Distinguishing Quantum Operations : LOCC vs Separable operators

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Distinguishability of Quantum states

Distinguishing Quantum States of a single system

Task of distinguishing an ensemble

$$\xi = \{|\psi_i\rangle, p_i\}$$

of pure quantum states of single system is equivalent of optimizing the probability

$$\sum_i p_i \langle\psi_i|M_i|\psi_i\rangle$$

of identify a state chosen at random from ξ in terms of finding optimum measurement process $\mathbb{M}=\{M_i\}$.

It has been found that in most of all cases the measurement

$$\{M_i = p_i \rho |\psi_i\rangle\langle\psi_i|\}$$

suffices to identify $|\psi_i\rangle$ from ξ.

Distinguishability of mixed states of a single system

Problem of distinguishing an ensemble of mixed states

$$\xi = \{\rho_i, p_i\} \quad ; \quad \rho_i = \sum_j \lambda_j^{(i)} \left|\psi_j^{(i)}\right\rangle\left\langle\psi_j^{(i)}\right|$$

is turned into distinguishing the supports of different states ρ_i.

Distinguishing multipartite states

Reflects power and limitations of different classes of quantum operations(POVM), v.i.z., LOCC class address difference between nonlocality and entanglement.

In 2000, Walgate et.al. showed, any two orthogonal pure states of multipartite system are perfectly distinguishable by one-way LOCC, by representing them in suitable basis

$$\left|\psi^1\right\rangle_{AB} = \left|1\right\rangle_A \left|\mu^1\right\rangle_B + \left|2\right\rangle_A \left|\mu^2\right\rangle_B + \cdots + \left|n\right\rangle_A \left|\mu^n\right\rangle_B$$

$$\left|\psi^2\right\rangle_{AB} = \left|1\right\rangle_A \left|\upsilon^1\right\rangle_B + \left|2\right\rangle_A \left|\upsilon^2\right\rangle_B + \cdots + \left|n\right\rangle_A \left|\upsilon^n\right\rangle_B$$

Nonlocality associated with concept of distinguishability

- The set of 9 pure, product states forming a basis of 3×3 system,

$$
\begin{aligned}
|\psi_0\rangle &= |1\rangle \otimes |1\rangle, \\
|\psi_1^{\pm}\rangle &= |0\rangle \otimes |0 \pm 1\rangle, \\
|\psi_2^{\pm}\rangle &= |2\rangle \otimes |1 \pm 2\rangle, \\
|\psi_3^{\pm}\rangle &= |1 \pm 2\rangle \otimes |0\rangle, \\
|\psi_4^{\pm}\rangle &= |0 \pm 1\rangle \otimes |2\rangle
\end{aligned}
$$

is not distinguishable by LOCC.

Distinguishability under different classes of operators

- A set of quantum states S={ρ_1, ρ_2,…, ρ_n} is said to be distinguishable by the POVM

