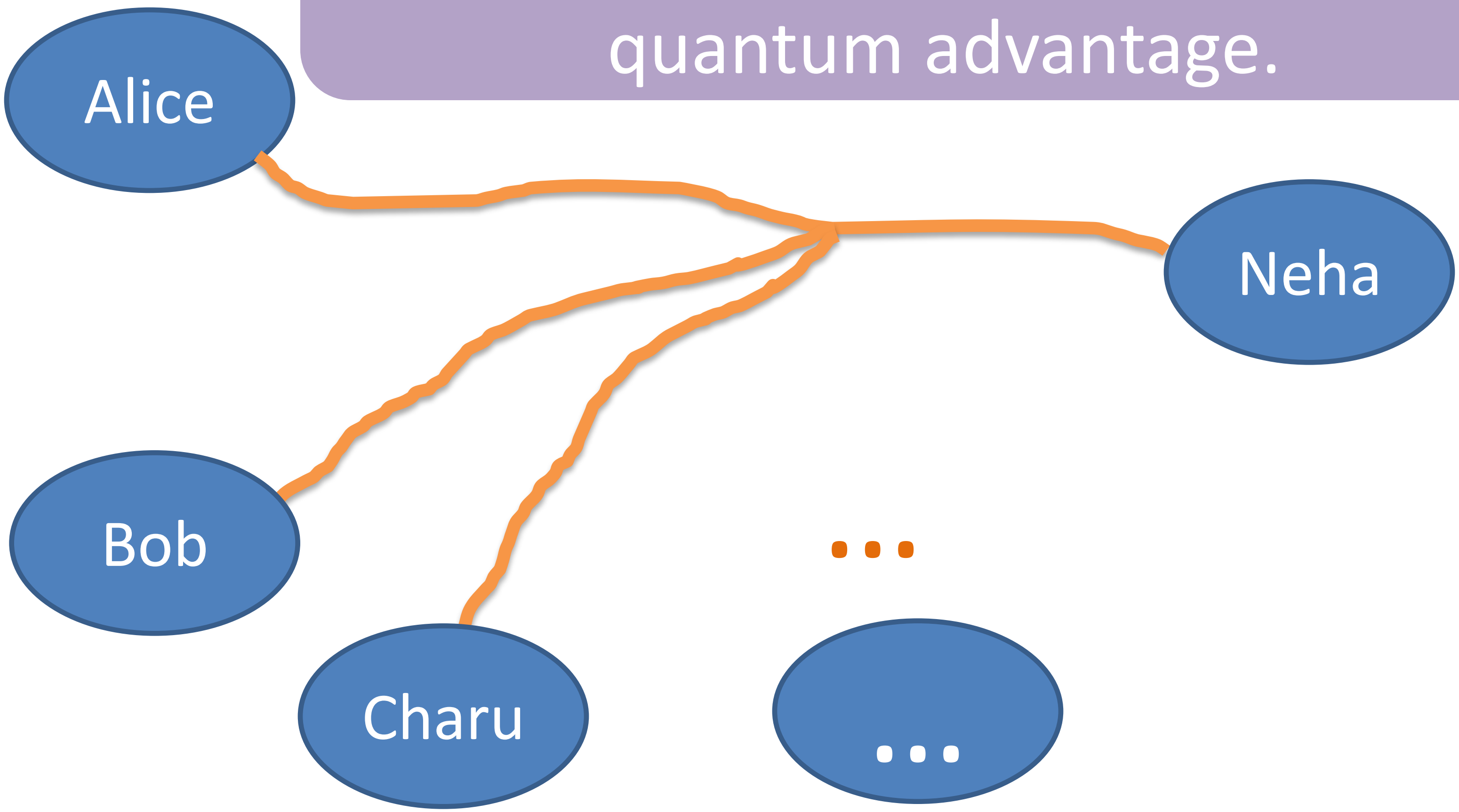
Exclusion pple for Q Dense Coding

Alice wishes to perform dense coding with some of the other parties.



Exclusion pple for Q Dense Coding

For every shared multiparty q state, **at most one** channel from Alice has a quantum advantage.



Exclusion pple for Q Dense Coding

For every shared multiparty q state, **<u>at most one</u>** channel from Alice has a quantum advantage.

