Lossy quantum data compression

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Joe Renes and Renato Renner

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- **arXiv:1304.2336**

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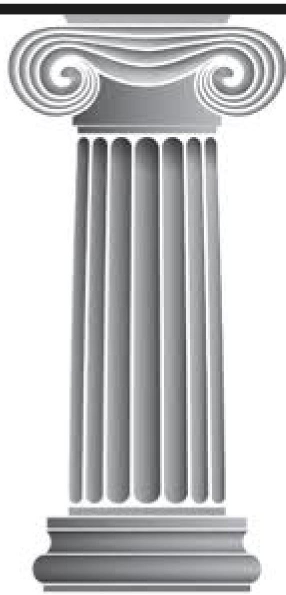
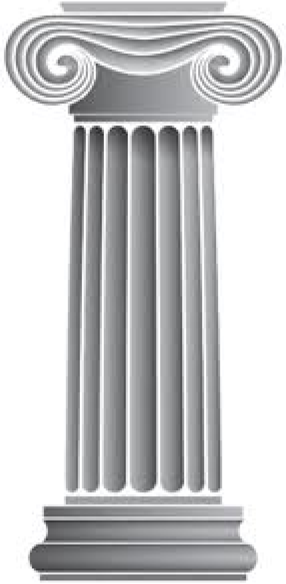
Classical Information Theory





Shannon's Source Coding Theorem

Compression of information emitted by a source

Shannon's Noisy Channel Coding Theorem

Transmission of information through a noisy channel

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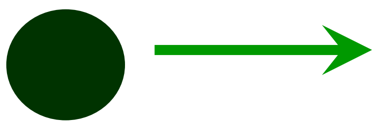
Shannon's Coding Theorems

- **Source Coding Theorem**



Memoryless *information source*

$$X \sim p_X$$

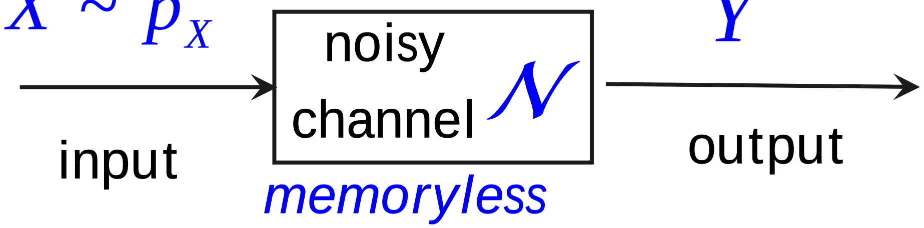
Data compression limit

= *Shannon entropy of the source:*

$$H(X) = -\sum_x p(x) \log p(x)$$

- **Channel Coding Theorem**

$X \sim p_X$



Y

$p_{Y|X}$

Capacity of the channel

$$C(\mathcal{N}) = \max_{p_X} I(X:Y)$$

mutual information

$$(= H(X) + H(Y) - H(X,Y))$$

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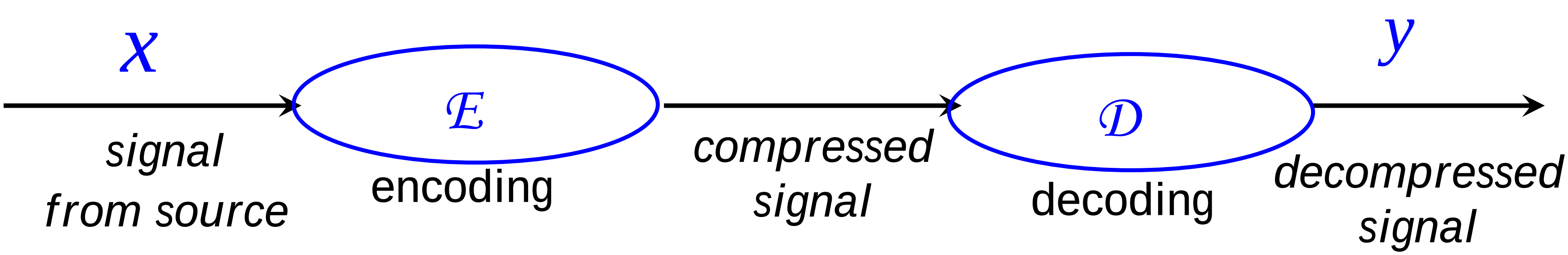
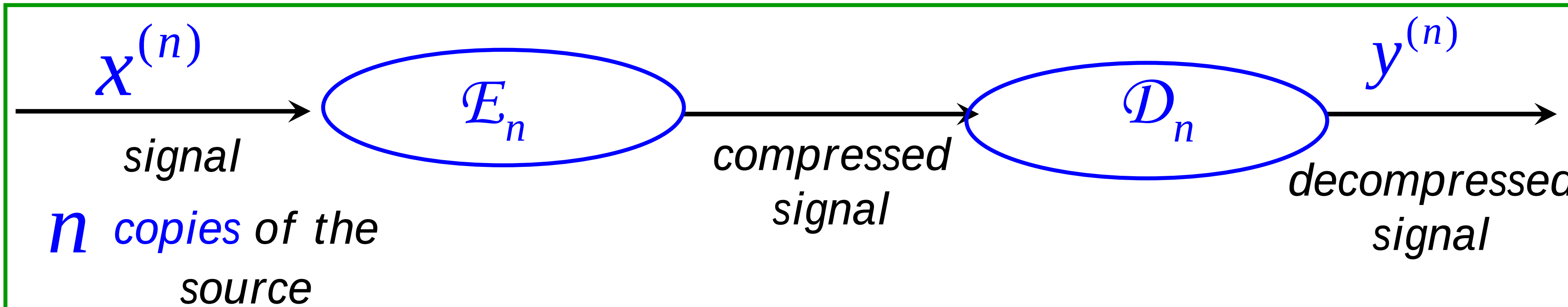
Lossless Data Compression



If $y = x$ -- **lossless**

Shannon's Source CodingTheorem: *asymptotically lossless data compression*



$$\Pr(y^{(n)} \neq x^{(n)}) \to 0 \quad \text{as} \quad n \to \infty$$

Lossless Compression - often too stringent a condition

- for cases of multimedia data;

  e.g. audio, video and still images

- or when storage space is insufficient
- or for continuous variable sources

- **Why ?**

Typically, a substantial amount of data can be discarded before the information is sufficiently degraded to be noticeable!

original image

84% less information

Lossy Data Compression

- a data compression scheme :

- recovered data $\neq$ original data

instead

- recovered data $\overset{D}{\approx}$ original data

D: the maximum allowed distortion

0000000000000000
0100000000100000

Rate Distortion Theory $=$ the theory of Lossy Data Compression

- rate of compression ⟵ tradeoff ⟶ the allowed distortion

→ fundamental limit on the asymptotic rate of data compression for a given maximum distortion D:

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Rate Distortion Function

$R(D)$: optimal rate of data compression for a given maximum distortion D.

A simple example: Random bit source

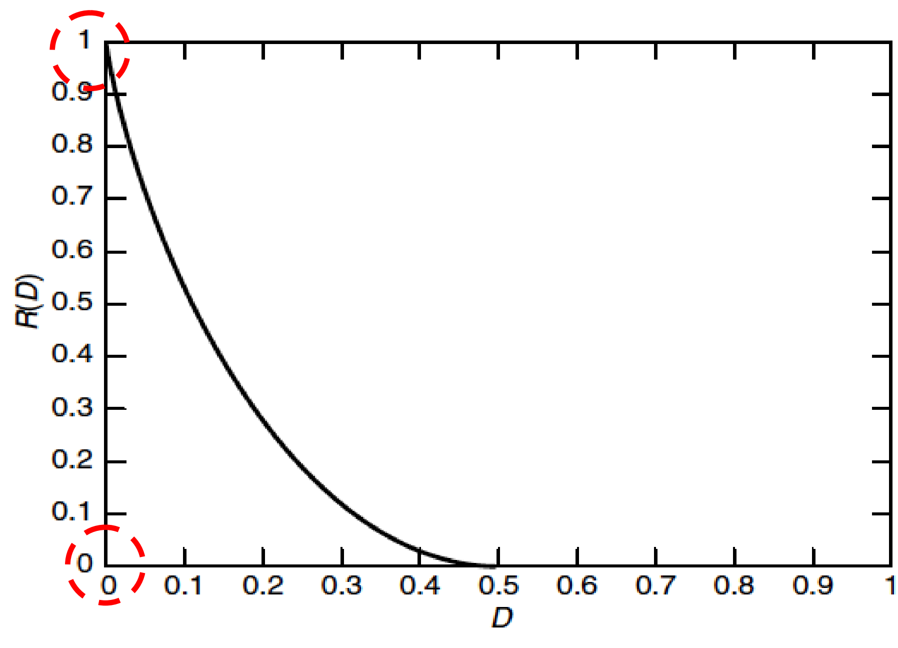
- Source alphabet: $\{0,1\}$ $\Pr(X=0)=\frac{1}{2}$; $\Pr(X=1)=\frac{1}{2}$