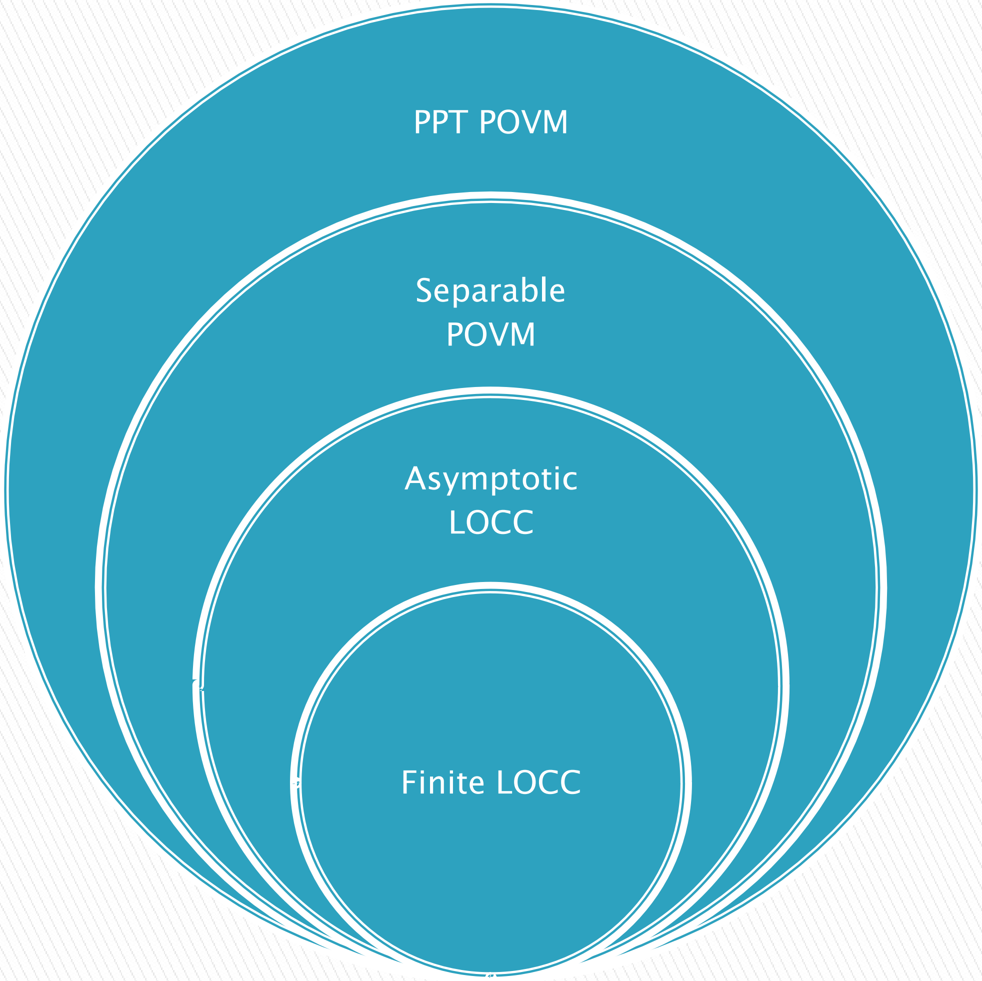
$$M = \left(\Pi_k\right)_{k=1}^{n} \quad \text{if} \quad \mathrm{tr}\left(\Pi_\mathrm{k}\rho_i\right) = \delta_{ik} \ \ \forall \ \mathrm{k,i} \in \{1,2,\cdots,\mathrm{n}\}$$

- If each Π_k is *PPT (/ Separable / LOCC)*, then

M is said to be *PPT (/ Separable / LOCC)* POVM.

Finite LOCC : Finite times of communication

Separable POVM : $\rho_i = \Pi_\mathrm{i} / tr(\Pi_\mathrm{i}) \, \forall i$ is separable



Relation between the classes of operations

In context of distinguishability,
which of the inner border lines do really exists?

Some Results

- A collection of mn-1 number of mutually orthogonal pure states of $m \times n$ system, is perfectly distinguishable by PPT POVMs if and only if they are distinguishable by separable POVMs.
- For systems with Schmidt rank greater than 2, there is no complete orthonormal basis that is perfectly distinguishable by LOCC.
- In 3×3 system there exists set of states that are perfectly distinguishable by Separable POVMs while asymptotic LOCC class is unable to distinguish the states.

- An orthonormal basis S of bipartite system is perfectly distinguishable by Separable(as well as by PPT) POVM if and only if S is a product basis.