Alice

Only two options possible:

C C C ……. C

OR

Q C C ……. C

Charu

…

Exclusion pple for Q Dense Coding

Note that
albeit of a
shown to b

This monogamy is stricter than that of quantum correlations.
W state have qc in AB ***and*** AC for whatever qc u may choose!

a

Only two options possible:

C C C ……. C

OR

Q C C ……. C

Charu

Prabhu R, Pati, Sen(De), US, Phys. Rev. A 2013

Outline of proof for 3-party states

- C_{AB} is quantum if $S_B - S_{AB} \geq 0$.
- This is becoz $C_{AB} = \log d + S_B - S_{AB}$,
  and log d can b achieved by classical means.
- And, C_{AC} is quantum if $S_C - S_{AC} \geq 0$.
- But, SSA dictates that sum of the LHSs ≤ 0.
- So, either both ≤ 0 or only one ≥ 0.

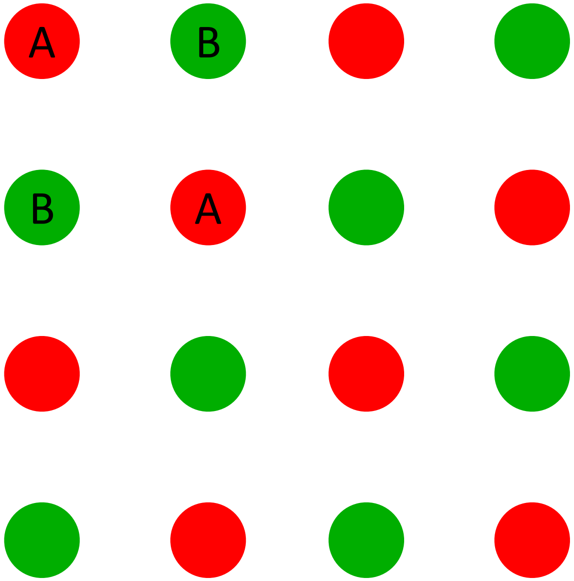
- For larger # of parties, suppose that two (or more channels) r quantum.
- Eg. let AB and AC be quantum for $\rho_{ABC \ldots N}$.
- Then ρ_{ABC} is such that AB and AC r quantum.
- This contradicts the 3-party result.

Outline

- What is strong subadditivity?
- SSA and the Holevo bound
- SSA and multiport quantum dense coding
- SSA and resonating valence bond states

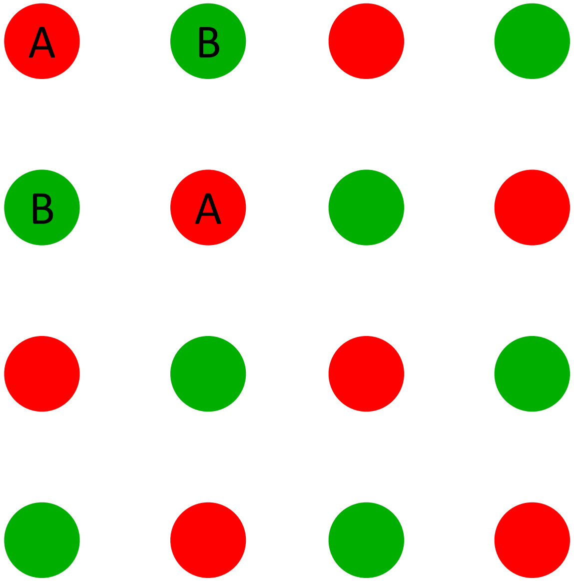
Resonating-Valence-Bond states

- $|\psi\rangle = \Sigma\, h(i_1,\ldots i_N;\, j_1,\ldots j_N)\ |(i_1, j_1)\rangle \ldots |(i_N, j_N)\rangle$



