# Quantum correlations in a work extraction protocol

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# Outline

Work extraction in a cyclic process

Maximal work extraction and entanglement

Quantum discord in bipartite and multipartite systems

Maximal work extraction and discord

Conclusions

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# Work extraction in a cyclic process

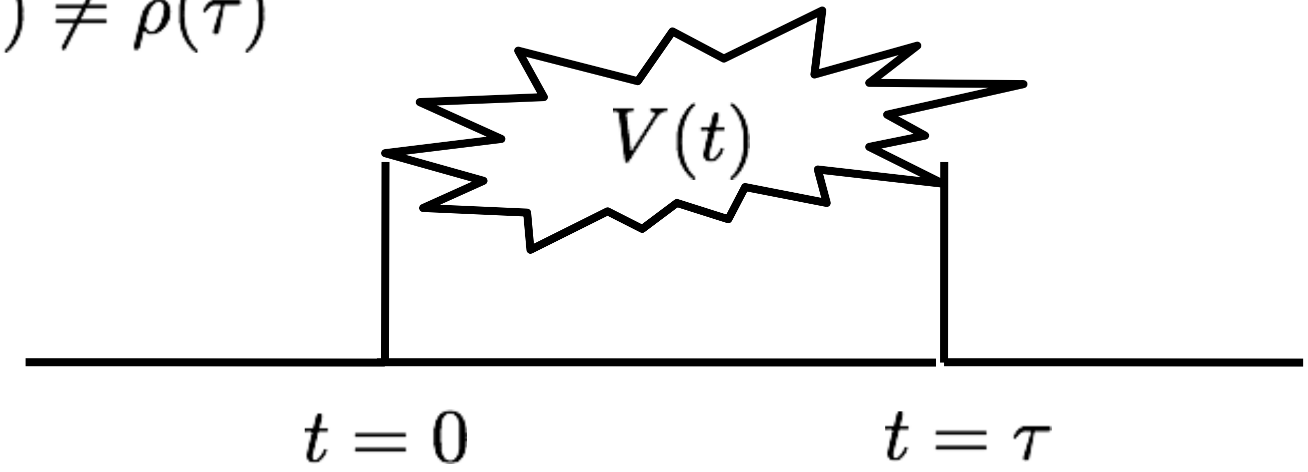
S

$$i\hbar\dot{\rho} = [H(t), \rho(t)]$$

$$H(t) = H + V(t)$$

S interacts with an external field
internal exchange allowed

$$\rho(0) \neq \rho(\tau)$$



# Work extraction in a cyclic process

Work extracted during the cycle

$$\mathcal{W} = \mathrm{Tr}[\rho(0)\mathrm{H}] - \mathrm{Tr}[\rho(\tau)\mathrm{H}]$$

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Maximum extraction in the thermodynamic limit

$$\rho^{\mathrm{opt}}(\tau) = \rho_{\mathrm{eq}} = \frac{e^{-\beta H}}{\mathrm{tr}[e^{-\beta H}]}$$

$$\beta : \qquad S(\rho_{eq}) = S(\rho(0))$$

# Work extraction in a cyclic process

Work extracted during the cycle

$$\mathcal{W} = \mathrm{Tr}[\rho(0)\mathrm{H}] - \mathrm{Tr}[\rho(\tau)\mathrm{H}]$$

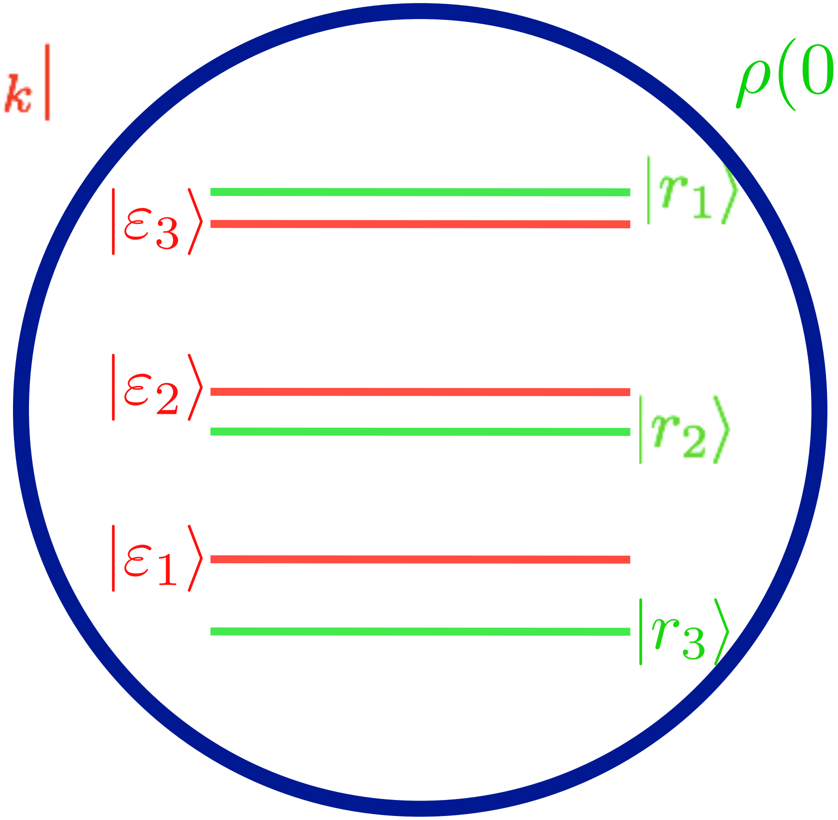
Maximal extraction for finite systems. The thermal equilibrium is not reached, only unitary transformations are allowed

$$\mathcal{W}_{\max} = \mathrm{Tr}[\rho(0)\mathrm{H}] - \min_{\mathrm{U}} \mathrm{Tr}[\mathrm{U}(\tau)\rho\mathrm{U}^{\dagger}(\tau)\mathrm{H}]$$