  $H(X)=\log 2 = 1$

- The rate distortion function:

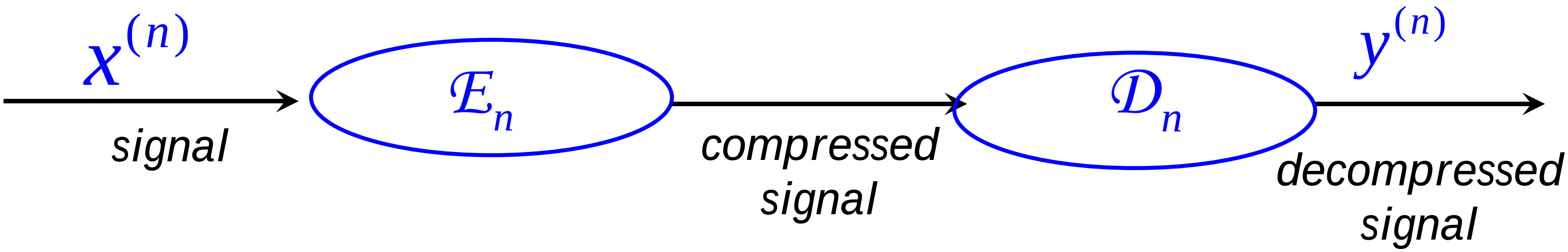
$$R(D)=\begin{cases}1-h(D) & \text{if } 0\le D<\frac{1}{2}\\ 0 & \text{if } \frac{1}{2}\le D\le 1\end{cases}$$

$h(D)=-D\log D-(1-D)\log(1-D)$
[binary entropy]



(Q) How is the distortion measured?

(A) On average:



$$x^{(n)} = \left(x_1, \ldots, x_i, \ldots, x_n\right); \qquad y^{(n)} = \left(y_1, \ldots, y_i, \ldots, y_n\right)$$

average distortion

$$\bar{d}\left(x^n, \mathcal{D}_n \circ \mathcal{E}_n\right) \equiv \bar{d}\left(x^n, y^n\right) := \frac{1}{n}\sum_{i=1}^{n} d\left(x_i, y_i\right)$$

single-letter distortion measure
(e.g. Hamming distortion)

$$d(x_i, y_i) = 0 \quad \text{if} \quad x_i = y_i$$
$$= 1 \quad \text{else}$$

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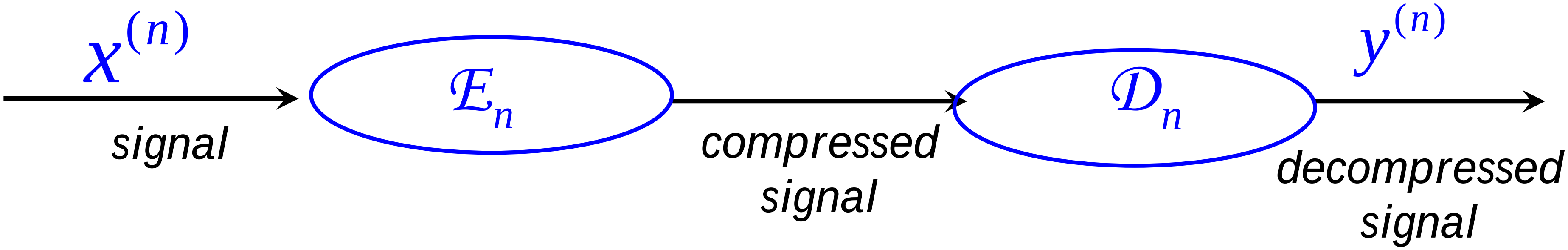
(Q) **How is the distortion measured?**

(A) On average:



average distortion

$$\bar{d}\left(x^{(n)}, \mathcal{D}_n \circ \mathcal{E}_n\right) \equiv \bar{d}\left(x^{(n)}, y^{(n)}\right) := \frac{1}{n}\sum_{i=1}^{n} d\left(x_i, y_i\right)$$

single-letter distortion measure

(e.g. Hamming distortion)

$R(D)$: One requires:

$\exists \mathcal{E}_n, \mathcal{D}_n$

$$\lim_{n\to\infty} \bar{d}\left(x^{(n)}, y^{(n)}\right) \leq D$$

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Classical Rate Distortion Theory (Shannon)

Theorem (Shannon): For a memoryless info. source $X \sim p_X$

rate distortion function

$$R(D) = \min_{\substack{p_{Y|X} \\ \mathbf{E}(d(X,Y)) \le D}} I(X:Y)$$

mutual information

Y : output of a stochastic map $p_{Y|X} : X \to Y$

$d(x, y)$: distortion measure

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Our Aim

To obtain rate distortion functions in the quantum realm

Motivation

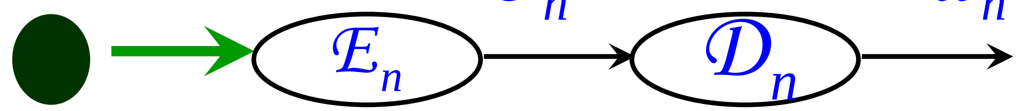
- when storage is insufficient
- to derive the quantum analogue of a cornerstone of Classical Information Theory

Scenario for Quantum Rate Distortion

for a fixed $0 \le D < 1$

- Storage setting

$\{\rho, \mathcal{H}_A\}$ memoryless quantum information source $\rho^{\otimes n}$



$\mathcal{E}_n$ $\quad \sigma_n \quad$ $\mathcal{D}_n$ $\quad \omega_n$

compressed state in $\mathcal{H}_c^n$

recovered signal

Rate of data compression $= R$ *if* $\dim \mathcal{H}_c^n \approx 2^{nR}$

For a fixed D:

R : *achievable rate* – *if* $\lim_{n\to\infty} \bar{d}(\rho^{\otimes n}, \mathcal{D}_n \circ \mathcal{E}_n) \le D$

$R \equiv R(D)$

average distortion of $\rho^{\otimes n}$ under $\mathcal{D}_n \circ \mathcal{E}_n$

Quantum Rate Distortion function

$$R_q(D) := \inf\{R : \text{achievable}\}$$

(minimum rate of compression under distortion D)

Average Distortion

$$\mathcal{E}_n : \mathcal{D}(\mathcal{H}^{\otimes n}) \to \mathcal{D}(\mathcal{H}_c^n) \qquad \mathcal{D}_n : \mathcal{D}(\mathcal{H}_c^n) \to \mathcal{D}(\mathcal{H}^{\otimes n})$$

$$\bar{d}(\rho^{\otimes n}, \mathcal{D}_n \circ \mathcal{E}_n) := \frac{1}{n}\sum_{i=1}^{n} d(\rho, \mathcal{F}_n^{(i)})$$

$$\mathcal{F}_n := \mathcal{D}_n \circ \mathcal{E}_n$$

$\mathcal{F}_n^{(i)} =$ marginal operation (CPTP) induced on the i^{th} copy of the source space

$A_1, A_2, \ldots, A_n$

$$\mathcal{F}_n^{(i)}(\rho) = \mathrm{Tr}_{A_1,\ldots A_{i-1}, A_{i+1},\ldots, A_n}\left(\mathcal{D}_n \circ \mathcal{E}_n\left(\rho^{\otimes n}\right)\right)$$