Resonating-Valence-Bond states

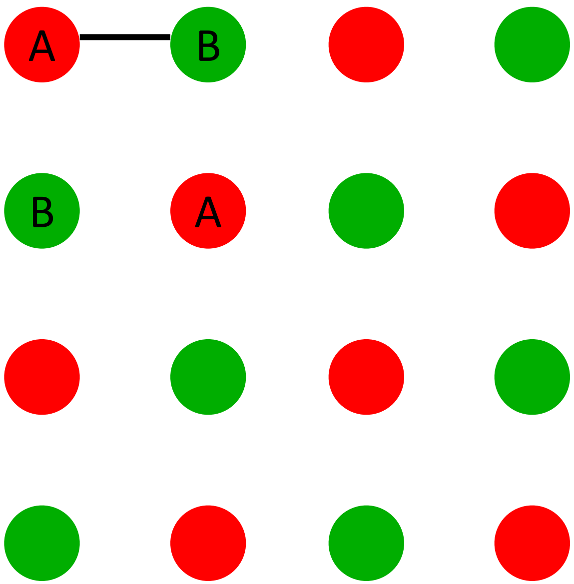
- $|\psi\rangle = \Sigma\, h(i_1,\ldots i_N;\, j_1,\ldots j_N)\; |(i_1, j_1)\rangle \ldots |(i_N, j_N)\rangle$



- $|(i, j)\rangle = \big(|\mathrm{up}\rangle\, |\mathrm{down}\rangle - |\mathrm{down}\rangle\, |\mathrm{up}\rangle\big)/2$

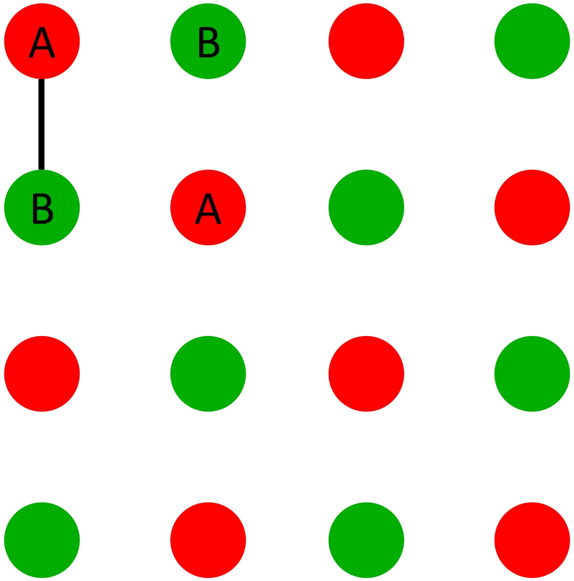
Resonating-Valence-Bond states

- $|\psi\rangle = \Sigma\, h(i_1,\ldots i_N;\, j_1,\ldots j_N)\ |(i_1, j_1)\rangle \ldots |(i_N, j_N)\rangle$



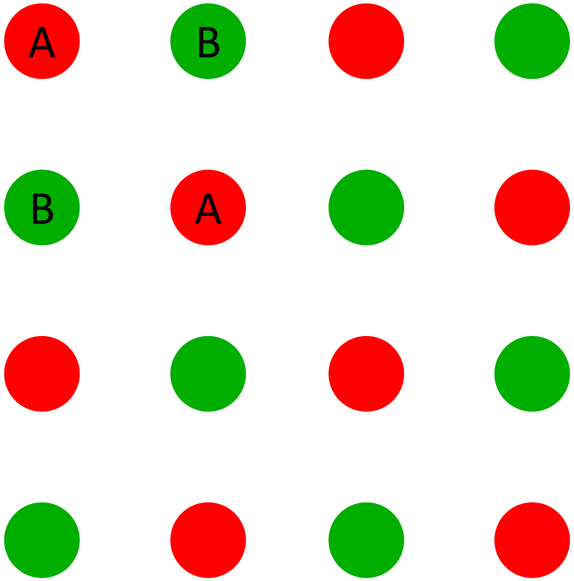
Resonating-Valence-Bond states

- $|\psi\rangle = \Sigma\, h(i_1,\ldots i_N;\, j_1,\ldots j_N)\ |(i_1,\, j_1)\rangle \ldots |(i_N,\, j_N)\rangle$



