Introduction to Quantum Metrology

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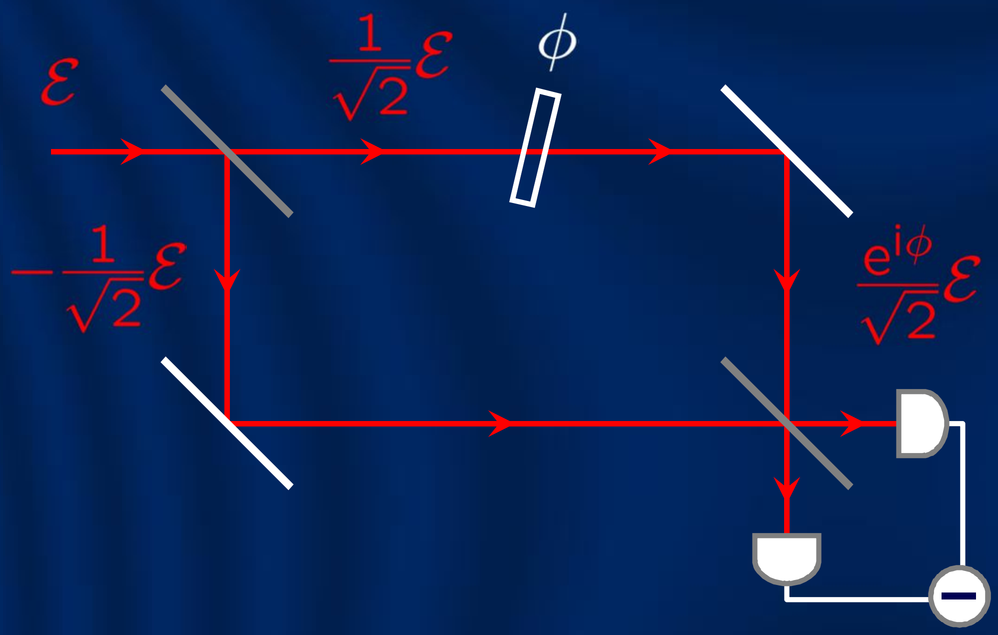
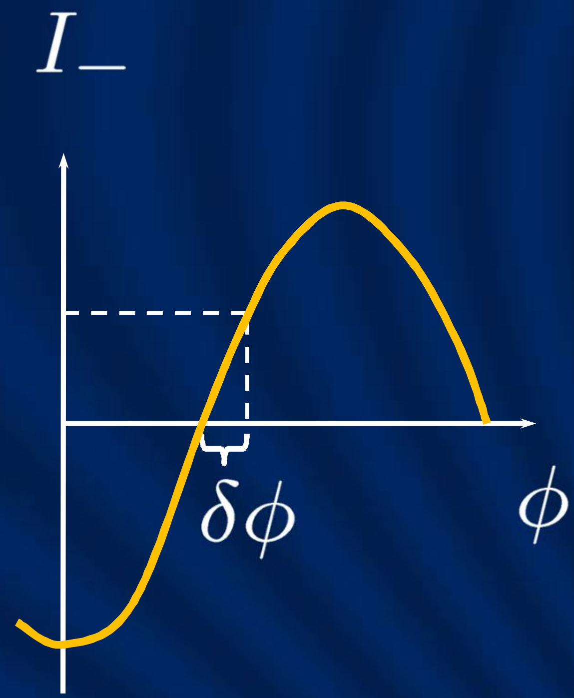
Bhubaneswar, 17-28 February 2014

Phase measurement



$$I_- = -|\mathcal{E}|^2 \cos\phi$$

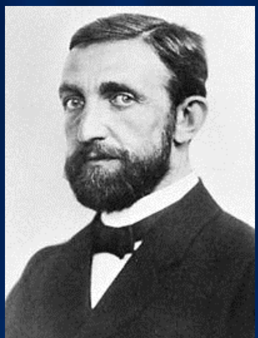


Photoelectric effect

9. ***Ueber die lichtelektrische Wirkung;***
von P. Lenard.

(Hierzu Taf. I, Figg. 1 u. 2.)

In einer früheren Mitteilung habe ich gezeigt, dass ultraviolettes Licht, das auf Körper trifft, Kathodenstrahlung aus denselben veranlassen kann.[1]) Diese Erzeugung von Kathodenstrahlen erwies sich unabhängig vom Vorhandensein eines Gases; sie ging, im Gegensatz zur früher allein bekannten Erzeugungsart in Entladungsröhren, auch im äussersten Vacuum vor sich.[2]) Charakteristisch war es, wie im Vacuum gefunden wurde, dass elektrische Kräfte diese Erzeugung nicht beeinflussten; ein Ansteigen der Kraft an der negativ ge-