$\forall \mathcal{N}:$ CPTP, $d(\rho, \mathcal{N}) = 1 - F_e(\rho, \mathcal{N})$

entanglement fidelity

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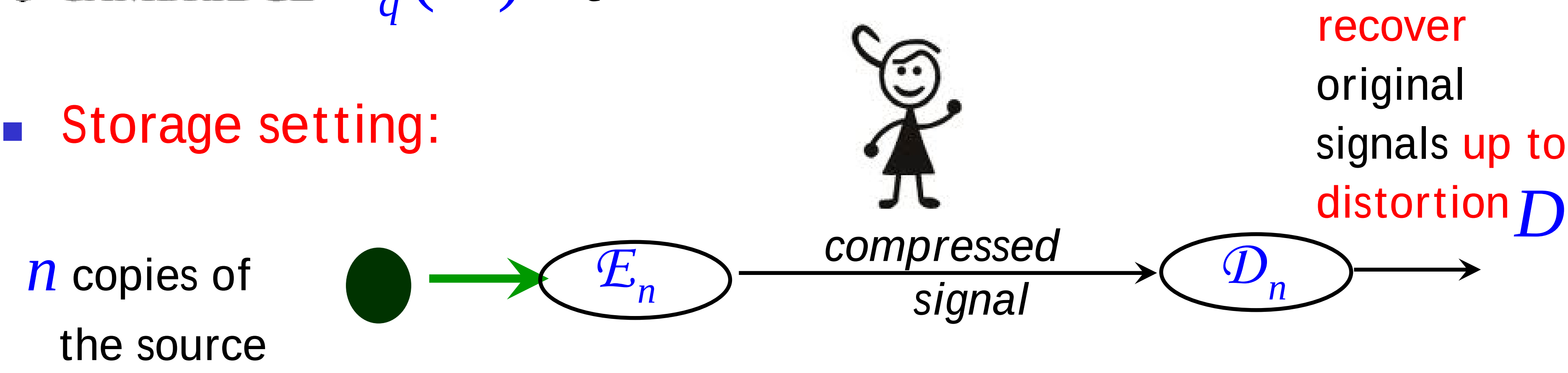
$R_q(D)$: Quantum Rate Distortion Function

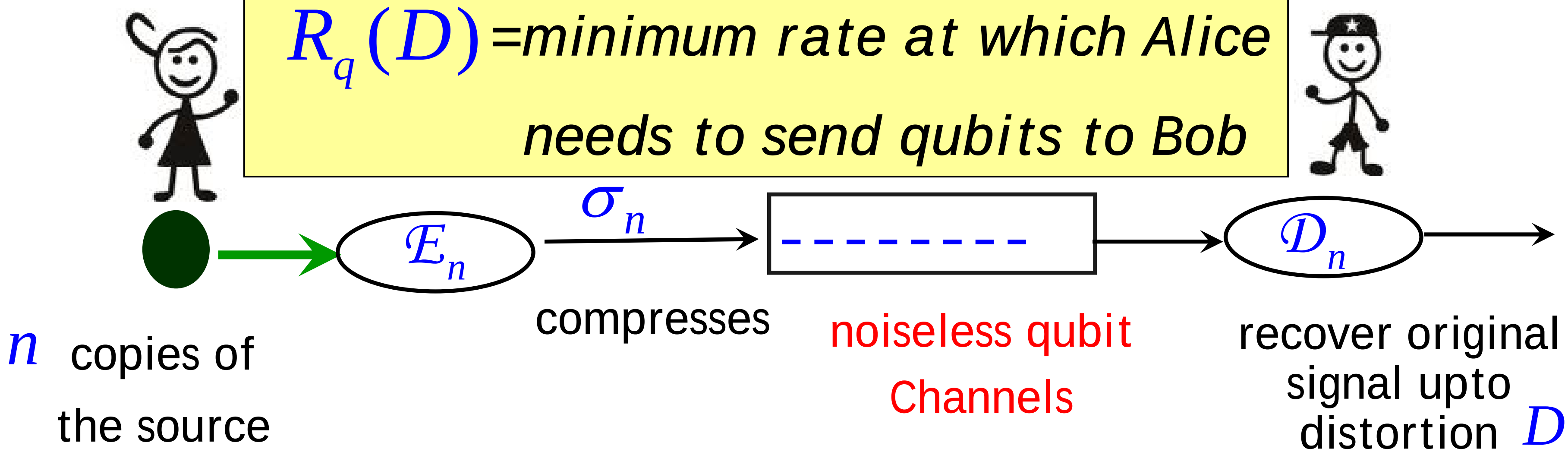
- Storage setting:



An Equivalent Scenario for Quantum Rate Distortion

- Communication setting:

$R_q(D)$ =*minimum rate at which Alice needs to send qubits to Bob*



- Aim: find an expression for $R_q(D)$

- For $D = 0:$ Schumacher's theorem

$$R_q(D) = S(\rho):$$ von Neumann entropy

- What is $R_q(D)$ for $D > 0?$

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- A guess for $R_q(D)$: *in analogy with the classical case*

Classical case

$$R(D) = \min_{\substack{p_{Y|X} \\ \mathbf{E}(d(X,Y)) \leq D}} I(X:Y)$$

stochastic map/ classical channel

A guess for $R_q(D)$: *in analogy with the classical case*

Classical case

$$R(D) = \min_{\mathcal{N}:\text{channel},\ \mathcal{N}(X)=Y,\ \mathbf{E}(d(X,Y)) \le D} I(X:Y)$$

Quantum case

$$R_q(D) = \min_{\mathcal{N}:\text{quantum channel},\ d(\rho,\mathcal{N}) \le D} I_c(\rho,\mathcal{N})$$

Classical capacity of a channel

$$C(\mathcal{N}) = \max_{p_X} I(X:Y)$$

$$\mathcal{N} \equiv \mathcal{D} \circ \mathcal{E} : x \to y$$

stochastic map: $X \to Y$

Quantum capacity of a q.channel

$$Q(\mathcal{N}) = \max_{\rho} I_c(\rho,\mathcal{N})$$

(degradable)

coherent information

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Quantum Rate Distortion Function

Howard Barnum

conjecture '98

$$R_q(D) = \min_{\substack{\mathcal{N}:\,\mathrm{CPTP} \\ d(\rho,N)\leq D}} I_c(\rho,\mathcal{N}) \quad \text{.......(1)}$$

coherent information

$$\omega_{RB} = \left(id_R \otimes \mathcal{N}_{A\to B}\right)\varphi_{RA}^{\rho};$$

purification

$$I_c(\rho,N) = S(\omega_B) - S(\omega_{RB})$$

- He proved: " $\geq$ " holds in (1)

problem!!

- the coherent information can be negative

BUT

- The rate of a data compression scheme is non-negative !!

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- What is $R_q(D)$ for $D > 0$?

Result - I

(ND, H,W: IEEE Jan. 2013)

- Theorem: An achievable rate for quantum rate distortion with parameter D:

$$\min_{\substack{\mathcal{N} \\ d(\rho,\mathcal{N})\leq D}} E_p(\rho,\mathcal{N})$$

Entanglement of purification

[Terhal, M.Horodecki, Leung, DiVincenzo]

$$\omega_{RB} = \left(id_R \otimes \mathcal{N}_{A\to B}\right)\varphi_{RA}^{\rho};$$

$|\omega_{RBE}\rangle$: a purification of ω_{RB}; $\quad \omega_{BE} = \mathrm{Tr}_R |\omega_{RBE}\rangle\langle\omega_{RBE}|$

$$E_p(\rho,\mathcal{N}) = \min_{\Lambda_E:CPTP} S\Big(\big(id_B \otimes \Lambda_E\big)\omega_{BE}\Big) \geq 0$$

- **(Q):** Is the rate:

$$\min_{\substack{\mathcal{N} \\ d(\rho,\mathcal{N})\leq D}} E_p(\rho,\mathcal{N})$$

optimal too?

- If so, we would have: for a given distortion parameter

$$R_q(D) = \min_{\substack{\mathcal{N} \\ d(\rho,\mathcal{N})\leq D}} E_p(\rho,\mathcal{N})$$

- **(A) No!** We can do better by regularizing.

