Upper bound on singlet fraction of mixed entangled two qubit states

Satyabrata Adhikari
Indian Institute of Technology Jodhpur, Rajasthan

Collaborator: Dr. Atul Kumar

Definitions

- *A pure state* $|\psi\rangle_{AB} \in H_A \otimes H_B$ *is separable iff*

$$|\psi\rangle_{AB} = |e\rangle_A \otimes |f\rangle_B$$

where $|e\rangle_A$ *and* $|f\rangle_B$ *denotes the basis of* H_A *and* H_B *respectively.*

- *A mixed state* ρ *is entangled iff it cannot be represented as*

$$\rho = \sum_i p_i |e_i, f_i\rangle\langle e_i, f_i| \,,\ \sum_i p_i = 1 \,,\ p_i \geq 0$$

Introduction

- Quantum entanglement has been used as an efficient resource for several quantum communication protocols.
- In general, if a state is maximally entangled then the optimal success of a communication protocol is a certainty.
- In an open system it is practically not possible to keep the state with cent percent purity.
- We have to deal with mixed entangled resources for quantum information processing.

Relation between teleportation and singlet fraction

- A mixed two-qubit entangled state useful for teleportation if the singlet fraction is greater than ½.

- Singlet fraction:

$$F(\sigma) = \max_{|\psi_{ME}\rangle} \langle \psi_{ME} | \sigma | \psi_{ME} \rangle$$

- Teleportation fidelity:

$$f_T(\sigma) = \frac{2F(\sigma)+1}{3}$$

M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A 60, 1888 (1999).

Result of Badziag et.al.

- Badzaig and Horodecki have shown that there exist mixed states with fidelity smaller than 1/*2, for which* local trace preserving protocols exist that transform this state into a state with fidelity larger than 1/*2* *without the* help of classical communication.
- Is it possible to show that any entangled state is useful for teleportation?

P. Badziąg, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A 62, 012311 (2000).

Result of Verstraete and Verschelde

- Proved that the optimal trace preserving protocol for maximizing the singlet fraction of a given state always belongs to a class of one-way communication (1-LOCC).
- Shown that any entangled two-qubit mixed state can be used as a resource for quantum teleportation using certain trace preserving local operations and classical communications.

F. Verstraete and H. Verschelde, Phys. Rev. Lett. 90, 097901 (2003)

Result of Verstraete and Verschelde

- A filter is constructed in such a way so that the cost function defined below is maximal.
- Cost function (K):

$$K = p_{AB}\, F(\sigma_f) + \frac{(1-p_{AB})}{2},$$

$$\text{where} \quad \sigma_f = \frac{(A\otimes I)\,\sigma\,(A\otimes I)^{+}}{p_{AB}}, \quad p_{AB} = Tr\,[(A\otimes I)\,\sigma\,(A^{+}\otimes I)]$$

- The optimal filter and singlet fraction F^{*} can be found by solving the convex semidefinite program.

F. Verstraete and H. Verschelde, Phys. Rev. Lett. 90, 097901 (2003)

Result of Verstraete and Verschelde

- Convex optimization problem:

$$F^*(\sigma) = Max[\frac{1}{2} - Tr(X\sigma^\Gamma)]$$

$$Subject\ to\quad 0 \le X \le I_4$$

$$\frac{-I_4}{2} \le X \le \frac{I_4}{2}$$

$X = (A \otimes I_2)\left|\psi^-\right\rangle\left\langle\psi^-\right|(A^+ \otimes I_2)$, *and* A represents the filter.