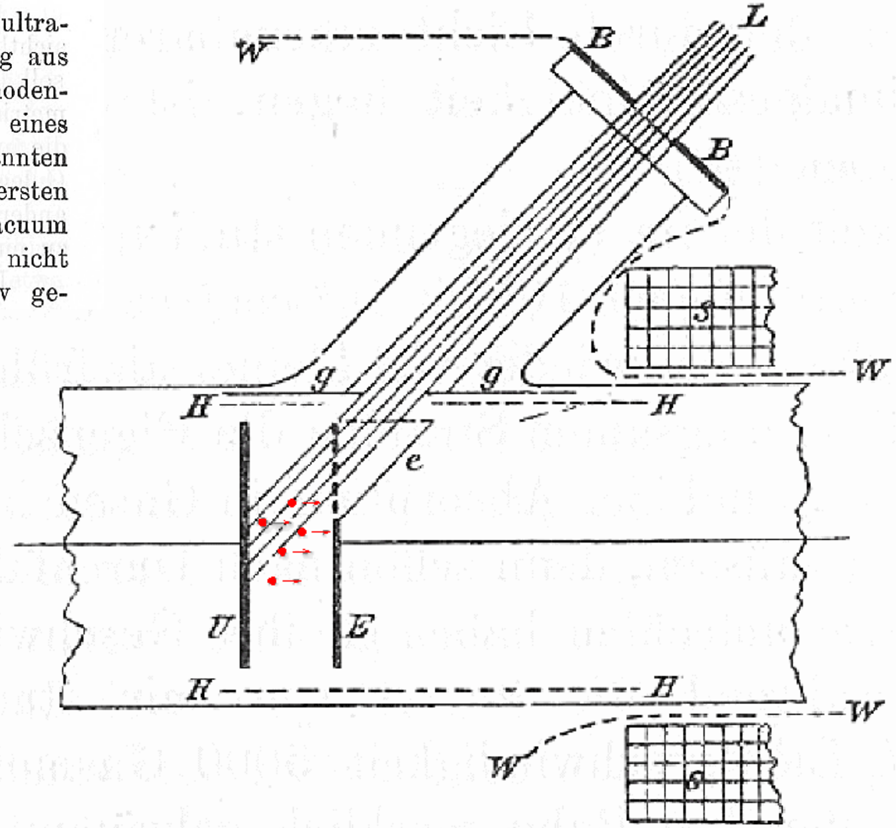
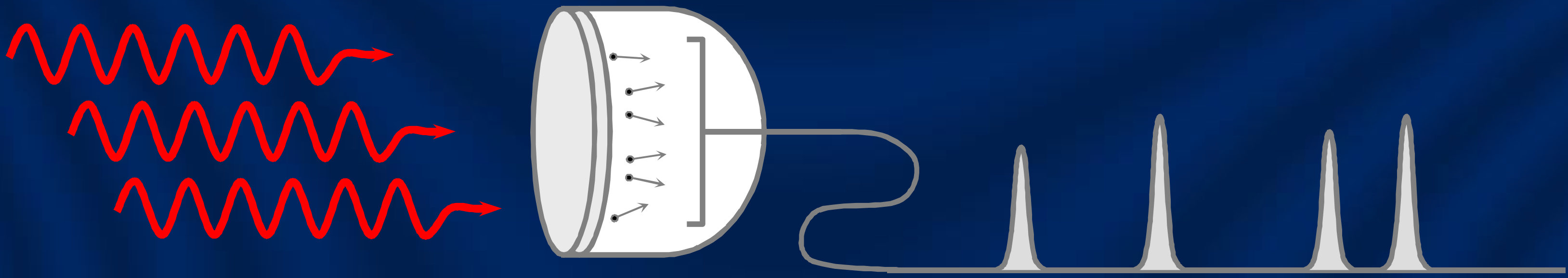


Fig. 1.

P. Lenard,
Ann. Physik **8**, 149 (1902)

Photons



6. *Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt;* *von A. Einstein.*

Zwischen den theoretischen Vorstellungen, welche sich die Physiker über die Gase und andere ponderable Körper gebildet haben, und der Maxwellschen Theorie der elektromagnetischen Prozesse im sogenannten leeren Raume besteht ein tiefgreifender formaler Unterschied. Während wir uns nämlich den Zustand eines Körpers durch die Lagen und Geschwindigkeiten einer zwar sehr großen, jedoch endlichen Anzahl von Atomen und Elektronen für vollkommen bestimmt ansehen, bedienen wir uns zur Bestimmung des elektromagnetischen Zustandes eines Raumes kontinuierlicher räumlicher Funktionen, so daß also eine endliche Anzahl von Größen nicht als genügend anzusehen ist zur vollständigen Festlegung des elektromagnetischen Zustandes eines Raumes. Nach der