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Result - II

Theorem: Quantum rate distortion function for $\{\rho, \mathcal{H}_A\}$:

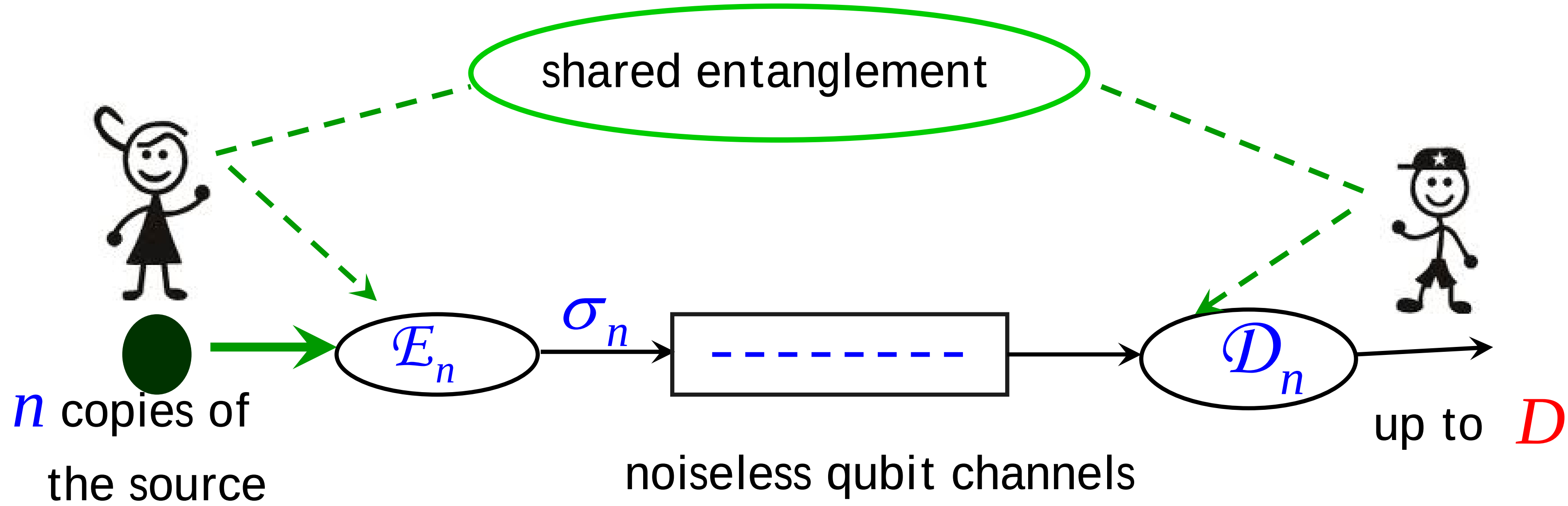
$$R_q(D) = E_p^{\infty}$$ regularized entanglement of purification

$$R_q(D) = \lim_{n\to\infty} \frac{1}{n} \min_{\substack{\mathcal{N}^{(n)} \\ \bar{d}(\rho^{\otimes n}, \mathcal{N}^{(n)}) \leq D}} E_p(\rho^{\otimes n}, \mathcal{N}^{(n)})$$

- **Advantage:** (unlike Barnum's)
  - Exact expression for $R_q(D)$; not just a lower bound
  - above expression always ≥ 0
- Disadvantage:
  - regularized formula – not tractable.

Another Scenario for Quantum Rate Distortion

- Communication setting: (entanglement assisted)



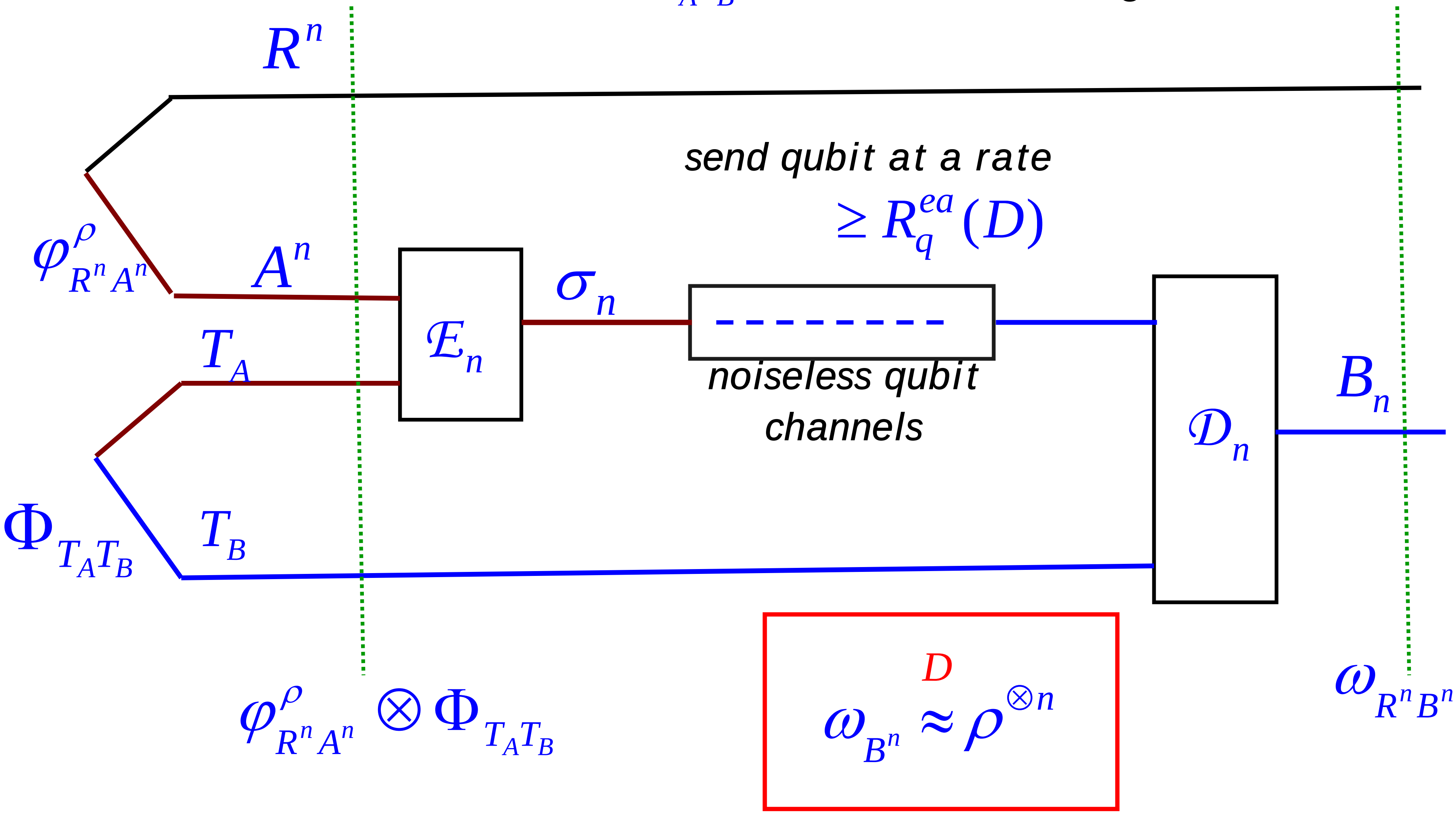
$$R_q^{ea}(D) =$$ *entanglement-assisted (EA) quantum rate distortion function*

= minimum rate at which Alice needs to send qubits to Bob in this case

Entanglement-assisted quantum rate distortion

- Source: $\{\rho, \mathcal{H}_A\}$; n copies: $\rho^{\otimes n}$; purification: $\varphi^{\rho}_{R^n A^n}$

$\Phi_{T_A T_B}$: shared entangled state



Result - III

- EA quantum rate distortion function:

$$R_q^{ea}(D) = \min_{\substack{\mathcal{N}:\,\text{CPTP} \\ d(\rho,N)\le D}} \frac{1}{2} I(R:B)_\omega \quad \text{..........(2)}$$

quantum mutual information

$$\omega_{RB} = \left(id_R \otimes \mathcal{N}_{A\to B}\right)\varphi_{RA}^{\rho};$$

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Result - II

Theorem: EA quantum rate distortion function:

$$R_q(D) \geq R_q^{ea}(D) = \min_{\substack{N:\,\mathrm{CPTP} \\ d(\rho,N)\leq D}} \frac{1}{2} I(R:B)_\omega \quad \text{.........(2)}$$

quantum mutual information

- We obtain a single-letter expression for $R_q^{ea}(D)$
- It provides a single-letter lower bound for $R_q(D)$
- Respects the analogy with the classical case

$$R(D) = \min I(X:Y)$$

$$C(\mathcal{N}) = \max_{p_X} I(X:Y)$$

$$R_q^{ea}(D) = \min \frac{1}{2} I(R:B)$$

$$Q_{ea}(\mathcal{N}) = \max_{\rho} \frac{1}{2} I(R:B)$$

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- **Example: Isotropic qubit source** $\rho = \frac{I}{2}$

Theorem III: EA quantum rate distortion function:

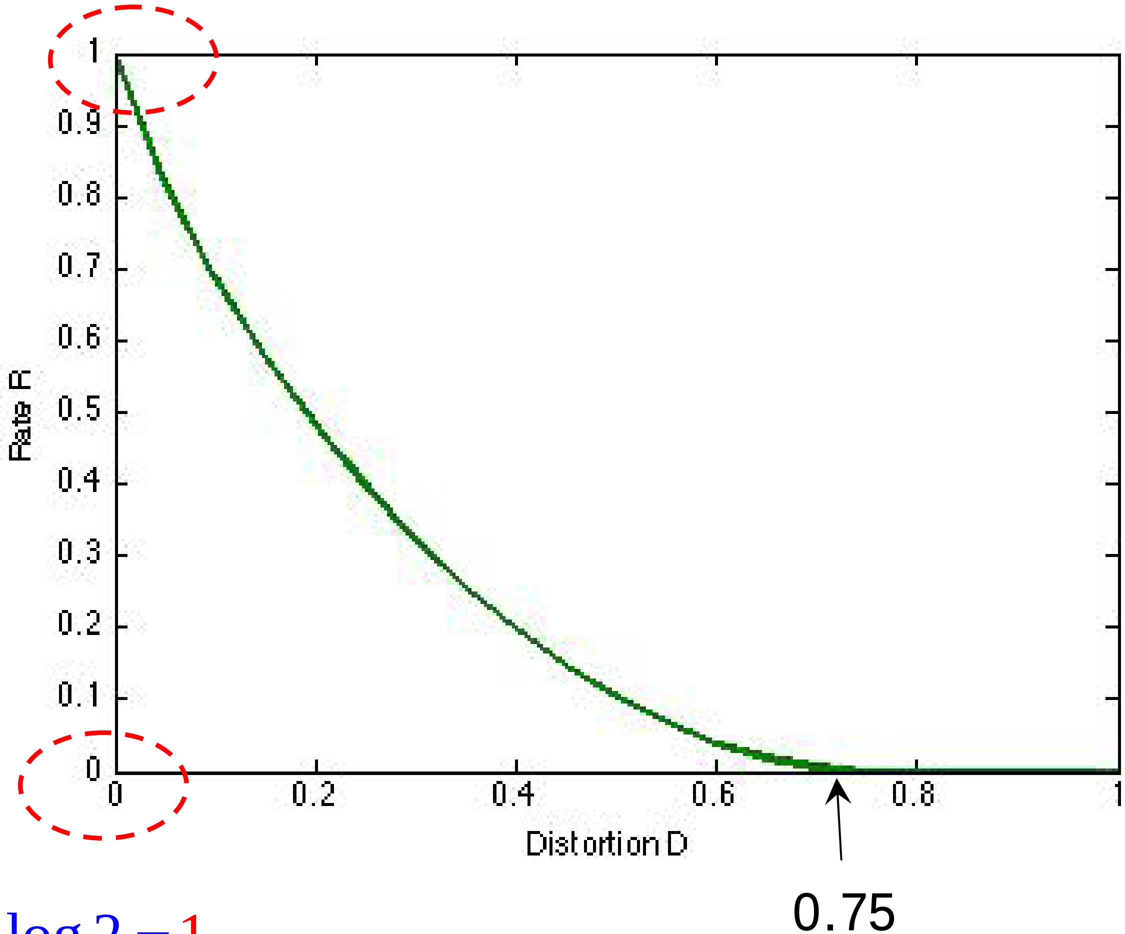
$$R_q^{ea}(D) = \min_{\substack{\mathcal{N}:\,\mathrm{CPTP} \\ d(\rho,\mathcal{N}) \le D}} \frac{1}{2} I(R:B)_\omega$$

quantum mutual information

$$R_q^{ea}(D) = \begin{cases} 1 - \frac{1}{2} H\left(1-D, \frac{D}{3}, \frac{D}{3}, \frac{D}{3}\right) & \text{if } 0 \le D \le \frac{3}{4} \\ 0 & \text{if } \frac{3}{4} < D \le 1 \end{cases}$$

■ **Example: Isotropic qubit source** $\rho = \frac{I}{2}$ contd.

$R_q^{ea}(D)$



$H(\rho) = \log 2 = 1$

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Summary of results: For a source $\{\rho, \mathcal{H}_A\}$:

Theorem I: An achievable rate for QRD, for a distortion D,

$$\min_{\substack{\mathcal{N} \\ d(\rho,\mathcal{N}) \le D}} E_p(\rho, \mathcal{N})$$

entanglement of purification

Theorem II: Quantum rate distortion function $\{\rho, \mathcal{H}_A\}$:

for

$$R_q(D) = E_p^{\infty}$$

regularized entanglement of purification

Theorem III: Entanglement Assisted quantum rate distortion fn:

$$R_q^{ea}(D) = \min_{\substack{N:\text{ CPTP} \\ d(\rho,N) \le D}} \frac{1}{2} I(R:B)_\omega \qquad \text{............(2)}$$

quantum mutual information

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Theorem I: An achievable rate for QRD, for a distortion D,

$$\min_{\substack{\mathcal{N}:\text{CPTP} \\ d(\rho,\mathcal{N})\leq D}} E_p(\rho,\mathcal{N})$$

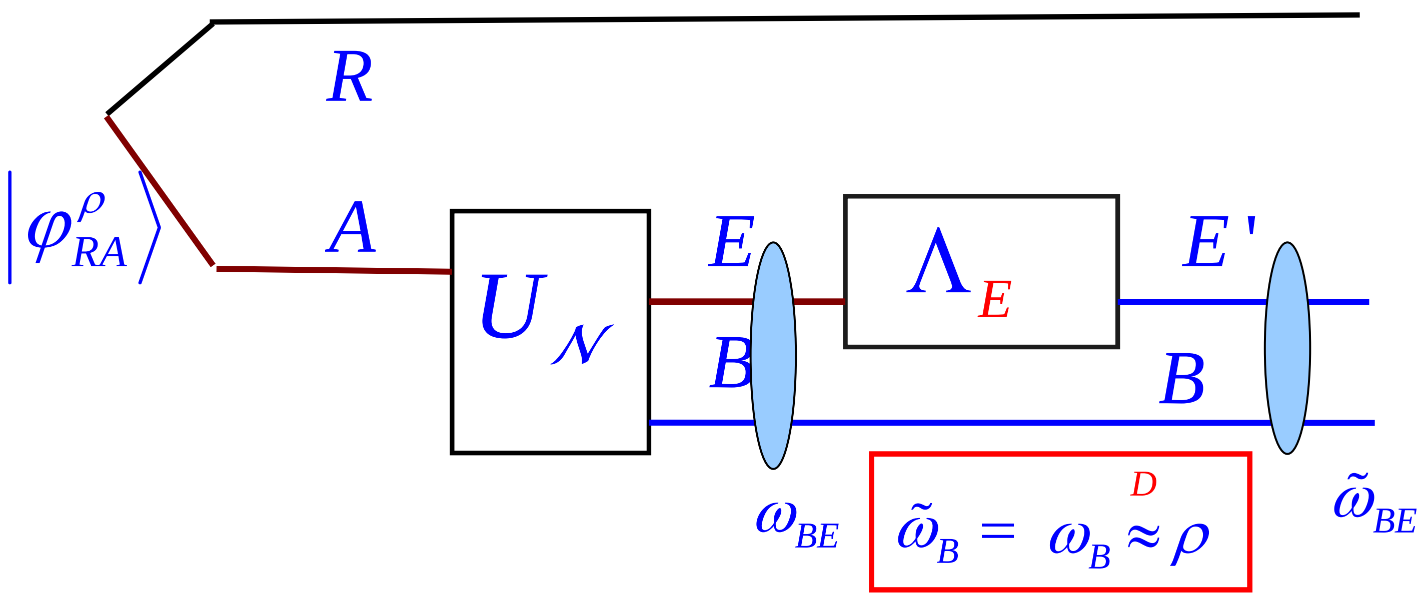
entanglement of purification

Sketch of Proof

Theorem I: $\min_{\substack{\mathcal{N} \\ d(\rho,\mathcal{N})\le D}} E_p(\rho,\mathcal{N})$ is an achievable rate

Sketch of proof:

- Find $\mathcal{N} = \mathcal{N}_{A\to B}$ which minimizes the above.
- Let $U_{\mathcal{N}} = U_{\mathcal{N}}^{A\to BE}$ be an isometric extension of it
- Alice has $\rho^{\otimes n}$; purification $\left|\varphi_{RA}^{\rho}\right\rangle^{\otimes n}$
- On each copy ρ she acts as follows; Λ_E :a CPTP map