# Work extraction in a cyclic process

$$H = \sum_k \varepsilon_k |\varepsilon_k\rangle\langle\varepsilon_k|$$

$$\rho(0) = \sum_j r_j |r_j\rangle\langle r_j|$$




$$\varepsilon_1 \leq \varepsilon_2 \leq \cdots$$

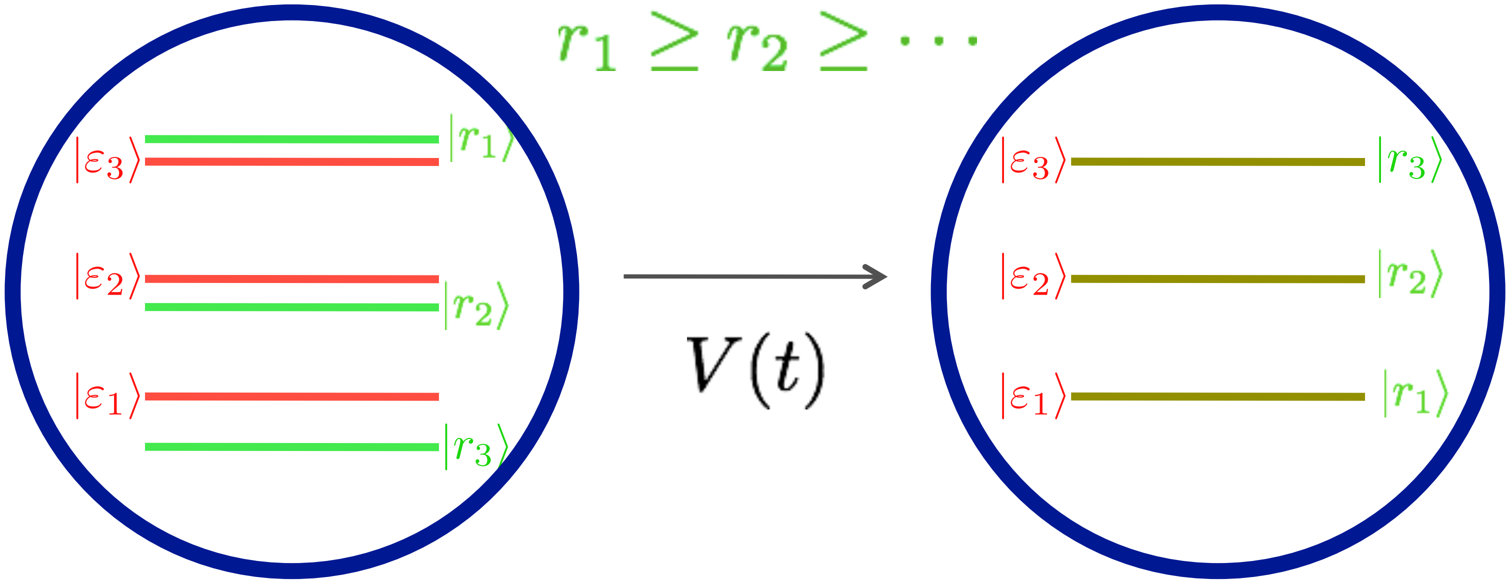
$$r_1 \geq r_2 \geq \cdots$$

# Optimal work extraction

(Allahverdyan et al., EPL 67, 565 (2004))

$t = 0$ $\qquad \varepsilon_1 \leq \varepsilon_2 \leq \cdots \qquad$ $t = \tau$

$r_1 \geq r_2 \geq \cdots$




Optimal transformation: $\rho(\tau) = \sum_j r_j |\varepsilon_j\rangle\langle\varepsilon_j|$

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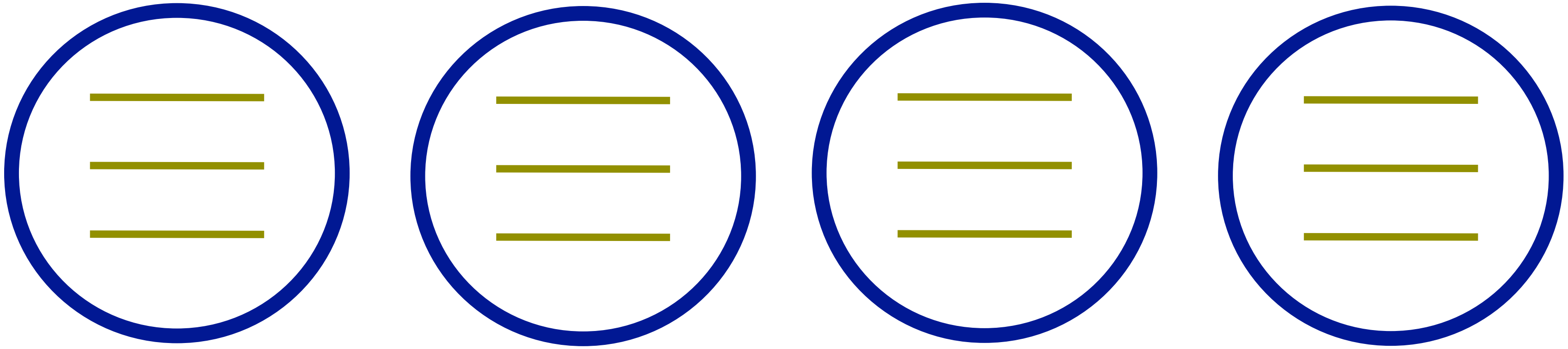
# N identical batteries

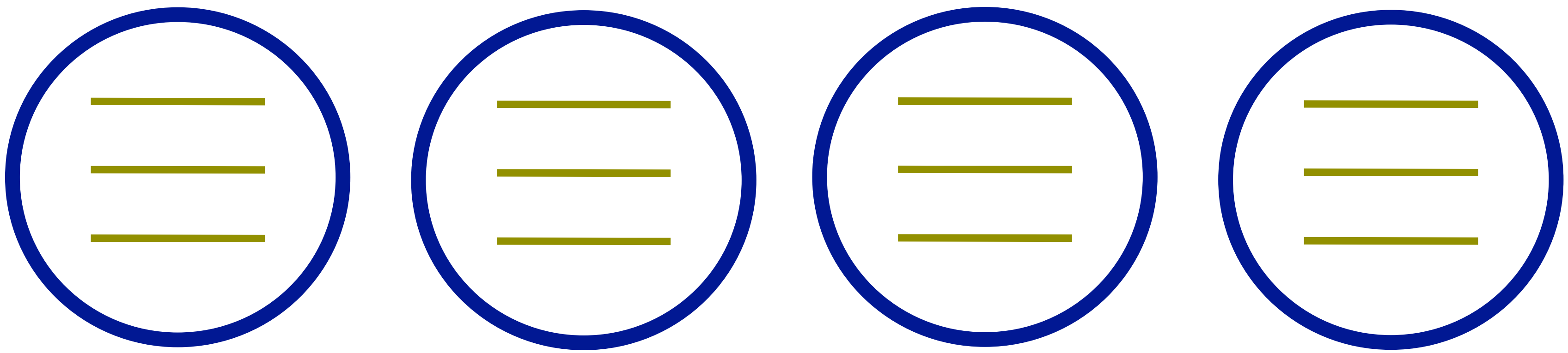


$$H^{(n)} = \sum_{j=1}^{n} \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes H_j \otimes \cdots \otimes \mathbb{1}$$