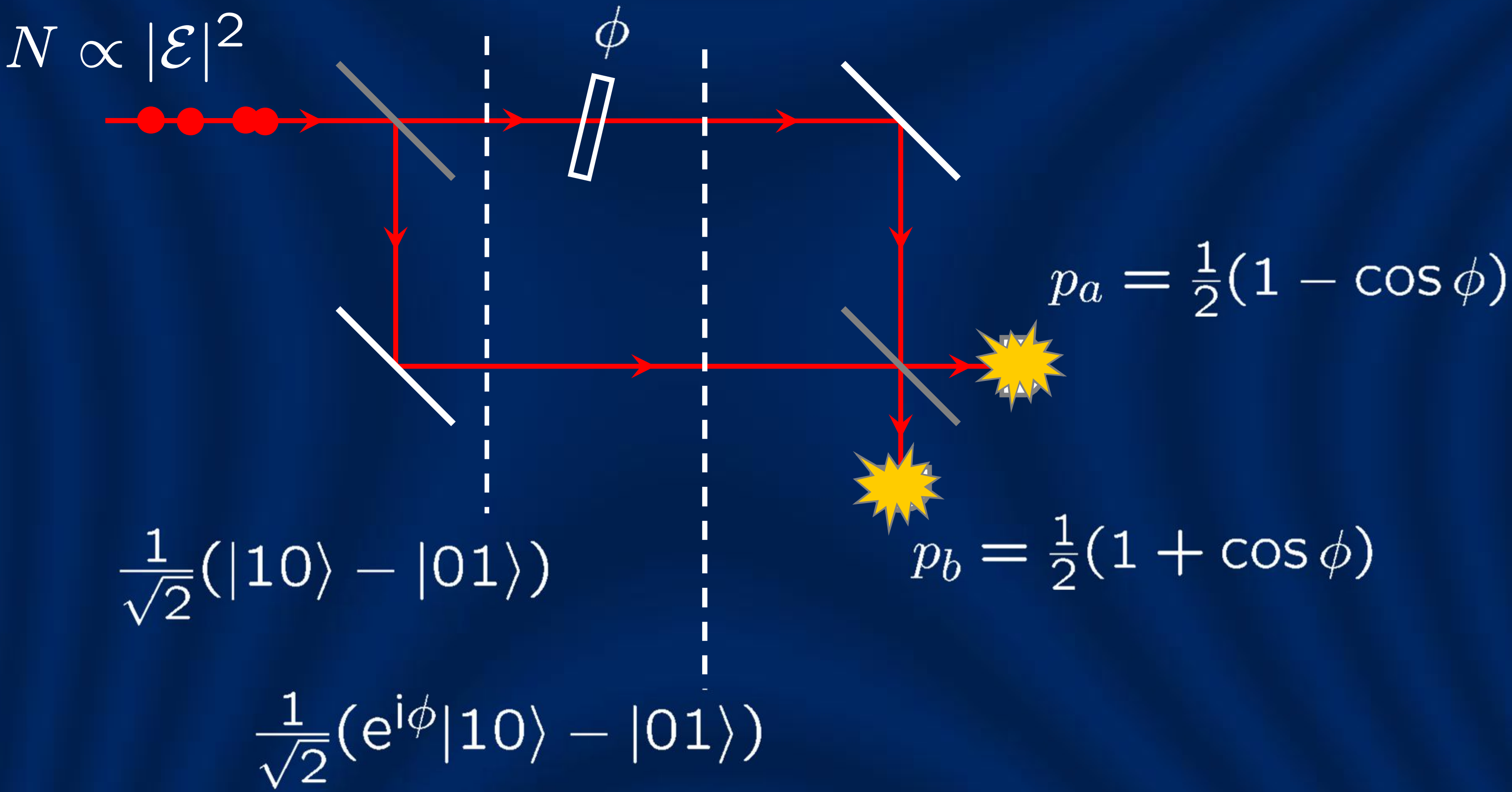
§ 8. Über die Erzeugung von Kathodenstrahlen durch Belichtung fester Körper.

Die übliche Auffassung, daß die Energie des Lichtes kontinuierlich über den durchstrahlten Raum verteilt sei, findet bei dem Versuch, die lichtelektrischen Erscheinungen zu erklären, besonders große Schwierigkeiten, welche in einer bahnbrechenden Arbeit von Hrn. Lenard dargelegt sind.[1])

Nach der Auffassung, daß das erregende Licht aus Energiequanten von der Energie $(R/N)\beta\nu$ bestehe, läßt sich die Erzeugung von Kathodenstrahlen durch Licht folgendermaßen auffassen. In die oberflächliche Schicht des Körpers dringen Energiequanten ein, und deren Energie verwandelt sich wenigstens zum Teil in kinetische Energie von Elektronen. Die einfachste Vorstellung ist die, daß ein Lichtquant seine ganze Energie an ein einziges Elektron abgibt; wir wollen annehmen, daß dies vorkomme. Es soll jedoch nicht ausgeschlossen sein, daß Elektronen die Energie von Lichtquanten nur teilweise aufnehmen. Ein im Innern des Körpers mit kinetischer Energie

1) P. Lenard, Ann. d. Phys. 8. p. 169 u. 170. 1902.

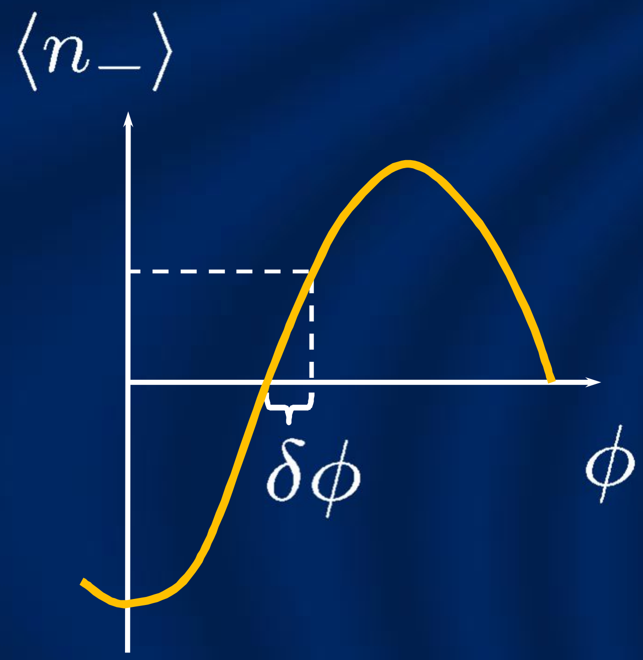
Quantum picture



Phase estimate

Let $\phi = \frac{\pi}{2} + \delta\phi$

Our task is to guess small $\delta\phi$.