F. Verstraete and H. Verschelde, Phys. Rev. Lett. 90, 097901 (2003)

Result

- Verstraete and Verschelde have shown that using trace preserving optimal local operations, the maximal achievable singlet fraction F^* for the family of states σ_f is

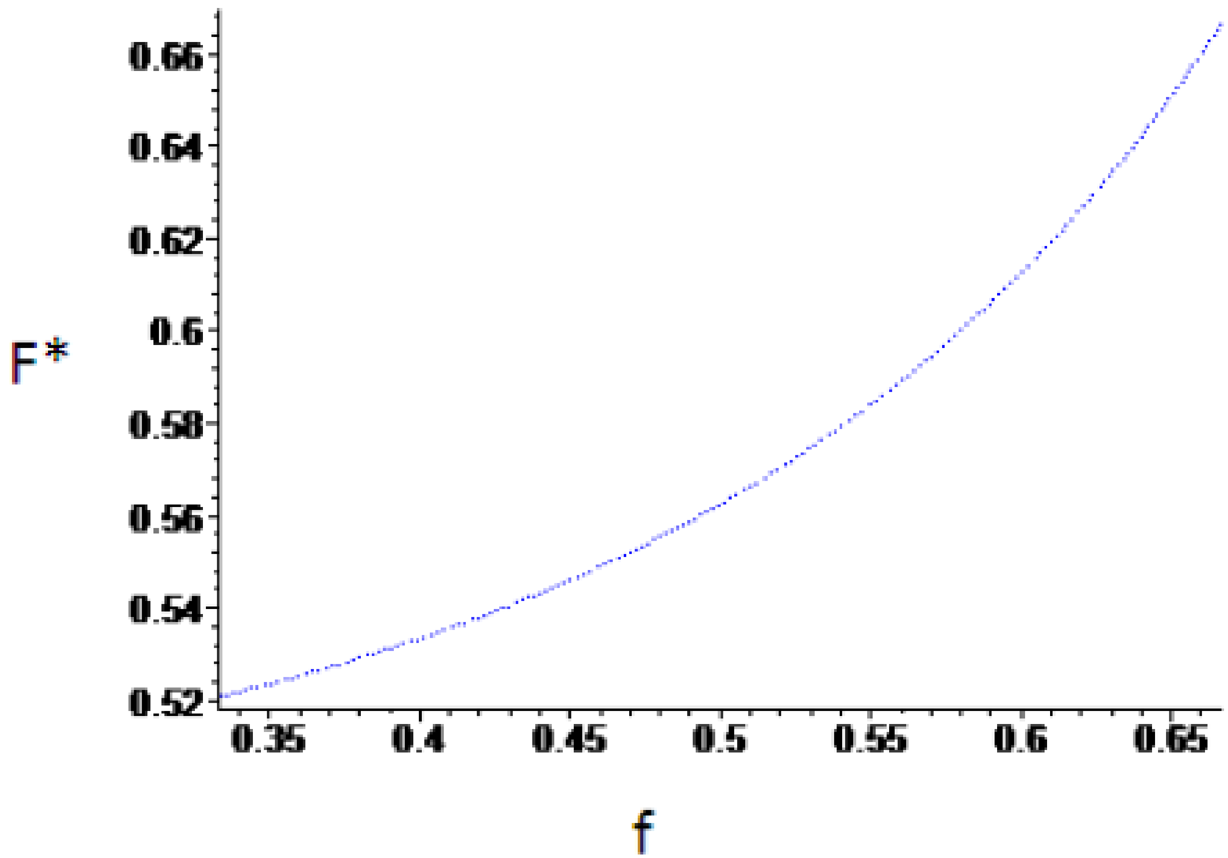
$$F^*(\sigma_f) = \frac{1}{2}[1 + \frac{f^2}{4(1-f)}]; \ \frac{1}{3} \le f \le \frac{2}{3}$$

$$F^*(\sigma_f) = f \qquad ; \ \frac{2}{3} \le f \le 1$$

$$\sigma_f = f|\psi\rangle\langle\psi| + (1-f)|01\rangle\langle 01|, \ |\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

F. Verstraete, and H. Verschelde, Phys. Rev. Lett. 90, 097901 (2003)

Motivation



Theorem

- For any real $n \otimes n$ matrix C and a positive semidefinite operator B, the following inequality holds

$$\lambda_1(\overline{C})\, Tr(B) \leq Tr(CB) \leq \lambda_n(\overline{C})\, Tr(B)$$

$where \quad \overline{C} = \dfrac{C + C^T}{2}$, $\lambda_n(\overline{C})$ is the nth eigenvalue of the matrix $\overline{C}$ and $\lambda_1 \leq \lambda_2 \leq \lambda_3 .... \leq \lambda_n$.