$$\rho^n(0) = \rho(0)^{\otimes^n}$$

Alicki and Fannes, PRE 87, 042123 (2013)
Hovhannisyan et al., PRL 111, 240401 (2013)

# N identical batteries

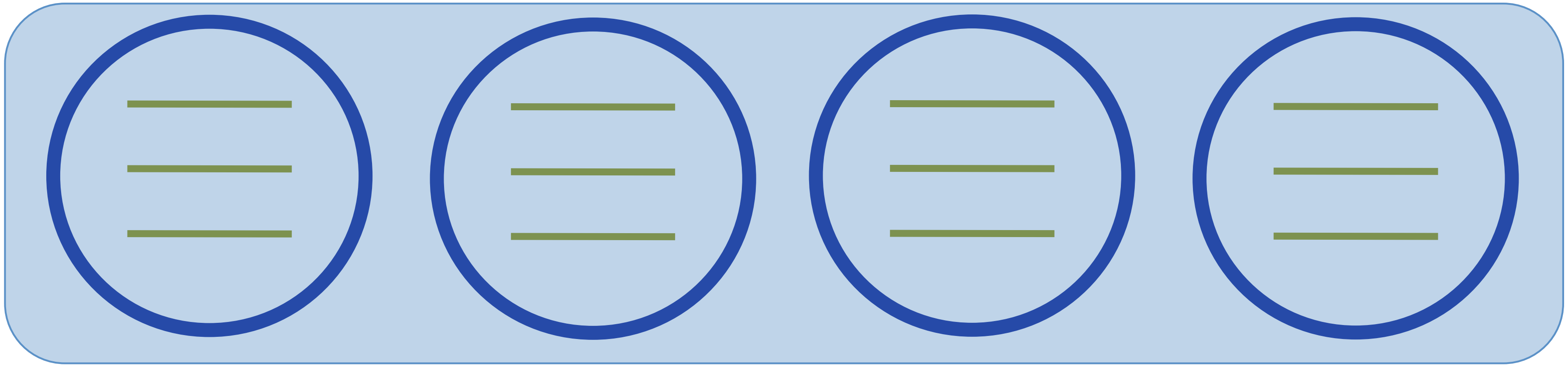


$$H^{(n)} = \sum_{j=1}^{n} \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes H_j \otimes \cdots \otimes \mathbb{1}$$

$$\rho^n(0) = \rho(0)^{\otimes^n}$$

Assumption: $|r_j\rangle \equiv |\varepsilon_j\rangle$

# N identical batteries



Global (entangling) unitary transformations help extract work

$$\mathcal{W}_n \geq n\mathcal{W}_1$$

Alicki and Fannes, PRE 87, 042123 (2013)

# N identical batteries: routes to maximal work extraction

During the time evolution we need to swap the elements of $\rho(0)$

A swap between two eigenstates with m different battery indices can be done in 1,3,…,2m -1 steps

Example: two qubits $|00\rangle \leftrightarrows |11\rangle$

Direct swap $|00\rangle \leftrightarrows |11\rangle$

Three steps $|00\rangle \leftrightarrows |01\rangle \quad |01\rangle \leftrightarrows |11\rangle \quad |01\rangle \leftrightarrows |00\rangle$

Hovhannisyan et al., PRL 111, 240401 (2013)

# Entanglement and work extraction

Two qubits, three steps

1° step $$U(t) = |0\rangle\langle 0| \otimes e^{-i\sigma_x \omega t} + |1\rangle\langle 1| \otimes \mathbb{1}$$

The evolution is not able to generate entanglement: the state remains factorized at any time

2° step …

3° step …

## No entanglement generation

Hovhannisyan et al., PRL 111, 240401 (2013)

# Entanglement and work extraction

Two qubits, one step $U(t) = \exp\left[-i\omega t(\sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y)/2\right]$

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remember $\rho^{(2)}(0) = (r_0|0\rangle\langle 0| + r_1|1\rangle\langle 1|)^{\otimes 2}$