Photocount difference:

$$n_- = n_a - n_b$$

Statistical average:

$$\langle n_- \rangle = -N \cos(\tfrac{\pi}{2} + \delta\phi) \approx N\delta\phi$$

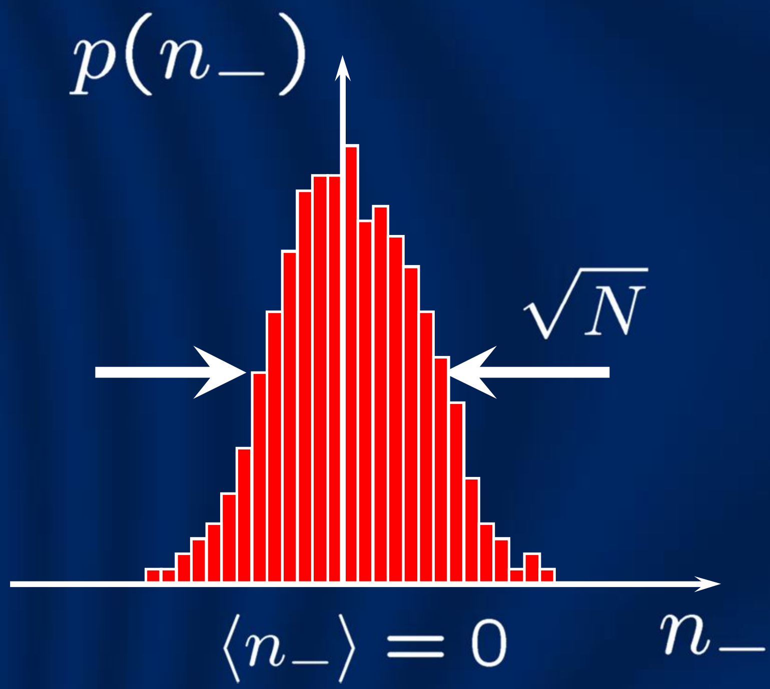
Estimation procedure:

$$\delta\phi = \frac{n_-}{N}$$

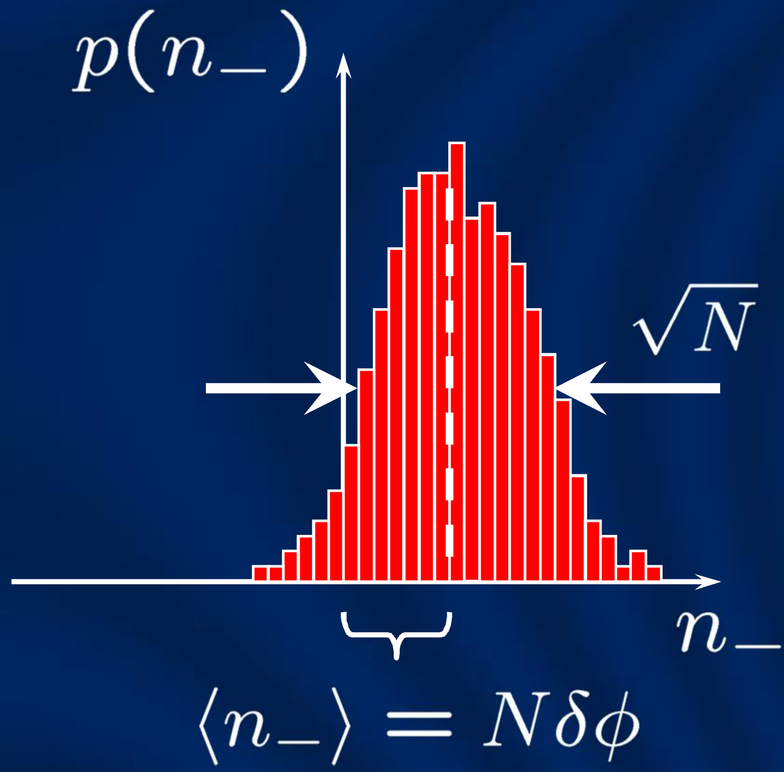
Individual realization of the experiment with $\bar{n}$ photons!

Shot noise

No phase shift $\delta\phi = 0$



Phase shift $\delta\phi \neq 0$



To identify
a phase shift $N\delta\phi \gtrsim \sqrt{N}$

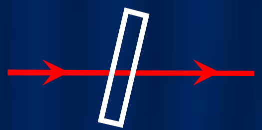
...hence the
phase resolution $\delta\phi \gtrsim \dfrac{1}{\sqrt{N}}$

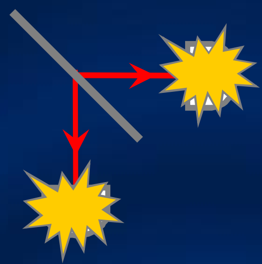
Estimation quality

Estimation procedure:

$$\Phi(n_a, n_b) = \frac{\pi}{2} + \frac{n_a - n_b}{N}$$

Actual value ϕ → $p(r|\phi)$ → Measurement result r → $\Phi(r)$ → Estimate $\tilde{\phi}$

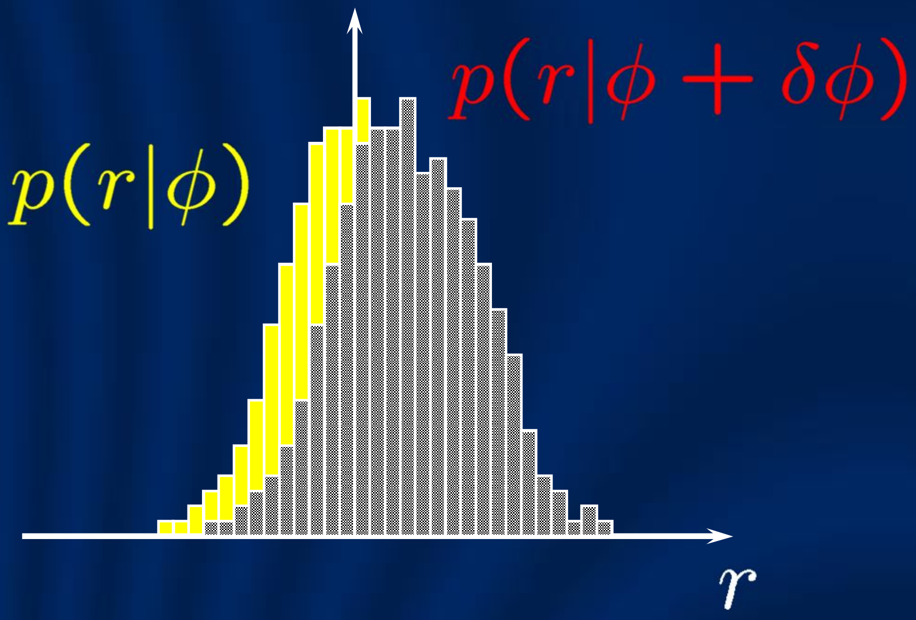




(n_a, n_b)