Distinguishability not so Perfect

- Minimum Error Distinguishability
- Unambiguous Distinguishability

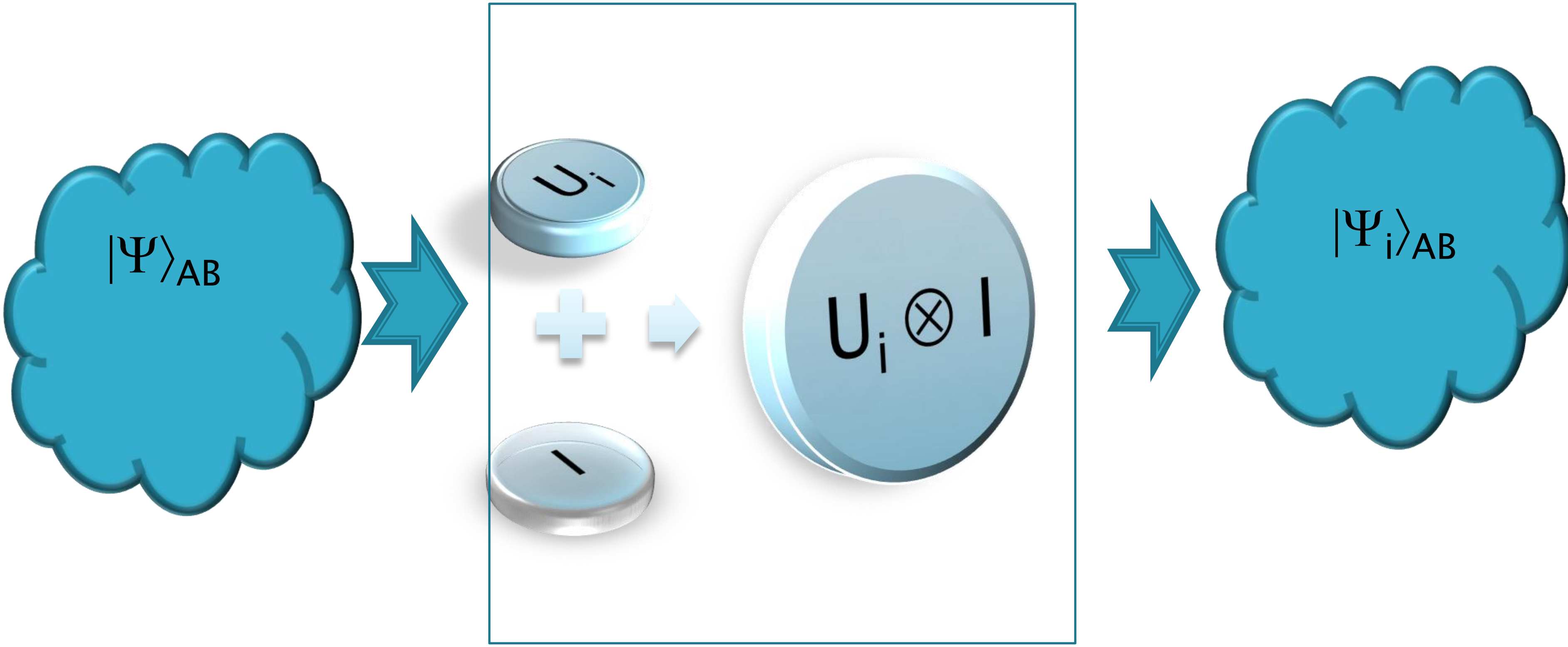
  - Both are optimization scheme
  - Both show a difference between the status of LOCC and Separable POVM

Distinguishability of Quantum Operations

- Given two quantum operations U_1 and U_2, distinct in the sense, there exist no real value of θ, satisfying $U_1 = e^{i\theta} U_2$. One of the operations is chosen at random. To identify the operation one search for a suitable state that will be transformed into two orthogonal states after the action of U_1 and U_2 .

$$U_i \left| \psi \right\rangle = \left| \psi_i \right\rangle, \quad i = 1,2 \ ; \qquad \left\langle \psi_1 \middle| \psi_2 \right\rangle = 0$$

- Then there exist a projective measurement on state space of this system, which by distinguishing the two orthogonal states, will subsequently distinguish the operations U_1 and U_2.



$|\Psi\rangle_{AB}$
U_i
I
$U_i \otimes I$
$|\Psi_i\rangle_{AB}$

- By all possible choice of the initial state, even chosen from a larger state space, the distance between the final states are to be maximized. That will optimize the possibility to identify the unknown quantum operation chosen from a known set.

Question : As accomplished through the distinguishability of quantum states, does the aspect of distinguishability of quantum operations will be equivalent with it ?

- Answer : **No**

- In 2001, Acin showed, any two unitary quantum operations are statistically distinguishable, i.e., **by a finite number of use** of the operation (chosen at random from the set of two unitary), on a suitably chosen initial state(may be **entangled**, in cases) of an extended system, the final states become mutually orthogonal.