Resonating-Valence-Bond states

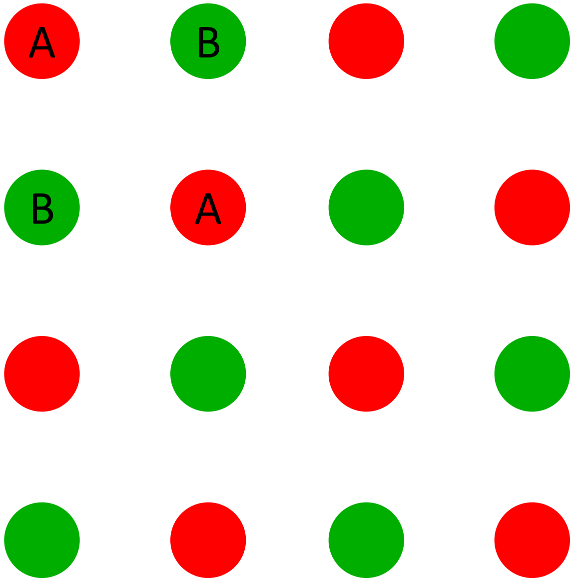
- $|\psi\rangle = \Sigma\, h(i_1,\ldots i_N;\, j_1,\ldots j_N)\ |(i_1,\, j_1)\rangle\ \ldots\ |(i_N,\, j_N)\rangle$



- h is assumed to be isotropic over the lattice.

Resonating-Valence-Bond states

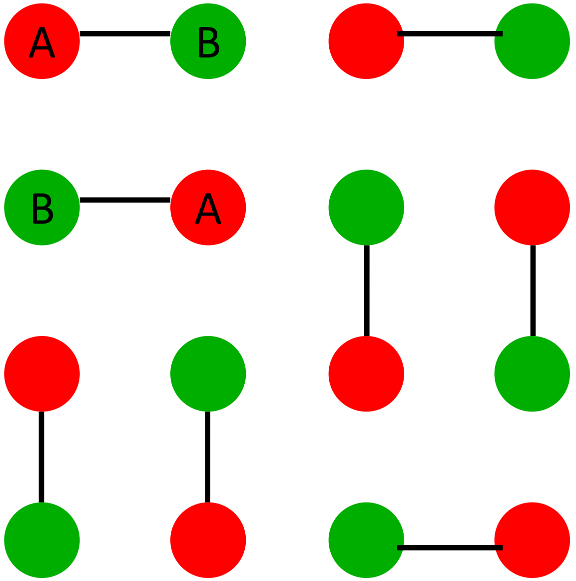
- $|\psi\rangle = \Sigma\, h(i_1,\ldots i_N;\, j_1,\ldots j_N)\, |(i_1, j_1)\rangle \ldots |(i_N, j_N)\rangle$



- Coverings of only nearest neighbor dimers.

Resonating-Valence-Bond states

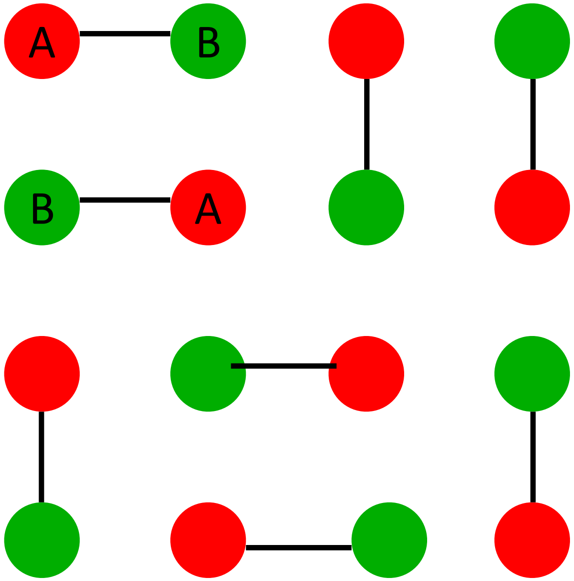
- $|\psi\rangle = \Sigma\, h(i_1,\ldots i_N;\, j_1,\ldots j_N)\, |(i_1, j_1)\rangle \ldots |(i_N, j_N)\rangle$



- Coverings of only nearest neighbor dimers.