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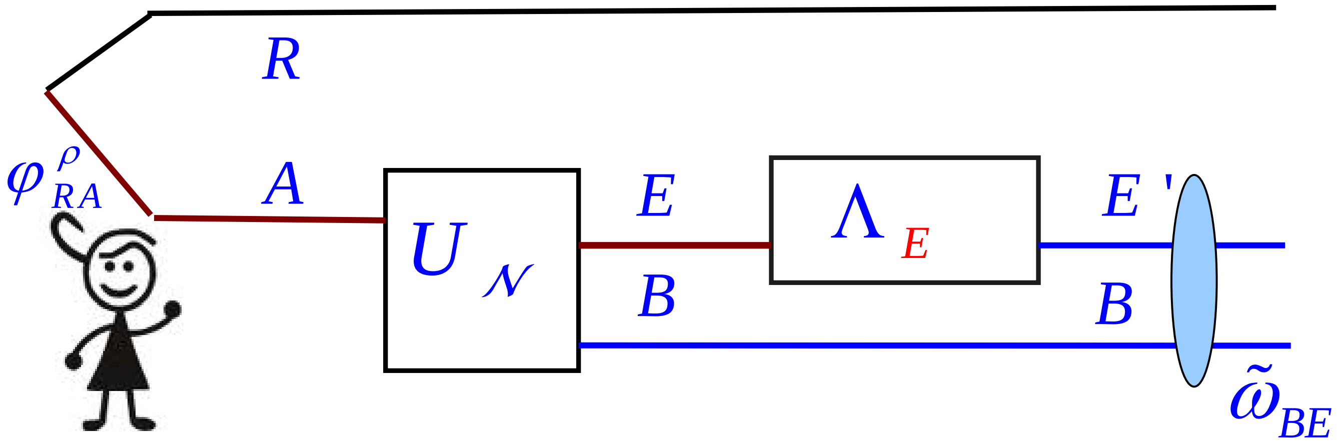
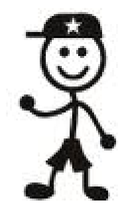
$\min_{\mathcal{N}:\, d(\rho,\mathcal{N}) \le D} E_p(\rho, \mathcal{N})$ is an achievable rate



- Alice generates $(\tilde{\omega}_{BE'})^{\otimes n}$; $\tilde{\omega}_{BE'} = \left(id_B \otimes \Lambda_E\right)\omega_{BE}$
- She Schumacher compresses $(\tilde{\omega}_{BE'})^{\otimes n}$ at a rate $S(\tilde{\omega}_{BE'})$
- Sends qubits at a rate $S\left(\tilde{\omega}_{BE'}\right)$ to Bob
- Bob decompresses and then traces out the E' systems
- He gets $\omega_B^{\otimes n}$ & hence $\omega_B \overset{D}{\approx} \rho.$
- Minimum rate $\min S(\tilde{\omega}_{BE'}) = \min_{\Lambda_E} S\left(\left(id_B \otimes \Lambda_E\right)\omega_{BE}\right) \equiv E_p(\rho, \mathcal{N})$



■

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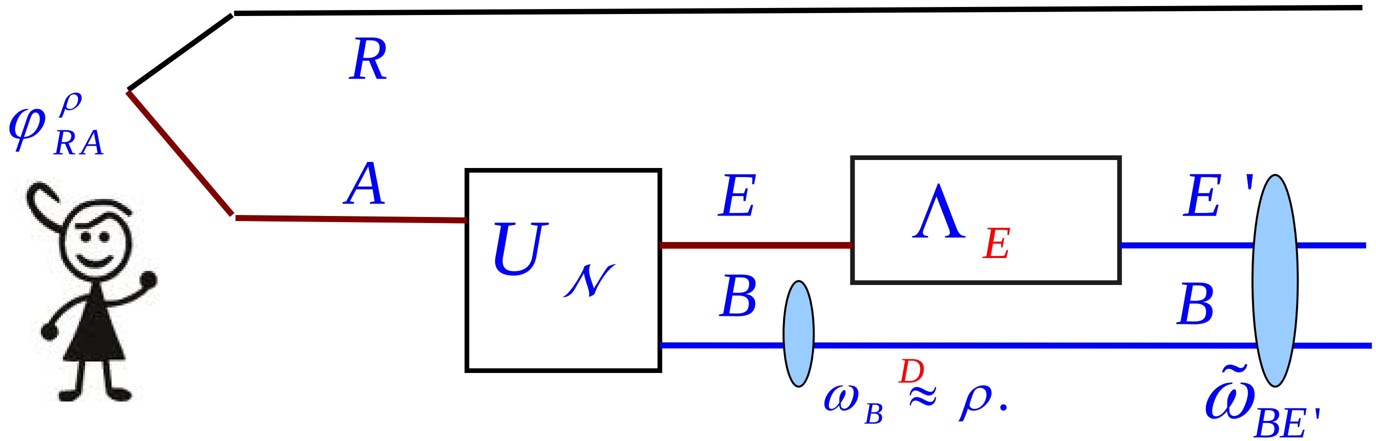
$\min\limits_{\substack{\mathcal{N} \\ d(\rho,\mathcal{N})\leq D}} E_p(\rho,\mathcal{N})$ is an achievable rate contd.



- Why do Λ_E ?
- Why not just send the systems B to Bob ?

  After all $\omega_B \overset{D}{\approx} \rho.$
- Schumacher compress $\omega_B^{\otimes n}$; rate = $S(\omega_B)$
  - That's because we could have

$$S(\tilde{\omega}_{BE'}) \leq S(\tilde{\omega}_B) = S(\omega_B)$$

(due to entanglement)

- Minimum rate $\min\limits_{\Lambda_E} S(\tilde{\omega}_{BE'}) \equiv E_p(\rho,\mathcal{N})$

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- **In the Entanglement-Assisted (EA) case:**

---- We obtain a more fundamental result corresponding

to a more realistic scenario

- Instead of **asymptotically many uses** of the source,

Alice is allowed to use the source only once!

- She shares an entangled state with Bob

(Q) For a fixed value of the distortion parameter D,

what is the minimum number of qubits that Alice needs

to send to Bob so that he can recover the signal emitted

by a single copy of the source, up to a maximum distortion D?

A characterization of : *one-shot EA lossy quantum data compression*

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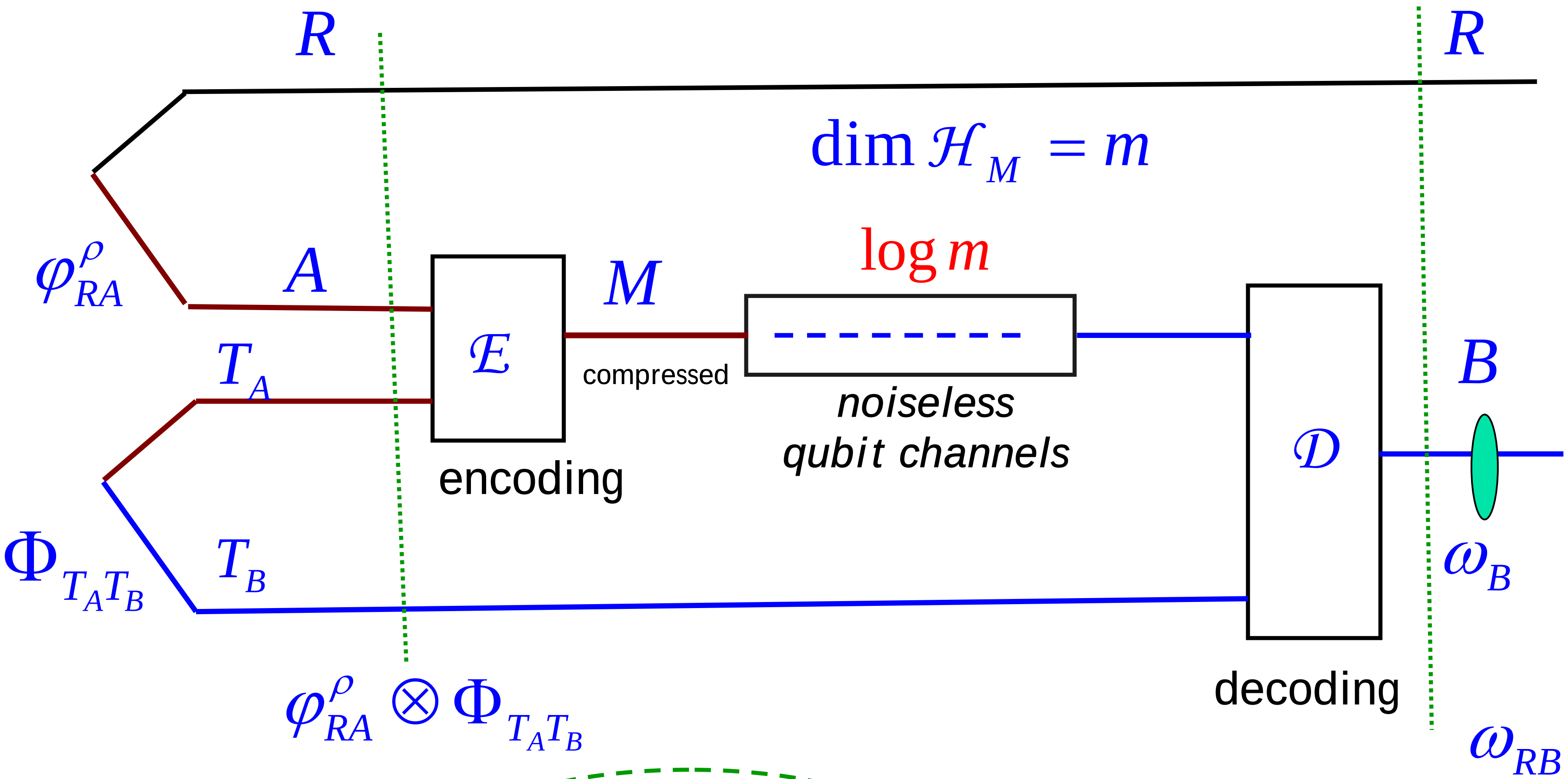
One-shot EA lossy quantum data compression