Kraus representation theorem

1. Action of any operator ξ acting on a quantum system ρ can be expressed in terms of linear operators $\{E_i\}$ as

$$\xi(\rho) = \sum_{i=0}^{n-1} E_i \rho E_i^\dagger \quad ; \qquad \sum_{i=0}^{n-1} E_i E_i^\dagger = I$$

2. For every ξ acting on ρ, there exists an unitary operator U acting on some enlarged system (including the state space and a suitably chosen environment space), on which the action of the operator ξ can be realized as

$$\xi(\rho) = Tr_{env}\left(U \rho_{system} \otimes \rho'_{enviornment} U^\dagger\right)$$

An Optimization process using entanglement

- To optimally distinguish the operations ξ_1 , ξ_2 an entangled input state ρ is chosen from a larger state space so that the output states $(\xi_1 \otimes I)\rho$, $(\xi_2 \otimes I)\rho$ can be distinguished optimally, in the sense of minimizing the probability of error

$$p_E = \tfrac{1}{2}\left(1 - \max_{\rho \in H \otimes K} \left\| p_1(\xi_1 \otimes I)\rho - p_2(\xi_2 \otimes I)\rho \right\|_1\right)$$

An example

$$\xi_1\left(|\psi\rangle_H\langle\psi|\right)=|\psi\rangle_H\langle\psi| \;\; \forall \; |\psi\rangle \in H$$ Identity map

$$\xi_2\left(|\psi\rangle_H\langle\psi|\right)=\frac{I_H}{d} \;\; \forall \; |\psi\rangle \in H$$ Depolarization map

By using entangled input state of a larger system probability of error decreases.

However for the case of distinguishing two unitary operations, use of entanglement not necessarily improve the situation.

Local distinguishability of two multipartite unitary operation

- Two unitary operations U_1 and U_2 acting on multipartite quantum system $\mathcal{H}$, can be distinguished by LOCC with single use of the operation and without any use entanglement in the process, if there exist a product state
  $$\left|\Psi\right\rangle = \otimes_{i=1}^{n}\left|\psi_i\right\rangle \in H$$ such that
  $$tr\left(U_1^{\dagger}U_2\left|\Psi\right\rangle\left\langle\Psi\right|\right) = 0$$
- Any two unitary operation, by using the finite times, can be distinguished locally.

Perfect distinguishability of two disjoint operators

- The operations with Kraus form,

$$\xi^k(\rho) = \sum_{i=0}^{n_k-1} E_{ik}\rho E_{ik}^{\dagger} \quad ; \qquad \sum_{i=0}^{n_k-1} E_{ik}E_{ik}^{\dagger} = I \qquad \text{for k} = 1,2$$

acting on $\mathcal{H}_1$ space, are said to be disjoint if there is a state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2$ space such that

$$Support\left(\left(I_{H_1} \otimes \xi^1_{H_2}\right)\rho\right) \cap Support\left(\left(I_{H_1} \otimes \xi^2_{H_2}\right)\rho\right) = \phi$$

A pair of disjoint operations can be distinguished by a single use of it, if

$$I \notin span\left\{E_{i1}E_{j2}^{\dagger}\right\}$$

Distinguishability through unitary realization in extended Hilbert space by Kraus 2nd representation

- Any two general operations in Kraus form,

$$\xi^k(\rho) = \sum_{i=1}^{n-1} E_{ik}\rho E_{ik}^{\dagger} \quad ; \qquad \sum_{i=1}^{n-1} E_{ik}E_{ik}^{\dagger} = I \qquad \text{for k} = 1,2$$

acting on state space $\mathcal{H}$ with same number(n) of Kraus operators $\{E_{11}, E_{21},\ldots, E_{n1}\}$ and $\{E_{12}, E_{22},\ldots, E_{n2}\}$.
The action of this operators can be implemented by some unitary in an enlarged space $\mathcal{H}\otimes\mathcal{K}$ as

$$\xi^k(\rho) = Tr_{env}\left(U_k \rho_{syst} \otimes \rho'_{env} U_k^{\dagger}\right) \text{ for } \text{k} = 1,2$$

Without any loss of generality environment can be prepared in the initial state $\rho'_{env} = |e_0\rangle\langle e_0|$.