Resonating-Valence-Bond states

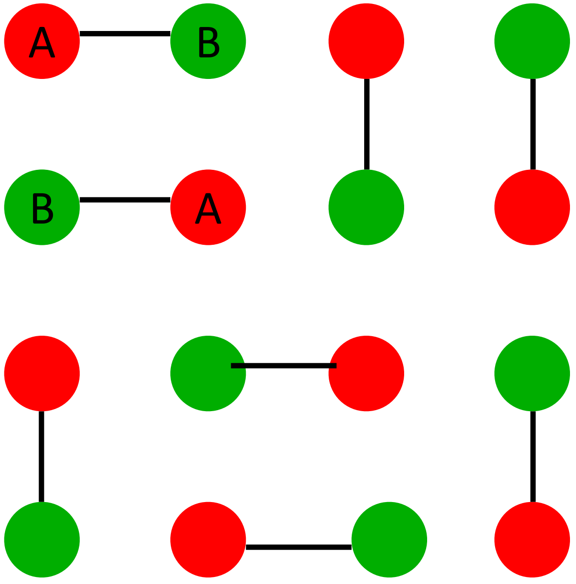
- $|\psi\rangle = \Sigma\, h(i_1,\ldots i_N;\, j_1,\ldots j_N)\ |(i_1,\, j_1)\rangle \ldots |(i_N,\, j_N)\rangle$



- Coverings of only nearest neighbor dimers.

Resonating-Valence-Bond states

- $|\psi\rangle = \Sigma\, h(i_1,\ldots i_N;\, j_1,\ldots j_N)\, |(i_1, j_1)\rangle \ldots |(i_N, j_N)\rangle$

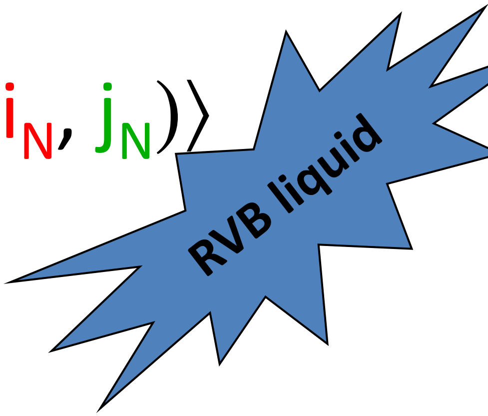


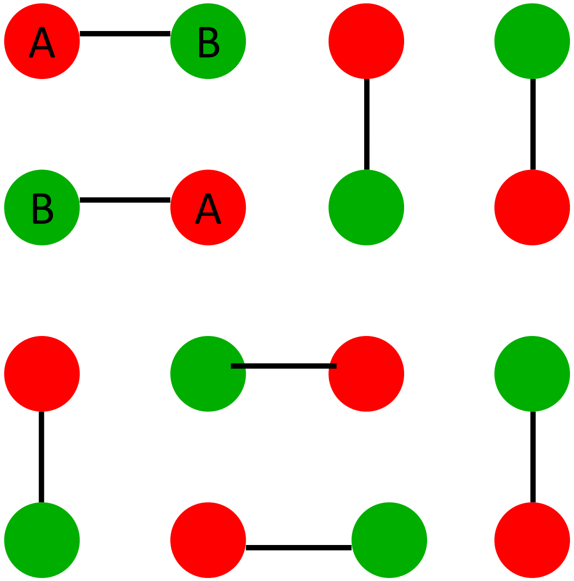
+ other nn coverings

- Coverings of only nearest neighbor dimers.

Resonating-Valence-Bond states

- $|\psi\rangle = \Sigma\, h(i_1,\ldots i_N;\, j_1,\ldots j_N)\ |(i_1, j_1)\rangle \ldots |(i_N, j_N)\rangle$





+ other nn coverings

- Coverings of only nearest neighbor dimers.