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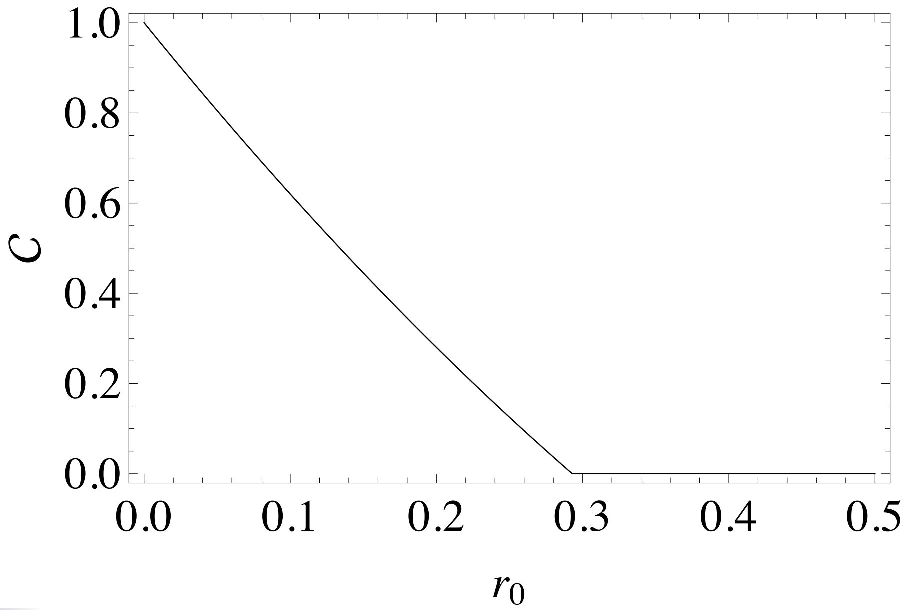
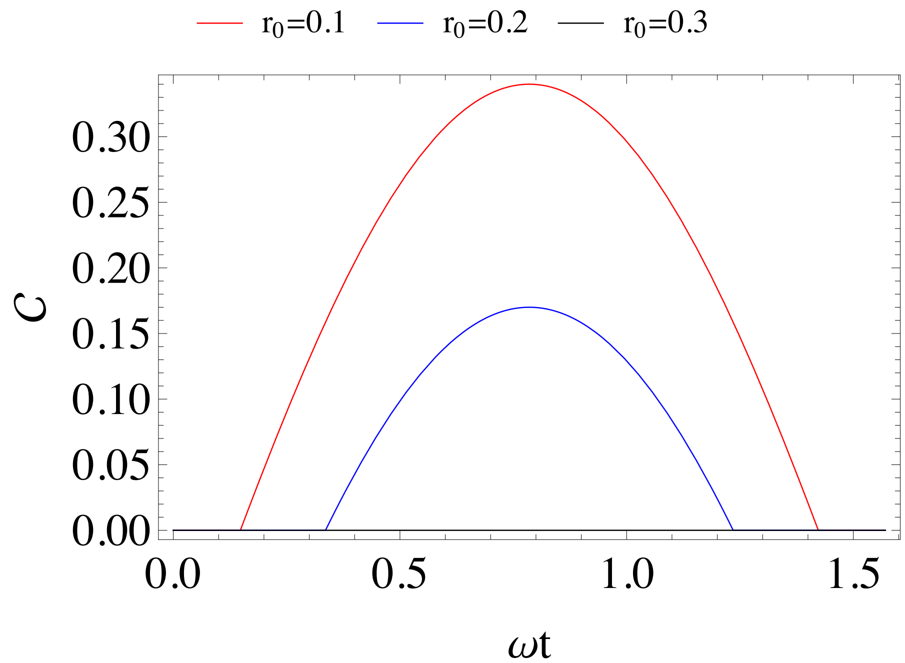
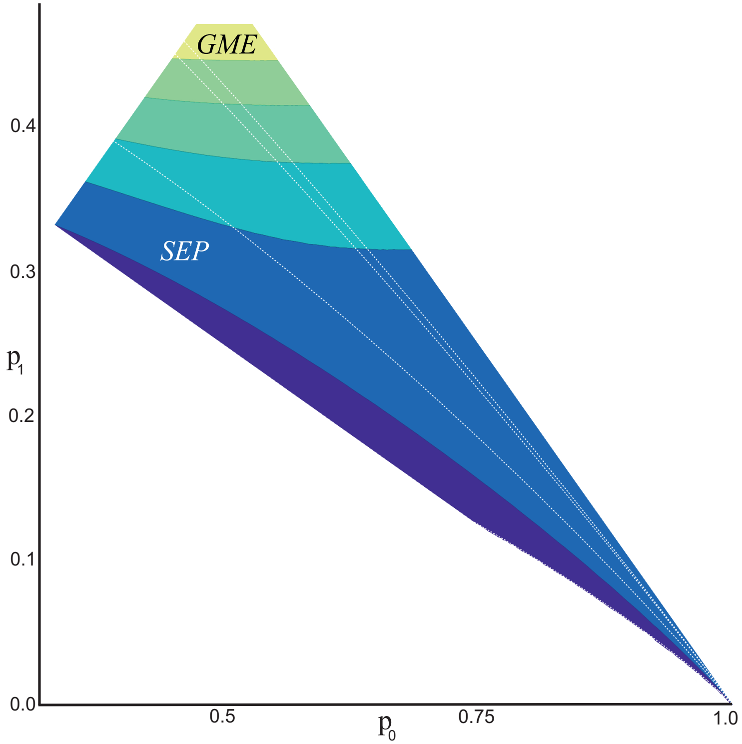


# Entanglement and work extraction

Many qudits: multipartite entanglement for 4 qutrits

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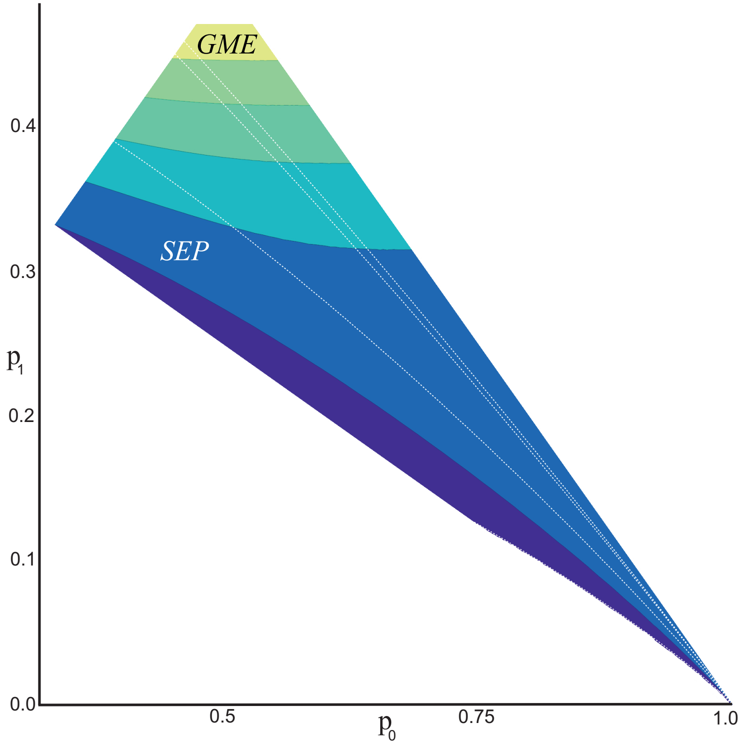


The maximal work extraction can be reached also without inducing any entanglement.
The higher the maximal work, the higher the probability of finding genuine multipartite entanglement

Hovhannisyan et al., PRL <u>111</u>, 240401 (2013)