Y. Fang, K. A. Loparo, and X. Feng, IEEE Transactions on Automatic Control 39, 2489 (1994).

Upper bound on singlet fraction

For $C = \frac{I_4}{2} - X^{\Gamma}$ *and* $B = \sigma$, we have

$$\lambda_1\left(\frac{I_4}{2} - X^{\Gamma}\right) \le F^* = Tr\left[\left(\frac{I_4}{2} - X^{\Gamma}\right)\sigma\right] \le \lambda_4\left(\frac{I_4}{2} - X^{\Gamma}\right)$$

where $X = (A \otimes I_2)\left|\psi^-\right\rangle\left\langle\psi^-\right|(A^+ \otimes I_2)$, *and* A represents the filter.

- The upper bound depends on the state parameter and hence, must have a maximum achievable value for every particular value of the state parameter. This value would be provided by Dembo's bound.

Dembo's Bound

Theorem: For any $n \otimes n$ positive semidefinite operator R_n with eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3 \le \dots\dots\dots\dots \le \lambda_n$, Dembo's bound can be given by

$$\frac{c+\eta_1}{2} + \sqrt{\left(\frac{c-\eta_1}{2}\right)^2 + b^* b} \le \lambda_n(R_n) \le \frac{c+\eta_{n-1}}{2} + \sqrt{\left(\frac{c-\eta_{n-1}}{2}\right)^2 + b^* b}$$

where $R_n = \begin{bmatrix} R_{n-1} & b \\ (b^*)^T & c \end{bmatrix}$, η_1 is the lower bound on the minimal eigenvalue of R_{n-1},

η_{n-1} is the upper bound on the maximal eigenvalue of R_{n-1} and b is an eigenvector of dimension $n-1$.

A. Dembo, IEEE Trans. Inform. Theory 34, 352 (1988).

Modified upper bound on singlet fraction

- Using Dembo's bound it can be easily shown that the upper bound on optimal singlet fraction is

$$F_D^* = \frac{c+\eta_3}{2} + \sqrt{\left(\frac{c-\eta_3}{2}\right)^2 + b^* b}, \quad \frac{I_4}{2} - X^{\Gamma} = \begin{bmatrix} (\frac{I_4}{2}\text{-}X^{\Gamma})_3 & b \\ (b^*)^T & c \end{bmatrix}$$

where η_3 is the upper bound on the maximal eigenvalue of $(\frac{I_4}{2}\text{-}X^{\Gamma})_3$.