- Considering an n dimensional space $\mathcal{K}$ the unitary operations can be realized as

$$U_k = \sum_{i=0}^{n-1} E_{ik} \otimes |e_i\rangle\langle e_0| \quad \text{for } \mathrm{k} = 1,2$$

- The unitary operators can be distinguished by acting on a product state $|\Psi\rangle = |\psi\rangle_H |\varphi\rangle_K, \langle e_0|\varphi\rangle = 0$
- The final states $|\Psi_i\rangle = U_i|\Psi\rangle,\ i = 1,2$ are mutually orthogonal and thus distinguishable, ensuring distinguishability of the operations ξ^1 and ξ^2 without using any resource of entanglement in initial state.

Class of Pauli Channels

- For example, the Pauli operations (considered earlier in optimizing schemes also)

$$\xi^k(\rho)=\sum_{i=1}^{n-1} q_i^{(k)}\sigma_i\rho\sigma_i\,; \quad \sum_{i=1}^{n-1}\left(q_i^{(k)}\right)^2=1 \quad \text{for } k=1,2$$

- Corresponding unitary operations are realized in 2×4 dimensional space as

$$U_k=\sum_{i=0}^{n-1}\sqrt{q_i^{(k)}}\sigma_i\otimes|e_i\rangle\langle e_0| \quad \text{for } k=1,2$$

- An arbitrary pair of operations will satisfy

$$U^{k\dagger}U^s=\sum_{i=0}^{3}\sqrt{q_i^{(k)}q_i^{(s)}}\; I\otimes|e_0\rangle\langle e_0|$$

- A pure product state $|\Psi\rangle = |\psi\rangle_H |\varphi\rangle_K, \langle e_0|\varphi\rangle = 0$ of 2×4 system will distinguish the operations.
- Instead of use of entanglement as necessary for optimizing schemes in a larger space the distinguishability of the arbitrary pair of operations accomplished.

Depolarizing (Entanglement-breaking) Channels

- Comparing two Depolarizing Channels

$$\xi^{k}(\rho) = p_k \rho + \frac{1-p_k}{3}\sum_{i=1}^{n-1}\sigma_i \rho \sigma_i \; ; \quad \text{for } \mathrm{k} = 1,2$$

- Corresponding unitary operations are realized in 2×4 dimensional space
- Here also a pure product state will distinguish the operations.

Class of General Pauli Channels

- Class of the General Pauli operations (considered earlier in optimizing schemes)

$$\xi^k(\rho) = \sum_{i=1}^{n-1} \sqrt{q_i^{(k)}} u_i^{(k)} \rho u_i^{(k)} ; \quad \sum_{i=1}^{n-1} q_i^{(k)} \left(u_i^{(k)}\right)^2 = I \quad \text{for k} = 1,2$$

can also be discriminated in the same way without use of entanglement.

Separable vs LOCC

- In the context of perfect distinguishability of quantum states the status of LOCC and separable operations are inequivalent. As there exists set of states that are not locally (even by asymptotic LOCC) distinguishable but distinguishable by separable operations.
- If we move on to distinguishability of quantum operations, allowing the operation (to be identified) to act only single time, then there exist pair Unitary Operators not distinguishable locally.

Example

- Consider a pair of Unitary operators acting on 2×2 system as

$$U_1 = |00\rangle\langle 00| + e^{-i\theta_1/2}|01\rangle\langle 01| + e^{-i\theta_2/2}|10\rangle\langle 10| + |11\rangle\langle 11|$$

$$U_2 = |00\rangle\langle 00| + e^{i\theta_1/2}|01\rangle\langle 01| + e^{i\theta_2/2}|10\rangle\langle 10| - |11\rangle\langle 11|$$

- Following the condition given in the work by Duan et.al,(PRL **100**, 020503, 2008) this two operations can not be distinguishable by LOCC, using the operation only one time). However globally(being unitary) they are distinguishable without use of entanglement, i.e., by Separable POVM.

- This difference in status of distinguishability by Separable vs LOCC POVM are until shown to exist only for single-use of the operation.

- An operational set-up, if prepared once, can be, without any extra cost, physically used a finite number of time to accomplish a given task.

To Distinguish Unitary Operations (even acting on Multipartite systems), use of Separable POVM gives not much advantage over LOCC

General quantum Operations

- In case of general operations, whether use of separable POVMs are advantageous over LOCC is an open problem till now.

Thank You