# Entanglement and work extraction

Final message: entanglement is not necessary for maximal work extraction

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The presence of entanglement is related to the "speed" (number of steps) of the process

Hovhannisyan et al., PRL 111, 240401 (2013)

# Entanglement and work extraction

Final message: entanglement is not necessary for maximal work extraction

The presence of entanglement is related to the “speed” (number of steps) of the process

Genuine n-partite entanglement can only occur if the swap involves the degrees of freedom of all the batteries at the same time

Hovhannisyan et al., PRL 111, 240401 (2013)

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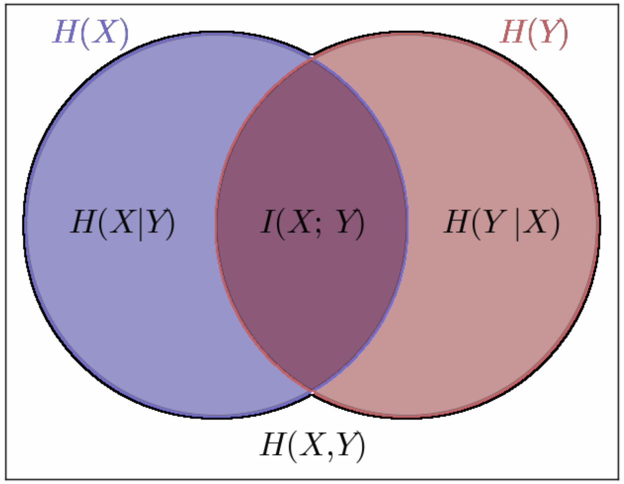
# Mutual information and classical correlations

Two random variables $$I(X;Y) = \sum_{y \in Y} \sum_{x \in X} p(x,y) \log \left( \frac{p(x,y)}{p_1(x)\, p_2(y)} \right)$$

Bayes's rule $$p(x|y) = p(x,y)/p(y)$$

Shannon entropy $$H(X) = -\sum_{x \in X} p(x) \log p(x)$$




Two equivalent forms

$$I(X;Y) = H(X) + H(Y) - H(XY)$$
$$I(X;Y) = H(X) - H(X|Y)$$

# Mutual information in quantum systems

Random variables $\longrightarrow$ density matrices

$$\varrho_{ab} \equiv \varrho, \;\; \varrho_a = \mathrm{Tr}_b \varrho, \;\; \varrho_b = \mathrm{Tr}_a \varrho$$

Shannon $\longrightarrow$ Von Neumann

$$S(\varrho) = -\mathrm{Tr}(\varrho \log \varrho)$$

Bayes' rule does not apply: the two classically equivalent definitions of mutual informationcbecome inequivalent

$$S(\varrho_a) + S(\varrho_b) - S(\varrho_{ab}) \neq S(\varrho_a) - S(\varrho_{a|b})$$

Furthermore, the conditional entropy is measurement-dependent
(in QM measurements modify states)

# Quantum discord

Quantum mutual information

$$I(a,b) = S(\varrho_a) + S(\varrho_b) - S(\varrho)$$

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Classical correlations

$$J(a,b) = \max_{\{\Pi_k\}} [S(\varrho_a) - \sum_k p_k S(\varrho_a^k | \Pi_k)]$$

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Classical correlations

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Discord

$$D(a,b) = I(a,b) - J(a,b)$$

H. Ollivier and W. H. Zurek, PRL 88, 017901 (2001)
L. Henderson and V. Vedral, JPA 34 6899 (2001)

# Quantum discord: a new paradigm for quantumness

Factorized state
$$\rho_{\mathrm{f}} = \sum_i p_i \rho_i^{(a)} \otimes \rho_i^{(b)}$$

Classical state
$$\rho_{\mathrm{cl}} = \sum_{i,j} p_{i,j} |\phi_i^{(a)}\rangle\langle\phi_i^{(a)}| \otimes |\psi_j^{(b)}\rangle\langle\psi_j^{(b)}|$$

Quantum-classical
$$\rho_{\mathrm{qc}} = \sum_i p_i |\phi_i^{(a)}\rangle\langle\phi_i^{(a)}| \otimes \rho_i^{(b)}$$

# Quantum discord

Why quantum discord? What is it useful for?

The power of one qubit (trace estimation)

E. Knill and R. Laflamme, PRL 81, 5672 (1998)
A. Datta *et al.,* PRL 100, 050502 (2008)
B. P. Lanyon *et al.*, PRL 101, 200501 (2008)

Quantum state discrimination

L. Roa *et al.*, PRL 107, 080401 (2011)

Remote state preparation

B. Dakić *et al.*, Nat. Phys. 8, 666 (2012)

………

# Genuine multipartite correlations

Given an n-partite system, a state has genuine n-partite correlations if it is nonproduct along any bipartite cut

Replace the mutual information with the total information

$$T(\rho) = \sum_{i=1}^{n} S(\rho_n) - S(\rho)$$

Genuine part of the total correlations among the n parts

$$T^{(n)}(\rho) = \min_{\{k\}} I(\rho_{\{k\}}, \rho_{\{\bar{k}\}})$$

In analogy, we can introduce a measure for m-partite correlations (m<n) considering any block of m subparties

GL Giorgi et al. PRL 107, 190501 (2011)

# Genuine multipartite correlations

Genuinely m-partite correlations (m<n)

$$T^{(n,m)} = \max_{\{m\}} \min_{\{k\}} I(\rho_{\{k\}}, \rho_{\{\bar{k}\}})$$

GL Giorgi et al. PRL 107, 190501 (2011)

# Genuine multipartite correlations

Genuinely m-partite correlations (m<n)

$$T^{(n,m)} = \max_{\{m\}} \min_{\{k\}} I(\rho_{\{k\}}, \rho_{\{\bar{k}\}})$$

Once established that genuine (or genuinely m-partite) correlations are equal to a bipartite mutual information, we can calculate their quantum and classical components (in analogy to bipartite systems)

$$T^{(n)}(\rho) = D^{(n)}(\rho) + J^{(n)}(\rho)$$

GL Giorgi et al. PRL 107, 190501 (2011)