Fisher information

$$\mathsf{F}(\phi) = \sum_r p(r|\phi) \left(\frac{\partial}{\partial \phi} \ln p(r|\phi) \right)^2$$



Cramér-Rao bound:
for unbiased estimators

$$\Delta\tilde{\phi} \geq \frac{1}{\sqrt{\mathsf{F}(\phi)}}$$

Proof

Cauchy-Schwarz inequality:

$$\left(\sum_r A_r^2\right)\left(\sum_r B_r^2\right) \geq \left(\sum_r A_r B_r\right)^2$$

Take

$$A_r = \sqrt{p(r|\phi)}[\Phi(r) - \phi], B_r = \frac{1}{\sqrt{p(r|\phi)}}\frac{\partial}{\partial\phi}p(r|\phi)$$

and on the RHS use the assumption of *unbiasedness*:

$$\sum_r p(r|\phi)\Phi(r) = \phi$$

Additivity

When variables are statistically independent

$$p(r_1, r_2|\phi) = p(r_1|\phi)p(r_2|\phi)$$

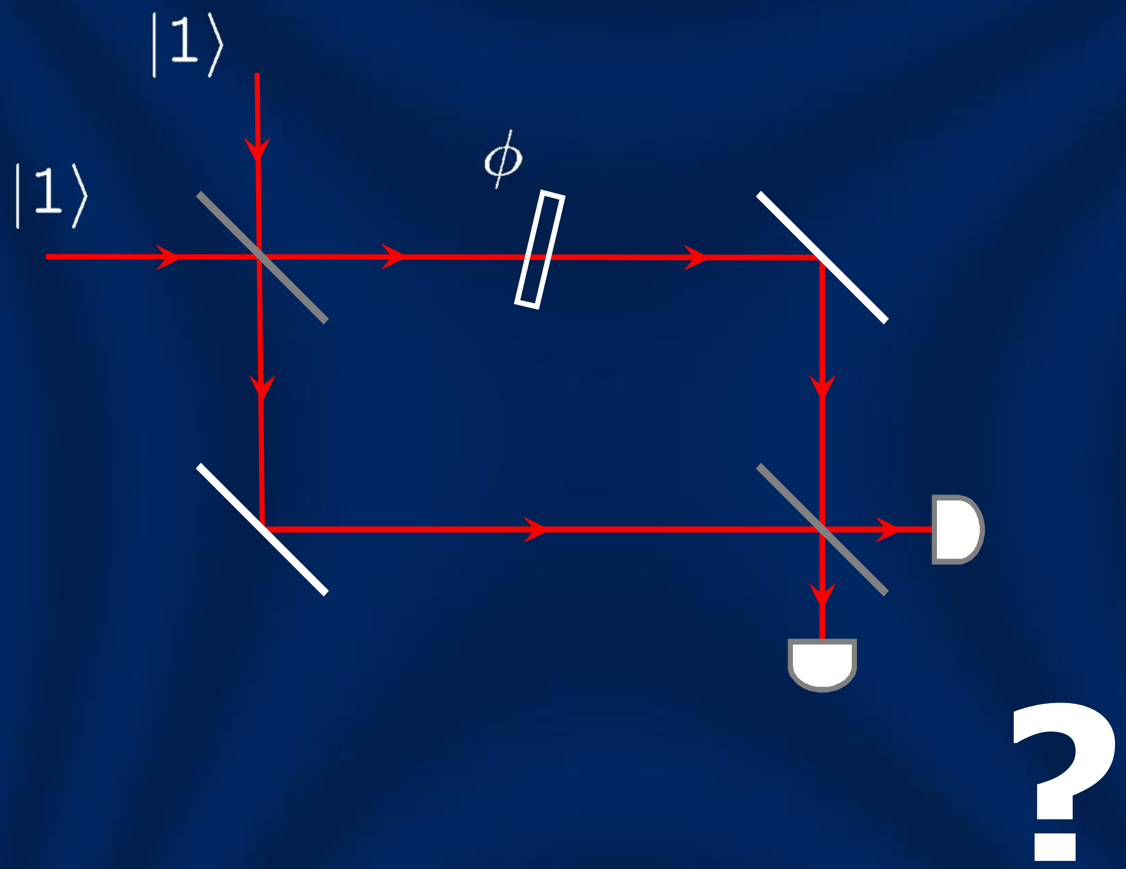
the Fisher information is additive:

$$\mathsf{F}(\phi) = \mathsf{F}_1(\phi) + \mathsf{F}_2(\phi)$$

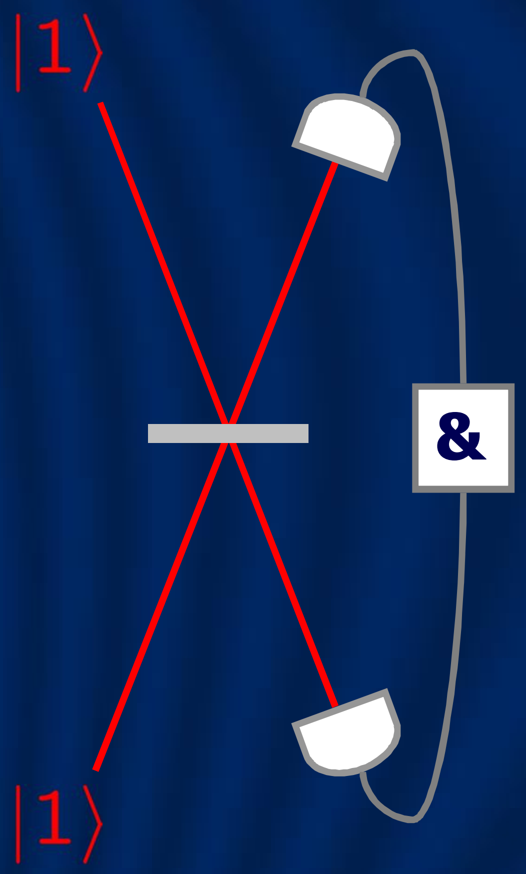
For one photon sent into the Mach-Zehnder interferometer $\mathsf{F}(\phi) = 1$. Using N photons yields $\mathsf{F}(\phi) = N$ and the precision is bounded by the *shot noise limit:*

$$\Delta\tilde{\phi} \geq \frac{1}{\sqrt{N}}$$

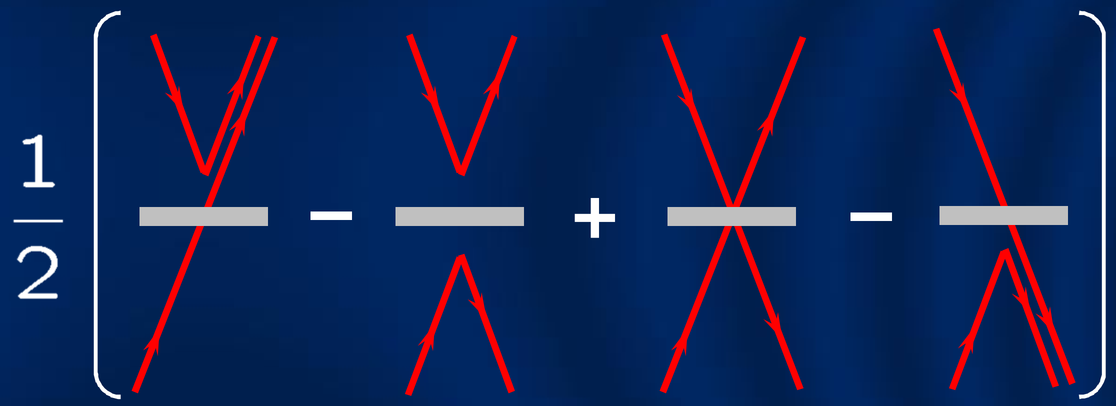
Two-photon interferometry



Two-photon interference



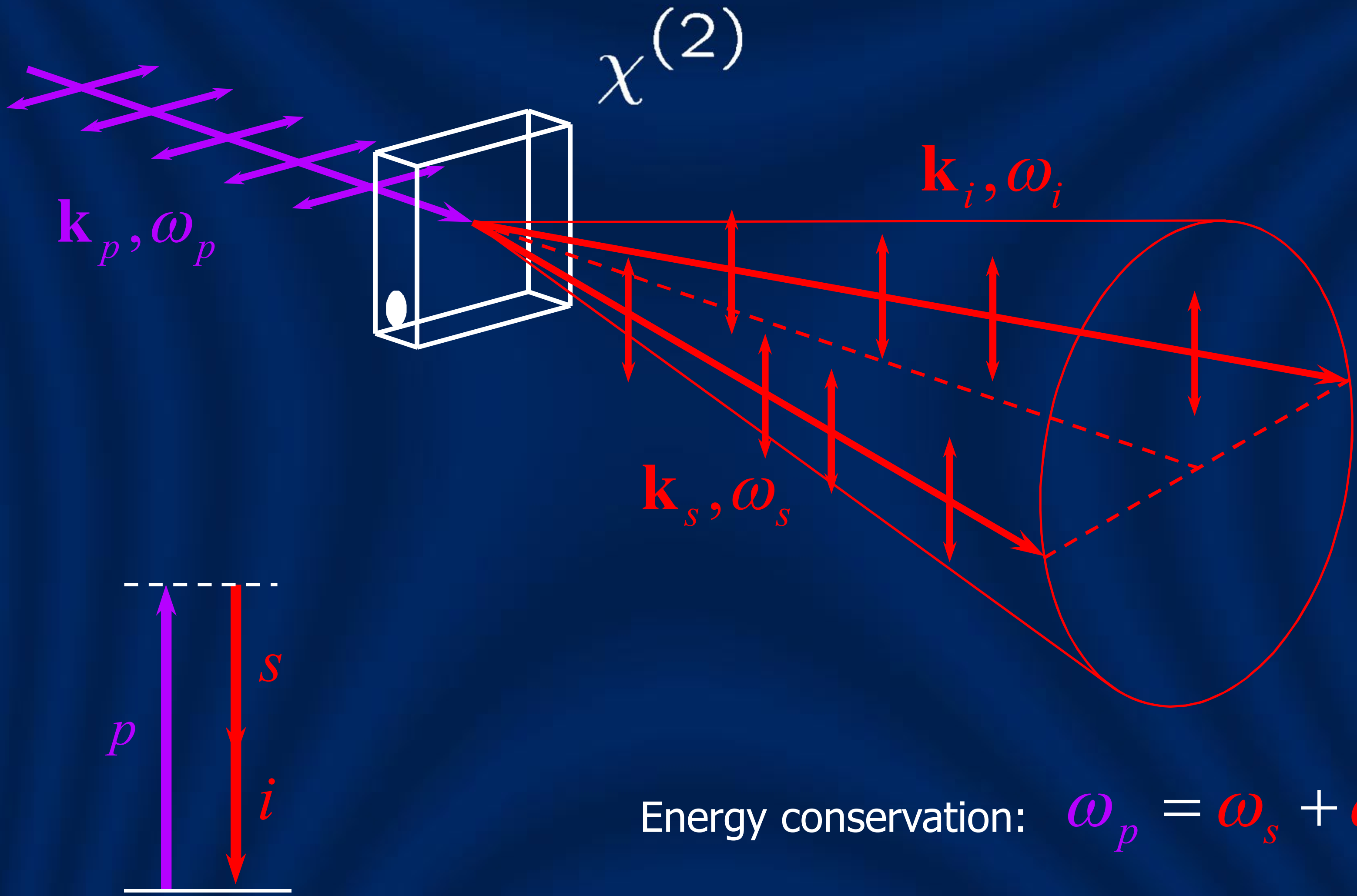
Probability amplitudes:



$$|11\rangle \rightarrow \frac{1}{\sqrt{2}}(|20\rangle - |02\rangle)$$

Only if photons are *indistinguishable!*

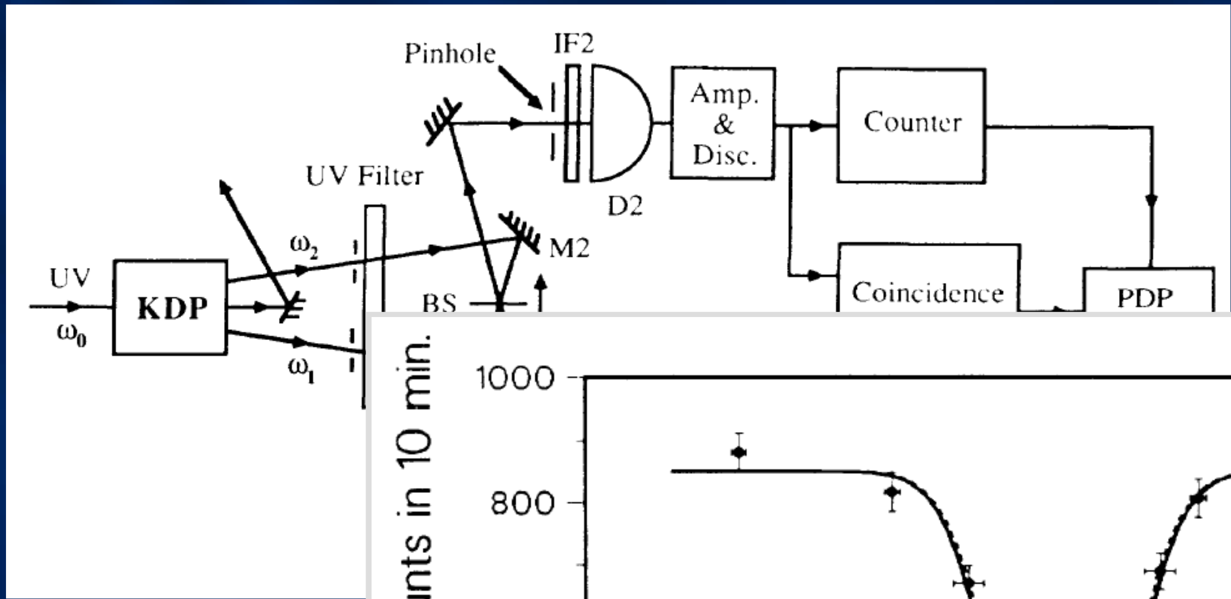
Parametric down-conversion



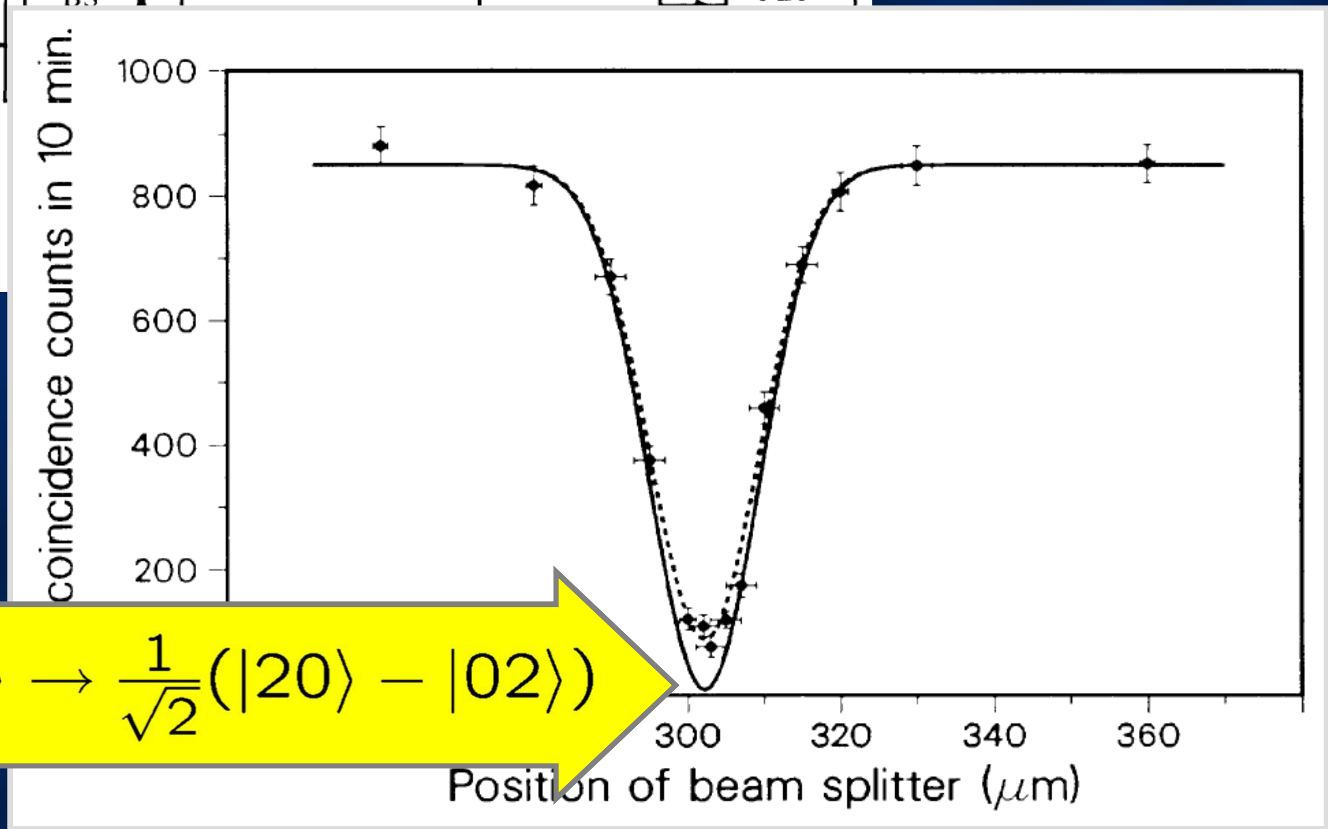
Energy conservation: $\omega_p = \omega_s + \omega_i$