- Alice has a single copy of the source



- Aim: Find $\min_{\{\mathcal{E},\mathcal{D}\}:} \{\log m\}$ such that $\omega_B \overset{D}{\approx} \rho$

[One-shot analogue of $R_q^{ea}(D)$]

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- In asymptotic, memoryless case: $\omega_{B^n} \overset{D}{\approx} \rho^{\otimes n}$

$$\lim_{n\to\infty} \bar{d}(\rho^{\otimes n}, \omega_B^{\otimes n}) \leq D$$

distortion

- In one-shot case :

-- requiring $d(\rho, \omega_B) \leq D$ too stringent

-- natural to allow for a non-zero error

-- instead require: $\Pr\big(d(\rho, \omega_B) > D\big) \leq \varepsilon$

$$\log m^* := \min_{\substack{\{\mathcal{E},\mathcal{D}\}:\\ \Pr(d(\rho,\omega_B)>D)\leq\varepsilon}} \{\log m\}$$

[One-shot analogue of $R_q^{ea}(D)$] $m^* \equiv m^*(D,\varepsilon)$

Result IV

Theorem IV: One-shot EA lossy quantum data compression:

Minimum number of qubits needed:

$$\log m^* \approx \min_{\substack{\mathcal{N}:\,\mathrm{CPTP} \\ \Pr\left(d(\rho,\omega_B) > D\right) \le \varepsilon}} \frac{1}{2} I_{\max}^{\varepsilon}\left(R:B\right)_{\omega}$$

smoothed max-information

$$\omega_{RB} = \left(id_R \otimes \mathcal{N}\right)\varphi_{RA}^{\rho};$$

- Compare with the asymptotic, memoryless result:

Theorem III: EA quantum rate distortion function:

$$R_q^{ea}(D) = \min_{\substack{\mathcal{N}:\,\mathrm{CPTP} \\ d(\rho,\mathcal{N}) \le D}} \frac{1}{2} I\left(R:B\right)_{\omega}$$

quantum mutual information

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Definition: max-information

- Quantum mutual information

$$I(R:B)_\omega := D(\omega_{RB} \| \omega_R \otimes \omega_B)$$

$$I(R:B)_\omega = \min_{\sigma_B} D(\omega_{RB} \| \omega_R \otimes \sigma_B)$$

- Quantum relative entropy

$$D(\rho \| \sigma) := \mathrm{Tr}\,(\rho \log \rho) - \mathrm{Tr}\,(\rho \log \sigma)$$

- Max- information

$$I_{\max}(R:B)_\omega := \min_{\sigma_B} D_{\max}(\omega_{RB} \| \omega_R \otimes \sigma_B)$$

Max-relative entropy

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▪ Definition: Max-relative entropy

[ND; 2008]

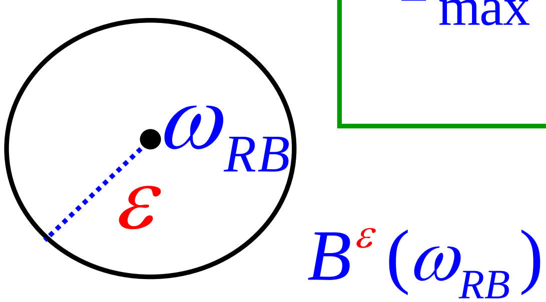
- The max- relative entropy of a state ρ & a positive operator σ is

$$D_{\max}(\rho \| \sigma) := \inf\{\gamma : \rho \le 2^{\gamma}\sigma\}$$

$$I_{\max}(R:B)_{\omega} := \min_{\sigma_B} D_{\max}(\omega_{RB} \| \omega_R \otimes \sigma_B)$$

- Smoothed Max- information

$$I^{\varepsilon}_{\max}(R:B)_{\omega} := \min_{\bar{\omega}_{RB} \in B^{\varepsilon}(\omega_{RB})} I_{\max}(R:B)_{\bar{\omega}}$$



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Theorem IV: $$\log m^* \approx \min \frac{1}{2} I^{\varepsilon}_{\max}(R:B)_{\omega}$$

[ND,R,R,W]

smoothed max-information

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One-shot result "more fundamental" : Why?

- One-shot result $\xrightarrow{n \to \infty}$ asymptotic memoryless result

(Theorem IV) (Theorem III)

$$\log m^* \approx \min \frac{1}{2} I_{\max}^{\varepsilon}(R:B)_{\omega} \xrightarrow{n \to \infty} R_q^{ea}(D) = \min \frac{1}{2} I(R:B)_{\omega}$$

because,

$$\forall \ 0 < \varepsilon < 1, \quad \lim_{n \to \infty} \frac{1}{n} I_{\max}^{\varepsilon}(R^n : B^n)_{\omega_{RB}^{\otimes n}} = I(R:B)_{\omega}$$

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- **In a nutshell……..**

What's exciting?

TIMELINE

Classical Case

- 1948 Shannon $D = 0$
- 1959 Shannon $D > 0$

Quantum Case

- 1995 Schumacher $D = 0$
- 1998 Barnum's conjecture $D > 0$

only 3 papers

- 2012-13 : a host of new results!

rejuvenation of a dormant field !

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Summary

- **Thm I:** An achievable rate for quantum rate distortion
  ---- in terms of entanglement of purification

- **Thm II:** Quantum rate distortion function, $R_q(D)$:
  ---- terms of regularised entanglement of purification

- **Thm III:**
  - entanglement-assisted quantum rate distortion function $R_q^{ea}(D)$:
  - single-letter formula
  - in terms of the quantum mutual information

- **Thm IV:**
  - characterization of One-shot EA lossy quantum data compression ----- in terms of smoothed max-information

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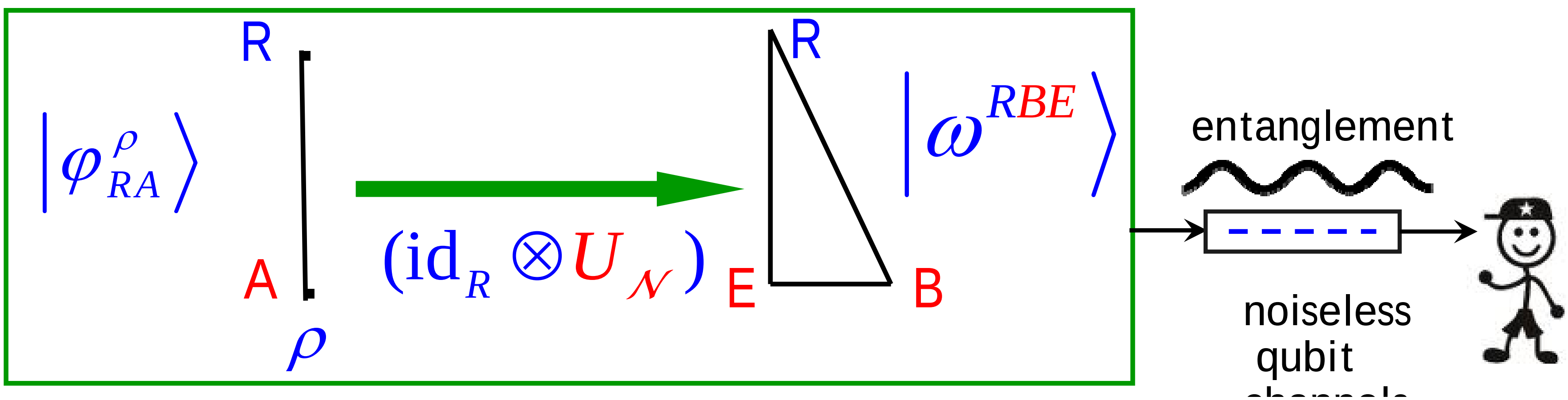
Theorem III: EA quantum rate distortion function

$$R_q^{ea}(D) = \min_{\substack{N:\,\mathrm{CPTP} \\ d(\rho,N)\le D}} \frac{1}{2} I(R:B)_\omega \quad \text{..........(2)}$$

$$\omega_{RB} = \left(id_R \otimes \mathcal{N}_{A\to B}\right)\varphi_{RA}^{\rho};$$