# Global discord (alternative definition)

$$\hat{\Pi}_k = \hat{\Pi}_{A_1}^{j_1} \otimes \cdots \otimes \hat{\Pi}_{A_N}^{j_N}$$

$$\Phi\left(\hat{\rho}_{A_1 \cdots A_N}\right) = \sum_k \hat{\Pi}_k \, \hat{\rho}_{A_1 \cdots A_N} \, \hat{\Pi}_k$$

$$\Phi_j\left(\hat{\rho}_{A_j}\right) = \sum_{j'} \hat{\Pi}_{A_j}^{j'} \, \hat{\rho}_{A_j} \, \hat{\Pi}_{A_j}^{j'}$$

$$\mathcal{D}\left(\hat{\rho}_{A_1 \cdots A_N}\right) = \min_{\{\hat{\Pi}_k\}} \left[ S\left(\hat{\rho}_{A_1 \cdots A_N} \parallel \Phi\left(\hat{\rho}_{A_1 \cdots A_N}\right)\right) - \sum_{j=1}^{N} S\left(\hat{\rho}_{A_j} \parallel \Phi_j\left(\hat{\rho}_{A_j}\right)\right)\right]$$

Relative entropy $S\left(\hat{\rho} \parallel \hat{\sigma}\right) = \mathrm{Tr}\left(\hat{\rho} \log_2 \hat{\rho} - \hat{\rho} \log_2 \hat{\sigma}\right)$

C. C.Rulli and M. S. Sarandy, PRA 84, 042109 (2011)

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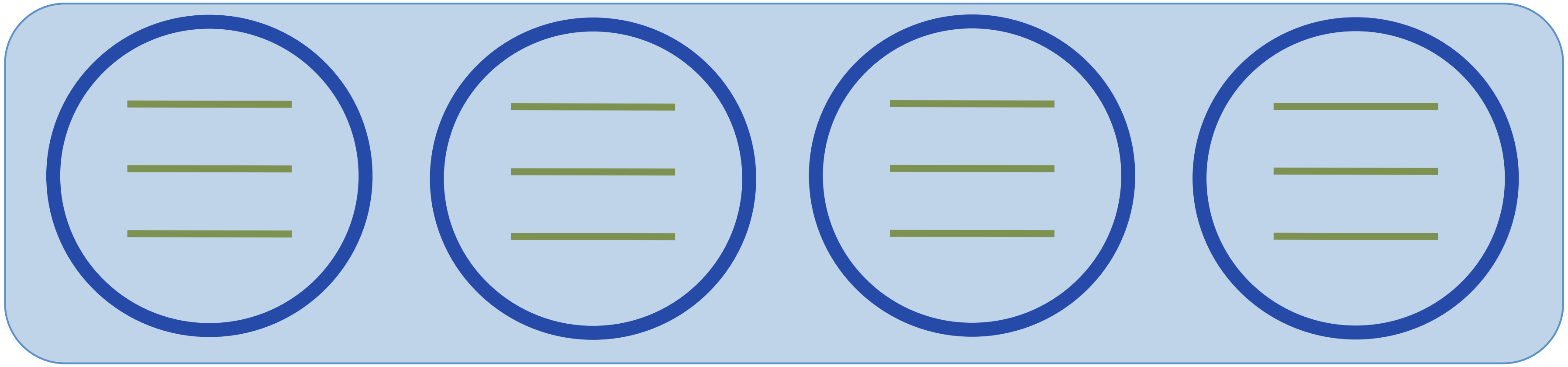
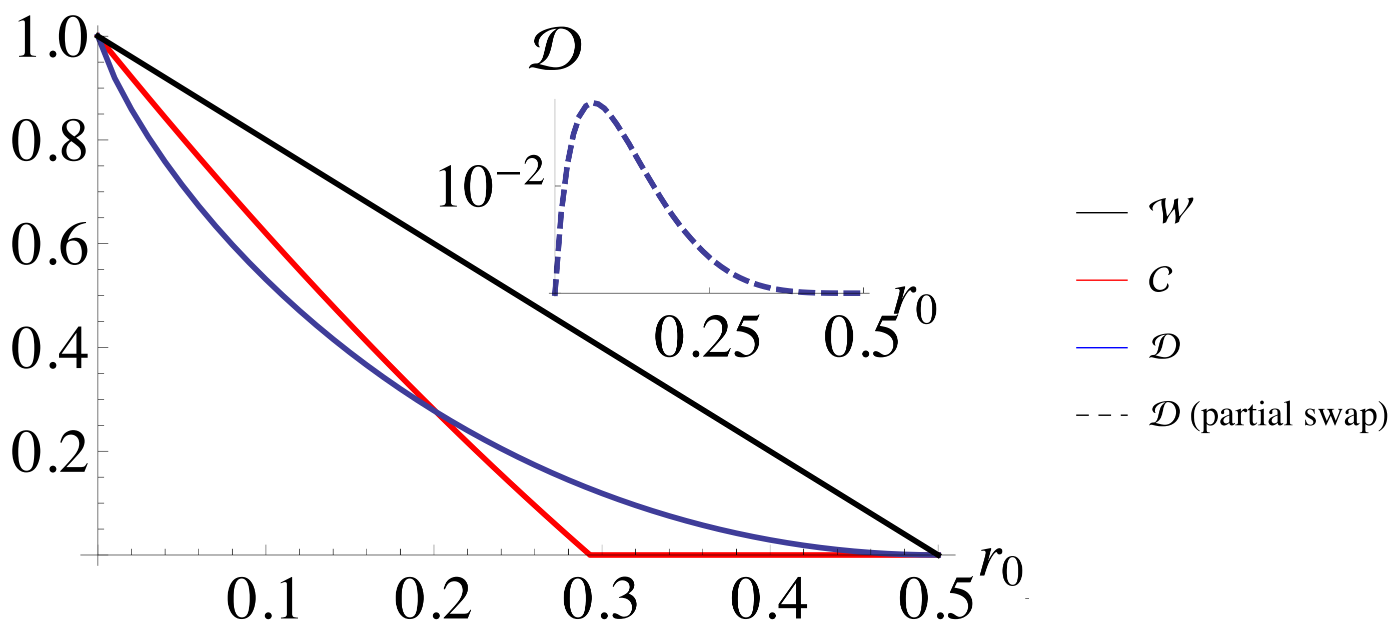
# n identical batteries



$$\rho(0) = (p_0|0\rangle\langle 0| + \cdots + p_{d-1}|d-1\rangle\langle d-1|)^{\otimes n}$$