Genuine multiparty entanglement

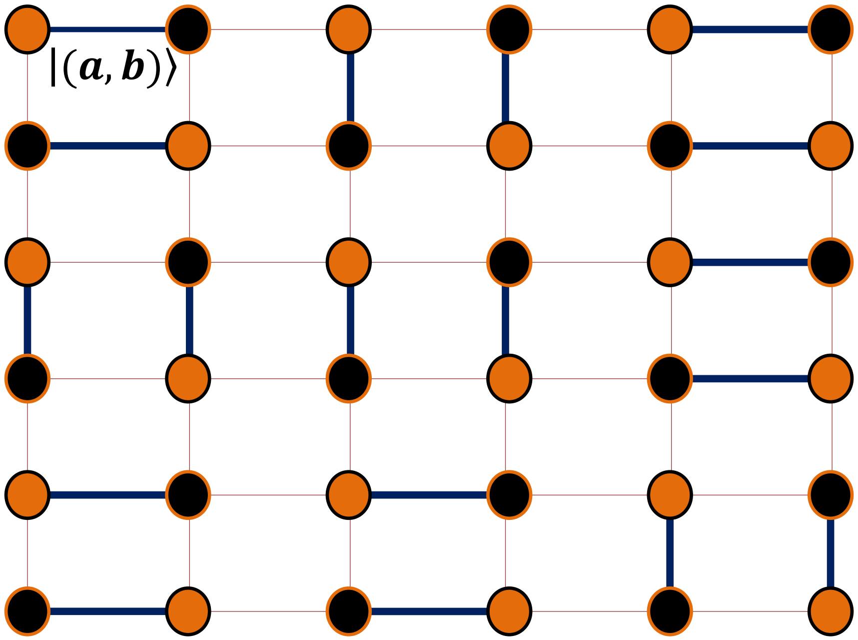
A multiparty pure quantum state is said to be genuinely multiparty entangled if

it is entangled across **every** bipartition.

ENTANGLEMENT PROPERTIES:

MULTIPARTITE ENTANGLEMENT IN ISOTROPIC RVB

CONSIDER AN ODD: REST BIPARTITE CUT



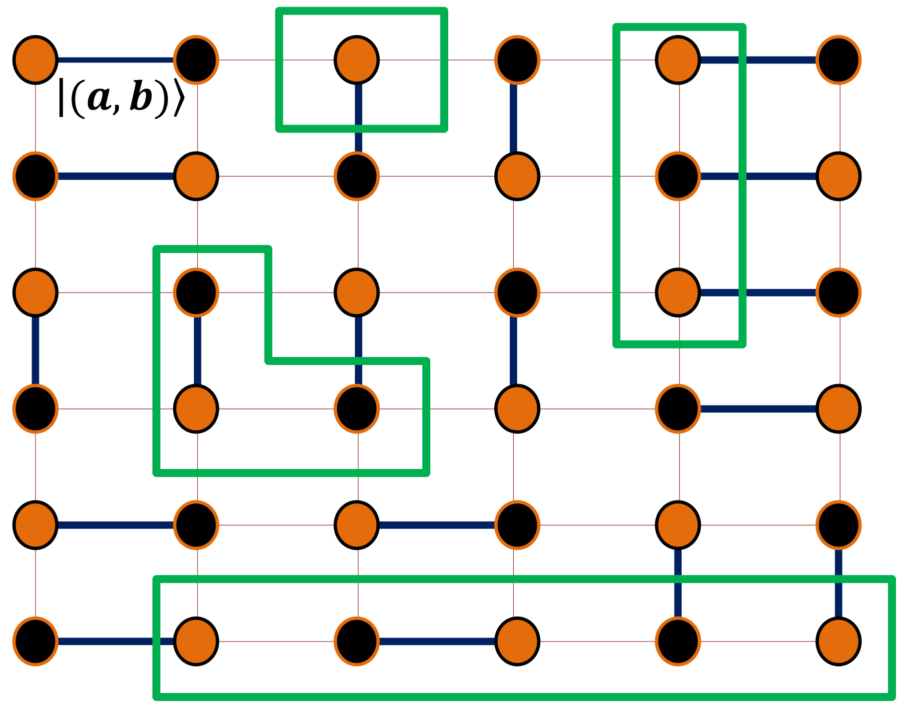
Chandran, Kaszlikowski, Sen(De), US, Vedral, PRL'07
Dhar, Sen(De), US, PRL'13

ENTANGLEMENT PROPERTIES:

MULTIPARTITE ENTANGLEMENT IN ISOTROPIC RVB

CONSIDER AN ODD: REST BIPARTITE CUT

$$\rho_{odd} = Tr_{rest}\left(|\psi\rangle\langle\psi|_{RVB}\right)$$



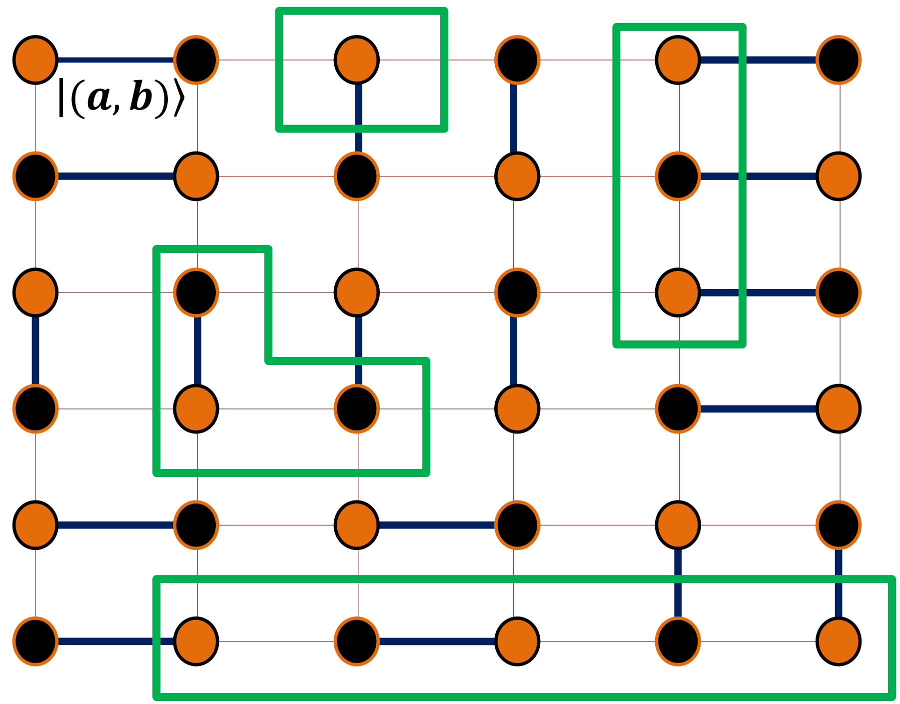
Chandran, Kaszlikowski, Sen(De), US, Vedral, PRL'07
Dhar, Sen(De), US, PRL'13

ENTANGLEMENT PROPERTIES:

MULTIPARTITE ENTANGLEMENT IN ISOTROPIC RVB

CONSIDER AN ODD: REST BIPARTITE CUT

$$\rho_{odd} = Tr_{rest}(|\psi\rangle\langle\psi|_{RVB})$$



There is no pure ρ_{odd} that is rotationally invariant.

(Since RVB is rotationally invariant, ρ_{odd} has to be so.)

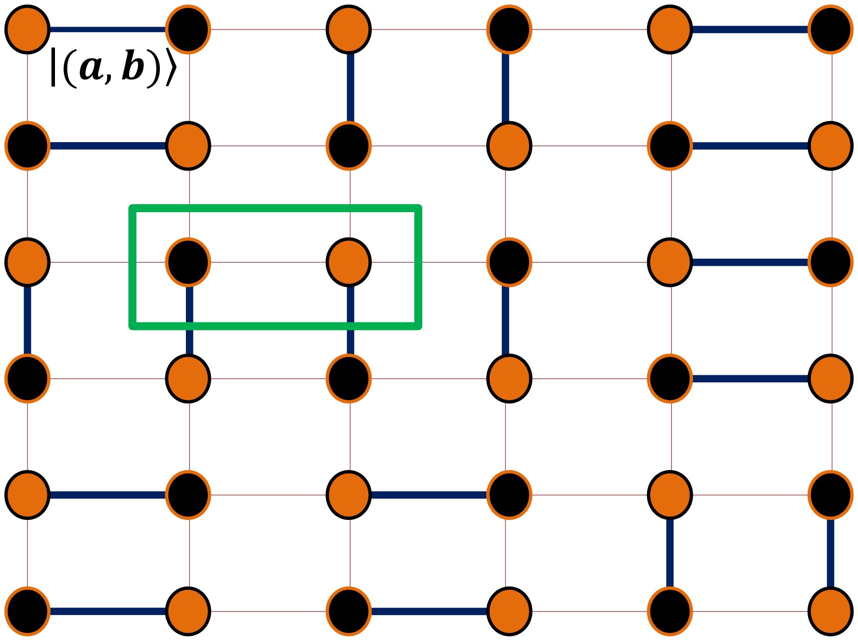
ρ_{odd} is mixed. The bipartition ODD: rest is entangled.