- *Sketch of proof of achievability*

- Let $\mathcal{N} =$ the minimizing CPTP map in (1)
- Let $U_{\mathcal{N}} : A \to BE$ an isometric extension of $\mathcal{N}$



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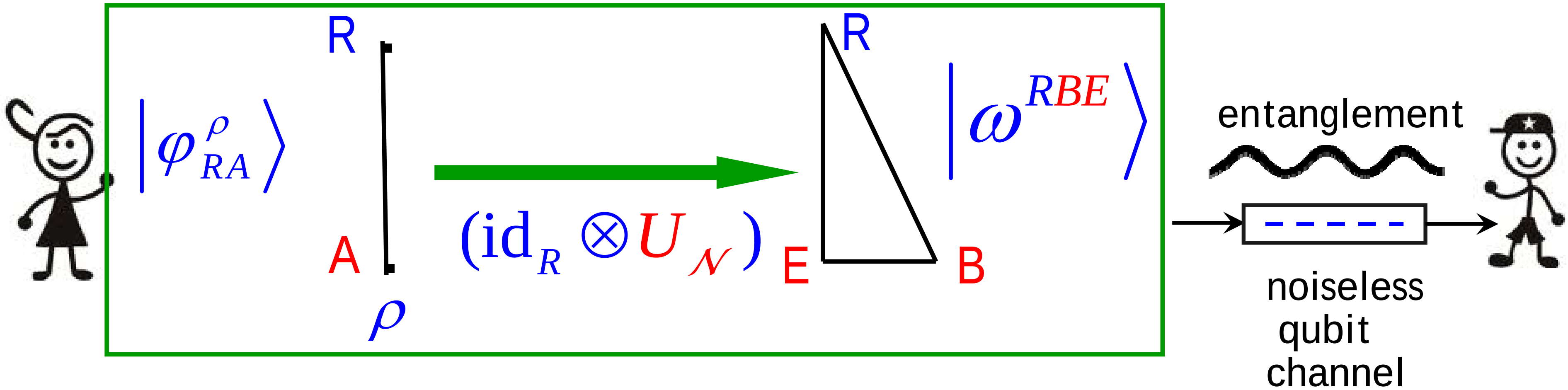


- Note: $\omega_B := \mathcal{N}(\rho)$ satisfies $d(\rho, \mathcal{N}) \leq D$

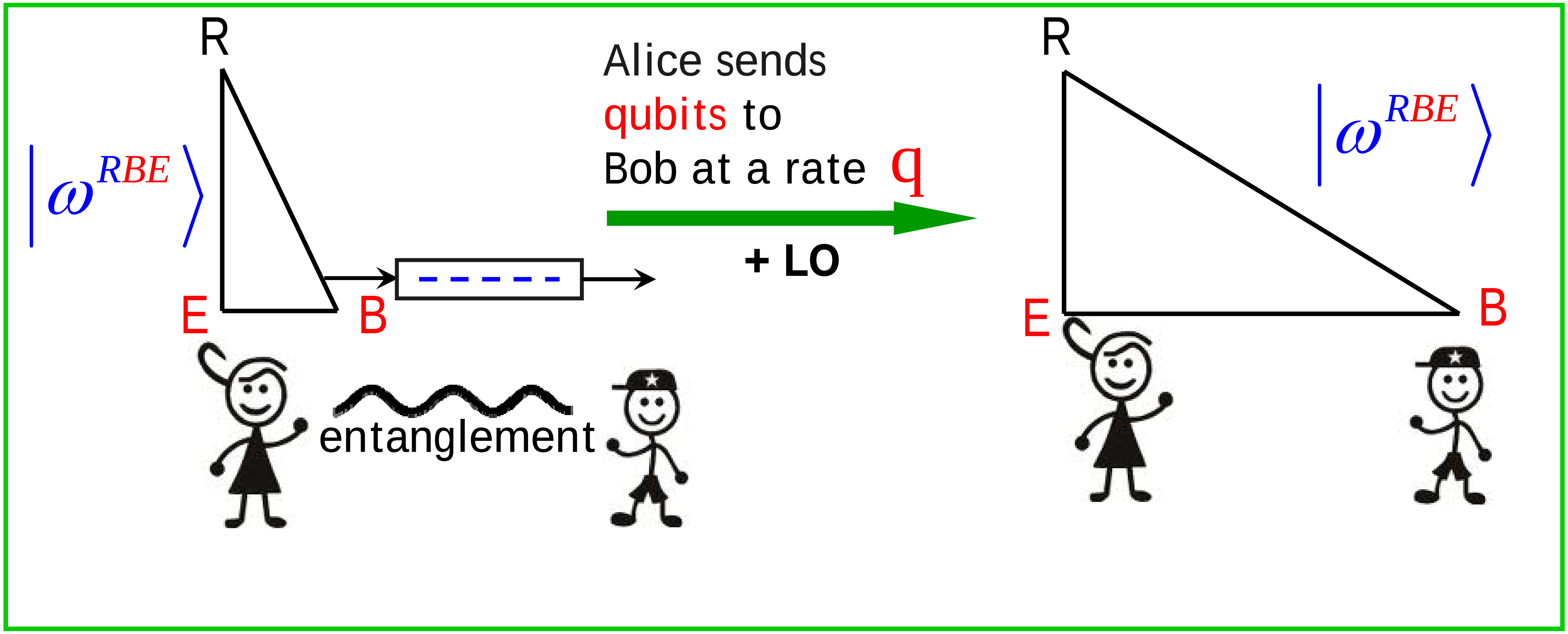
To prove: $R_q^{ea} = \min_{\substack{N:\,\mathrm{CPTP} \\ d(\rho,N)\leq D}} \frac{1}{2} I(R:B)_\omega$ is achievable

Suffices to prove:

Bob can get the state ω_B if Alice sends her qubits at a rate $\frac{1}{2} I(R:B)_\omega$

Quantum State splitting

- How can we transfer ω_B to Bob: ?



Bennett et al; Berta et al.

$$q = \frac{1}{2} I\left(R:B\right)_{\omega}$$

achievable rate !

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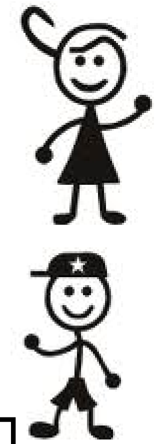
Asymptotic setting



(i) local preparation of $\rho^{\otimes n} \underset{U_{\mathcal{N}}^{\otimes n}}{\longrightarrow} \omega_{BE}^{\otimes n}$

(ii) State splitting with the help of shared entanglement such that $\approx \omega_B^{\otimes n}$

$$\omega_B = \mathcal{N}(\rho)$$

$\equiv$ simulating the (output of the) quantum channel $\mathcal{N}^{\otimes n}$ when the input is $\rho^{\otimes n}$ (using shared entanglement)

$=$ a special case of another protocol -- channel simulation

– Quantum Reverse Shannon Theorem **(QRST)**

$\Longrightarrow$ achievability of the expression for $R_q^{ea}(D)$

Open Questions

- Can one find a single-letter expression for the unassisted quantum rate distortion function $R_q(D)$?

- Can one find bounds on the one-shot analogue $(\log m^*)$ of the unassisted quantum rate distortion function ?

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Proof of converse bound:

- Assume: $\exists\, \mathcal{E}_n, \mathcal{D}_n;$ s.t. if Alice sends qubits at a rate r,

Bob can recover the state $\omega_{B^n} \underset{D}{\approx} \rho^{\otimes n}$ for n large enough.

- Prove that

$$r \geq \min_{\substack{N:\,\text{CPTP} \\ d(\rho,N)\leq D}} \frac{1}{2} I\left(R:B\right)_{\omega} \equiv R_q^{ea}(D)$$

- Tools used:

(i) entropic inequalities – e.g. data-processing inequality for the $I\left(R:B\right)_{\omega}$

(ii) Properties of the expression for $R_q^{ea}(D)$

e.g. it is a convex, non-increasing function of D, etc.