Calculation of Dembo's bound

The upper bound on singlet fraction F_D^* is

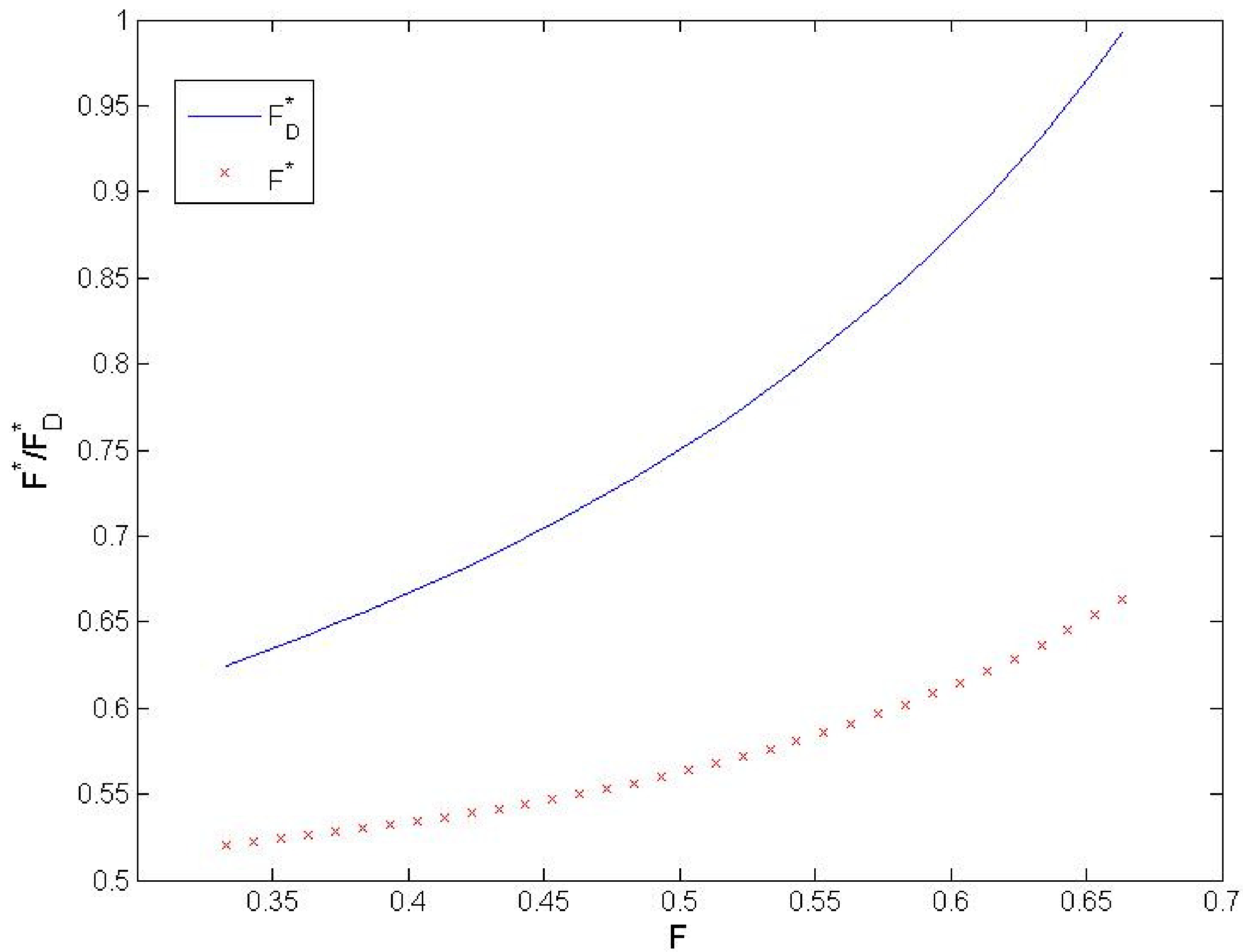
$$F_D^*(\sigma_f) = \frac{2-f}{4(1-f)};\ \frac{1}{3} \le f \le \frac{2}{3}$$

$$F_D^*(\sigma_f) = f;\ f \ge \frac{2}{3}$$

$$\text{where } \sigma_f = f\left|\psi\right\rangle\left\langle\psi\right| + (1-f)\left|01\right\rangle\left\langle 01\right|,\ \ \left|\psi\right\rangle = \frac{1}{\sqrt{2}}(\left|00\right\rangle + \left|11\right\rangle)$$

$$\frac{I_4}{2} - X^\Gamma = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{F}{4(1-F)} \\ 0 & \frac{1}{2} - \frac{F^2}{8(1-F)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{F}{4(1-F)} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Comparison of maximal singlet fraction F^* and upper bound on singlet fraction F_D^* obtained using Dembo's boun '



Local operations and classical communication

- Our next task is to find a way by which we can obtain this bound experimentally i.e. would it be possible to increase the value of optimal singlet fraction performing local operations and classical communication on the filtered state i.e. can we achieve the upper bound of singlet fraction given by Dembo's bound?
- Our results show that the bound is indeed achievable by performing local operations and classical communication.

Singlet fraction after second filtering operation

- In order to enhance the value of optimal singlet fraction F, we perform another filtering operation on the filtered state such that the singlet fraction of the output state can be given as

$$F_{opt}^{*} = p\,F^{*}(\sigma_{f}) + (1-p)\,F(\sigma_{f})$$

where p is the success probability multiplied with the optimal singlet fraction of the state coming out of the second filter.