Chandran, Kaszlikowski, Sen(De), US, Vedral, PRL'07
Dhar, Sen(De), US, PRL'13

ENTANGLEMENT PROPERTIES:

MULTIPARTITE ENTANGLEMENT IN ISOTROPIC RVB

CONSIDER A EVEN: REST BIPARTITE CUT

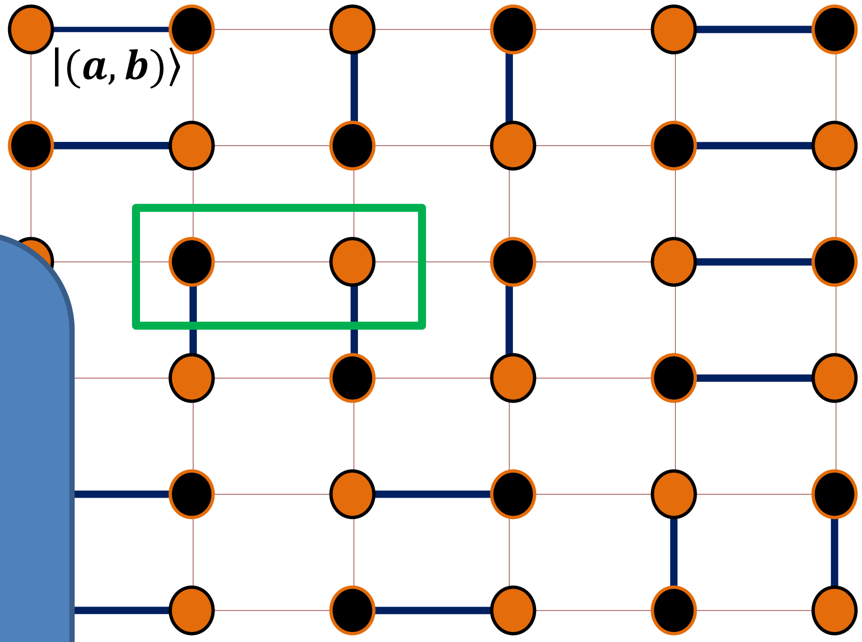


Dhar, Sen(De), US, PRL'13

ENTANGLEMENT PROPERTIES:

MULTIPARTITE ENTANGLEMENT IN ISOTROPIC RVB

CONSIDER A EVEN: REST
BIPARTITE CUT



Can be shown to be entangled by using "strong subadditivity"

Dhar, Sen(De), US, PRL'13

ENTANGLEMENT PROPERTIES:

MULTIPARTITE ENTANGLEMENT IN ISOTROPIC RVB

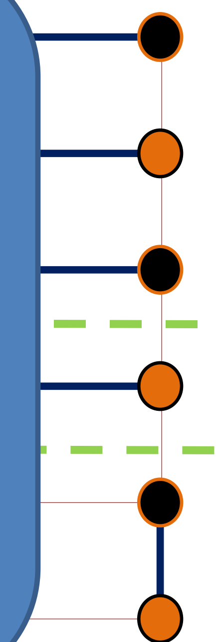
FOR INFINIT
INIFNITE:IN
BIPARTITIO

ASSUME EA
(GREEN) IS

WE CAN AL
STRIPS THA

WE CAN AGA
SINGLE SITE THAT IS
REQUIRED TO BE PURE

Similar proofs can show that INFINITE : INFINITE BIPARTITIONS ARE ALSO ENTANGLED



Dhar, Sen(De), US, PRL’13

In conclusion, …

- Strong subadditivity is very useful.
- Has a host of important applications in QIC.

Thank you!

QIC Group @ HRI, 2013

Pictures used may not be free, and so do not use them commercially without relevant permissions!

References r incomplete!