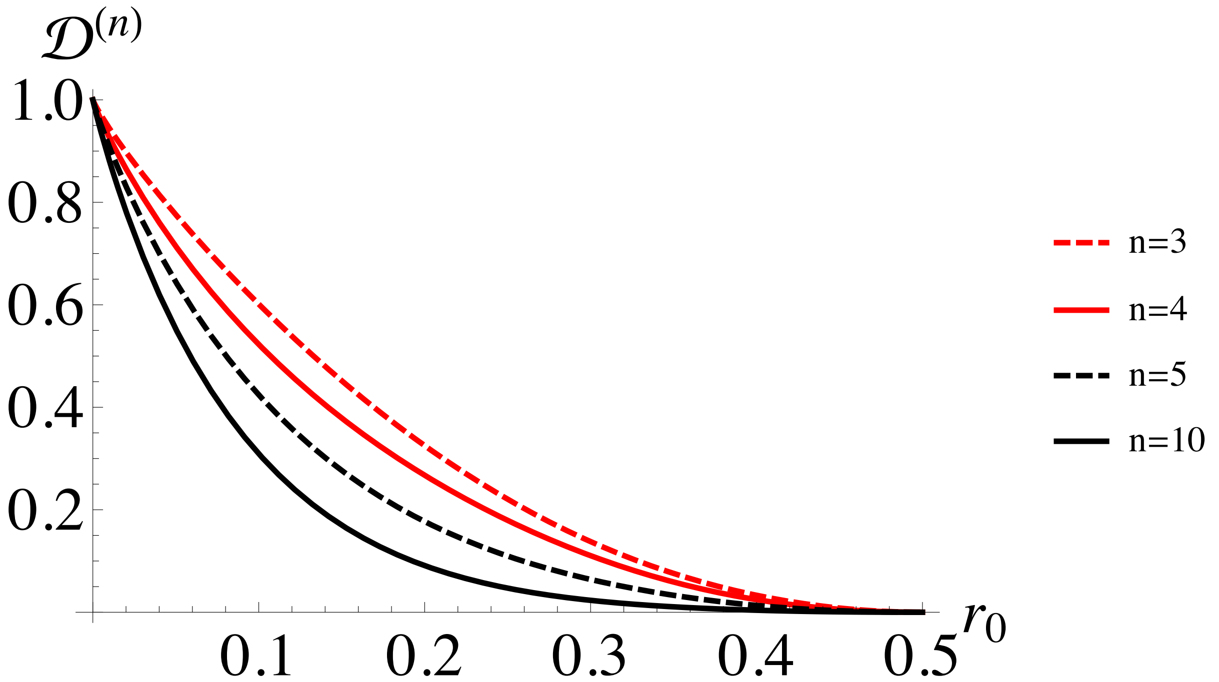
# 2 qubits: $\mathcal{W}$, discord, entanglement

Complete swap $$|00\rangle \leftrightarrows |11\rangle$$



# n qubits: genuine correlations

Complete swap: $$|0,0,\ldots,0\rangle \leftrightarrows |1,1,\ldots,1\rangle$$



# n qubits: global discord

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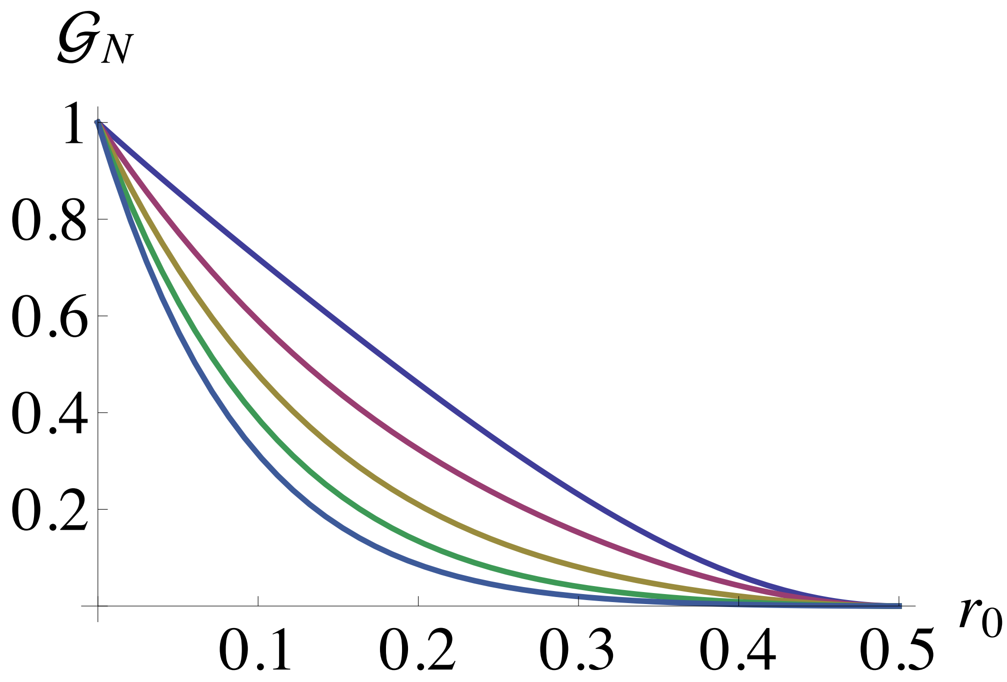
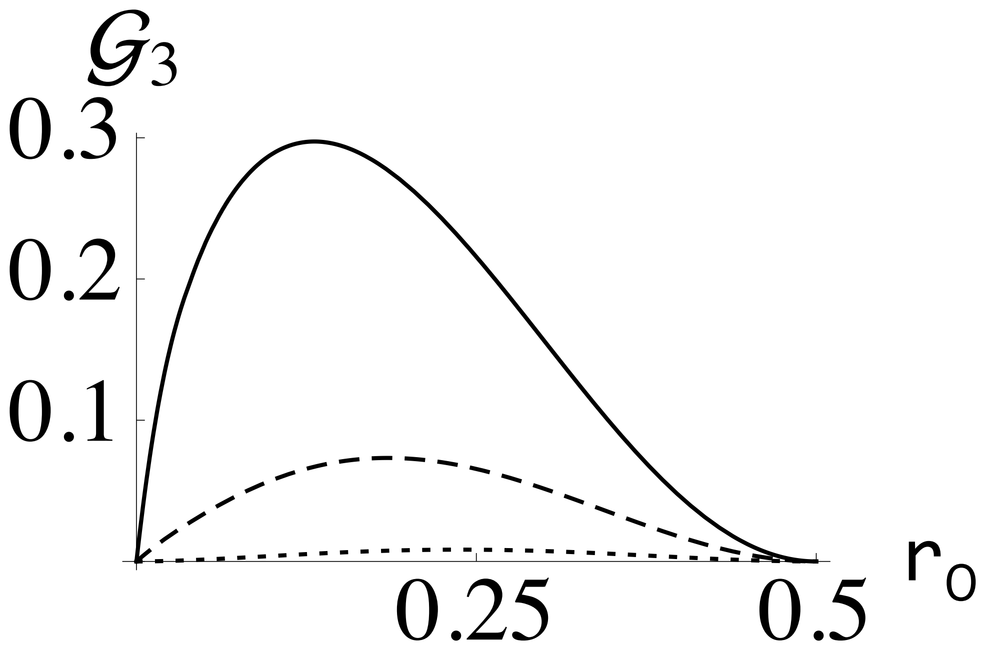