Singlet fraction after second filtering operation

- Define $1-p=p_{AB}$ where p_{AB} denotes the success probability of the first filter.

Then for $F(\sigma_f)=\langle\psi|\sigma_f|\psi\rangle$, where $\sigma_f=\dfrac{(A\otimes I_2)\sigma(A^+\otimes I_2)}{p_{AB}}$.

F_{opt}^* can be re-expressed as

$$F_{opt}^*=(1-p_{AB})F^*(\sigma_f)+tr[(A\otimes I)\sigma(A^+\otimes I)|\psi\rangle\langle\psi|]$$

Comparison between the singlet fraction obtained after the first and second filter

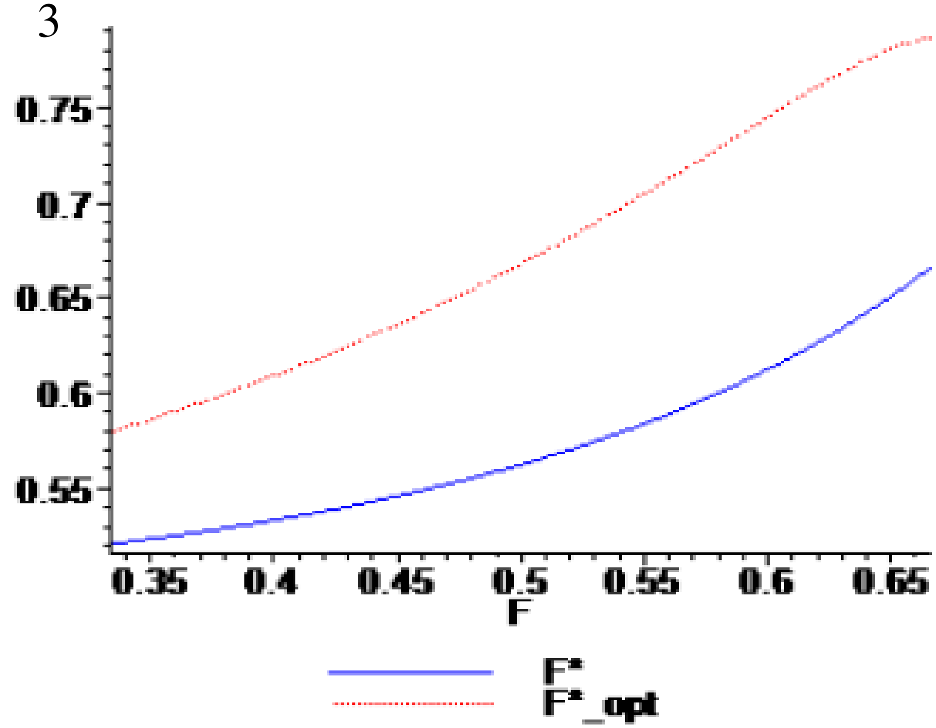
For the state described by the density operator

$$\sigma_f = f\left|\psi\right\rangle\left\langle\psi\right| + (1-f)\left|01\right\rangle\left\langle 01\right|,\quad \left|\psi\right\rangle = \frac{1}{\sqrt{2}}(\left|00\right\rangle + \left|11\right\rangle)$$

$$p_{AB} = \frac{f^3}{8(1-f)^2} + \frac{f^2}{4(1-f)} + \frac{f}{2},\quad \frac{1}{3} \le f \le \frac{2}{3}$$

$$F^*(\sigma_f) = \frac{1}{2}[1 + \frac{f^2}{4(1-f)}];\ \frac{1}{3} \le f \le \frac{2}{3}$$

$$A = \frac{f}{2(1-f)}\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|$$



Optimal value of the probability p_{AB}

- For p to be high, p_{AB} must be minimized.
- The minimum value of p_{AB} should be chosen in such a way that the value of singlet fraction for the second filter must not exceed Dembo's bound.
- The minimum value of p_{AB} would be

$$p_{AB}^{\min} = 1 - \frac{F_D^* - Tr\,[(A \otimes I)\,\sigma\,(A^T \otimes I)\,|\psi\rangle\langle\psi|]}{F^*(\rho_f)}$$

- With this minimum value of p_{AB}, we have

$$F_{opt}^* = F_D^*$$

- This shows that one can achieve the maximum singlet fraction equals to Dembo's bound by using the filtering operations twice.

Illustration

For the state

$$\rho_f = f\left|\psi\right\rangle\left\langle\psi\right| + (1-f)\left|01\right\rangle\left\langle 01\right|, \quad \left|\psi\right\rangle = \frac{1}{\sqrt{2}}(\left|00\right\rangle + \left|11\right\rangle)$$

The minimum success probability of the filter is

$$p_{AB}^{\min} = \frac{f^2}{2(1-f)(2-f)}$$

Hence the optimal singlet fraction would be

$$F_{opt}^{*} = \frac{2-f}{4(1-f)}$$

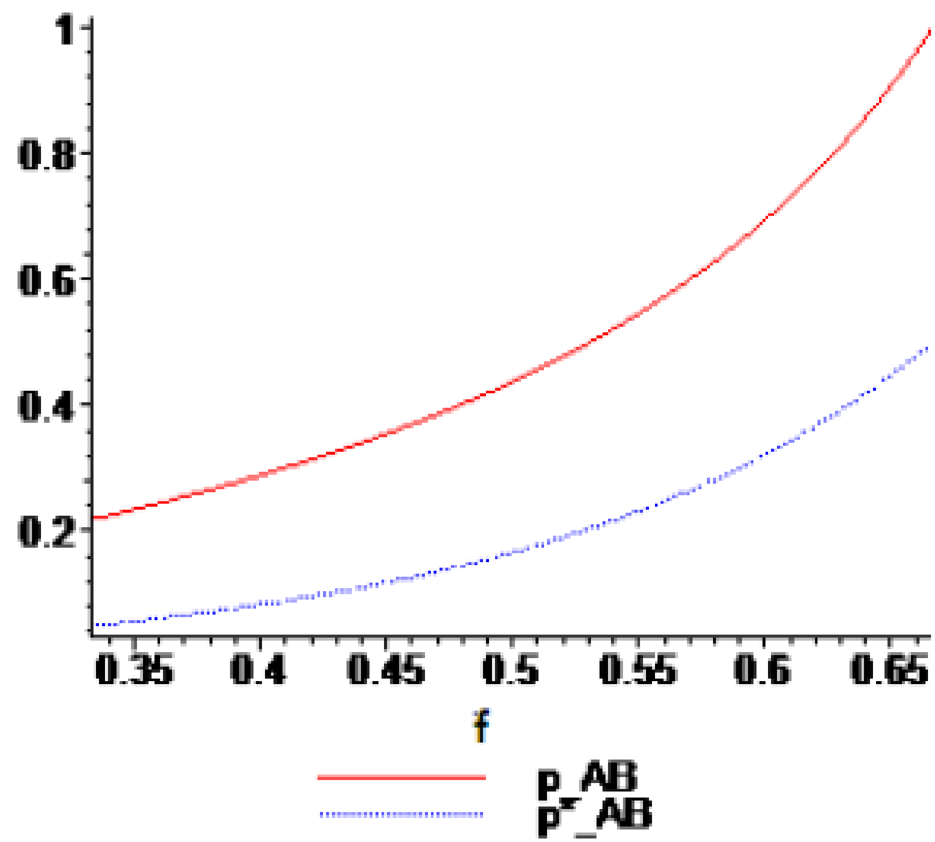
Illustration

- In the above example, we have

$$F_{opt}^{*} = F_{D}^{*}$$

- Applying the filter twice will always result in achieving the upper bound on singlet fraction for any two-qubit mixed density operator.

Comparison between the probabilities of success after passing through the First Filter



Summary

- We have established a relation between Dembo's upper bound and singlet fraction (and hence with teleportation fidelity) of a mixed two-qubit entangled state.
- This relation is used to demonstrate that any two-qubit mixed entangled state can be used as a resource to achieve maximum possible teleportation fidelity.
- we found that the maximum fidelity obtained earlier can be increased with additional local operations with certain non-zero probability.

THANK YOU