# 3 qubits: global discord, 5-step swap

$$|111\rangle \leftrightarrows |110\rangle; \quad |110\rangle \leftrightarrows |100\rangle; \quad |100\rangle \leftrightarrows |000\rangle$$



# Discord witness

Beyond the cases shown explicitly, is it possible to draw a general conclusion about the presence of discord?

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In the case of global discord, there exists a witness that can be tested: a state is not classical if

$$[\rho, \rho_1 \otimes \rho_2 \otimes \ldots, \rho_n] \neq 0$$

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$$[\rho, \rho_1 \otimes \rho_2 \otimes \ldots, \rho_n] \neq 0$$

As for genuine correlations, we shall inspect the shape of the density matrix

# Global discord witness

$$\rho = \mathrm{diag}(\rho'_{(i)}) + c_{\alpha,\alpha}(t)|\alpha\rangle\langle\alpha| + c_{\beta,\beta}(t)|\beta\rangle\langle\beta| + (c_{\alpha,\beta}(t)|\alpha\rangle\langle\beta| + h.c.)$$

Complete swap: $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$ is diagonal

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Complete swap: $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$ is diagonal

The commutator can be calculated explicitly

$$[\rho, \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n] = \begin{pmatrix} 0 & 0 \ \ldots & 0 & -c_{\alpha,\beta}(\tilde{\rho}_{\alpha,\alpha} - \tilde{\rho}_{\beta,\beta}) \\ 0 & 0 \ \ldots & 0 & 0 \\ & & \ldots & \\ & & \ldots & \\ c_{\alpha,\beta}(\tilde{\rho}_{\alpha,\alpha} - \tilde{\rho}_{\beta,\beta}) & 0 \ \ldots & 0 & 0 \end{pmatrix}$$

# Global discord witness

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# Global discord witness

Partial swap: natural bipartition
a= common set of indices
b= swapped indices

$$\rho = \sum\nolimits'_{i \in a,\ j \in b} c_{ij} |i, j\rangle\langle i, j| \\ + |\alpha_a\rangle\langle\alpha_a| \otimes [c_{\alpha,\alpha}(t)|\alpha_b\rangle\langle\alpha_b| + c_{\beta,\beta}(t)|\beta_b\rangle\langle\beta_b| + (c_{\alpha,\beta}(t)|\alpha_b\rangle\langle\beta_b| + h.c.)]$$

Two-step proof:
1) there are correlations between party a and party b
2) condition 1 implies that there is n-partite global discord

# Global discord witness

1) As a consequence of $[\rho, \rho_1 \otimes \rho_2 \otimes \ldots, \rho_n] \neq 0$

$$\rho_j = \sum_n p_n \Pi_j^n$$

The nondiagonal element prevents $\rho_b$ from having this form

# Global discord witness

1) As a consequence of $[\rho, \rho_1 \otimes \rho_2 \otimes \ldots, \rho_n] \neq 0$

$$\rho_j = \sum_n p_n \Pi_j^n$$

The nondiagonal element prevents $\rho_b$ from having this form

2) The definition of global discord is based on an optimization procedure over a set of orthogonal projectors. If we consider a block of sub-parties, we enlarge the space of possible operations. Any good global measure should obey this kind of criterion.

# Genuine discord witness

Complete swap

$$\rho = \mathrm{diag}(\rho'_{(i)}) + c_{\alpha,\alpha}(t)|\alpha\rangle\langle\alpha| + c_{\beta,\beta}(t)|\beta\rangle\langle\beta| + (c_{\alpha,\beta}(t)|\alpha\rangle\langle\beta| + h.c.)$$

Take any bipartition

$$\rho = {\sum}'_{i\in a,\ j\in b} c_{ij}|i,j\rangle\langle i,j| \\ + c_{\alpha,\alpha}(t)|\alpha_a,\alpha_b\rangle\langle\alpha_a,\alpha_b| + c_{\beta,\beta}(t)|\beta_a,\beta_b\rangle\langle\beta_a,\beta_b| + (c_{\alpha,\beta}(t)|\alpha_a,\alpha_b\rangle\langle\beta_a,\beta_b| + h.c.)$$

The state has a quantum form in any of the bipartitions

$$\rho \neq \sum_i p_i|\phi_i^{(a)}\rangle\langle\phi_i^{(a)}| \otimes \rho_i^{(b)}$$

$$\rho \neq \sum_i p_{i,j}|\phi_i^{(a)}\rangle\langle\phi_i^{(a)}| \otimes |\psi_j^{(a)}\rangle\langle\psi_j^{(a)}|$$

# Genuine discord witness

Direct inspection of the density matrix (not so complicated, only two nondiagonal terms)

Complete swap: genuine correlations

m-index swap: m-partite genuine correlations

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However, not always is entanglement created during the extraction

The role played by discord is completely different: it is necessary to create it

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*Thank you for your attention*