Momentum conservation: $\mathbf{k}_p \approx \mathbf{k}_s + \mathbf{k}_i$

Hong-Ou-Mandel experiment

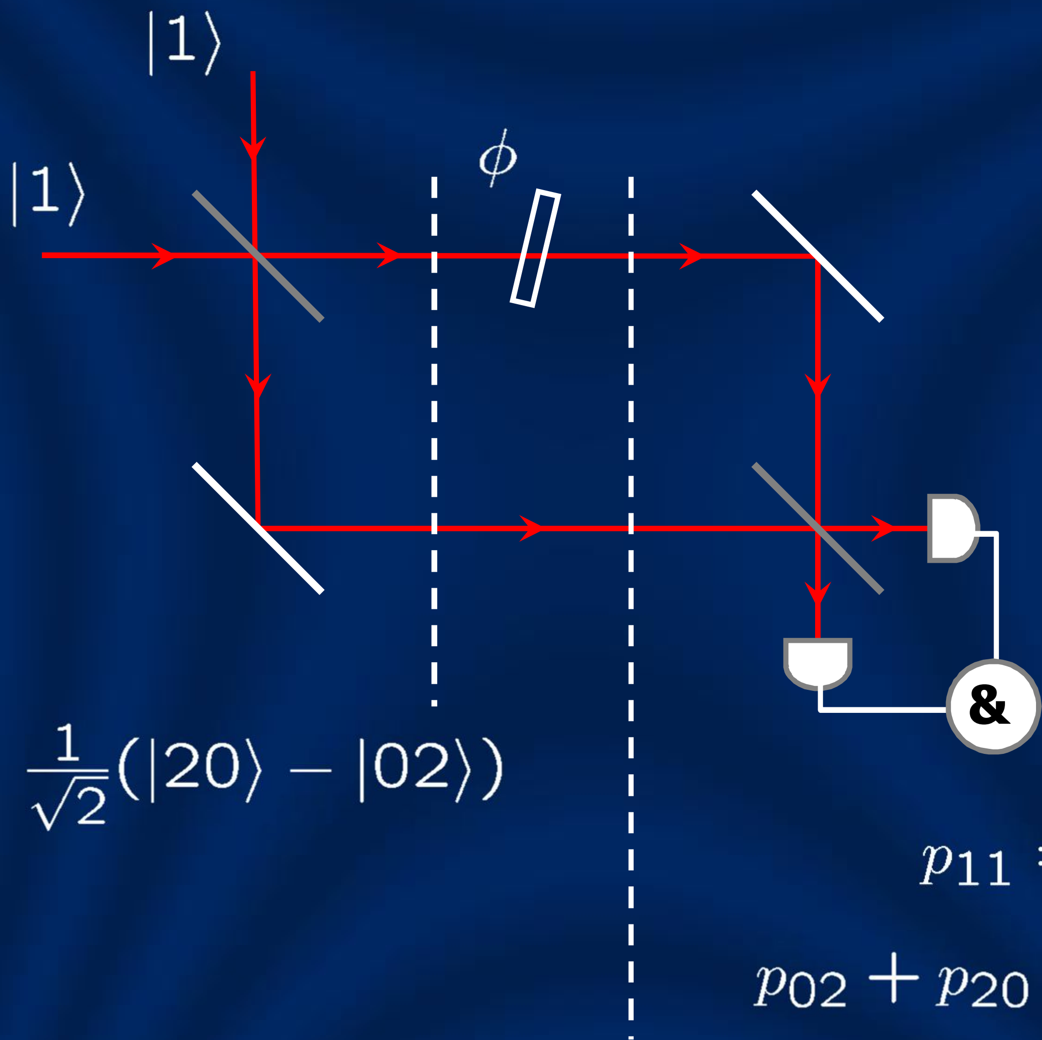


C. K. Hong, Z. Y. Ou, and L. Mandel, Phys. Rev. Lett. **59**, 2044 (1987)



$$|11\rangle \rightarrow \frac{1}{\sqrt{2}}(|20\rangle - |02\rangle)$$

Two-photon phase shift



$$\frac{1}{\sqrt{2}}(|20\rangle - |02\rangle)$$

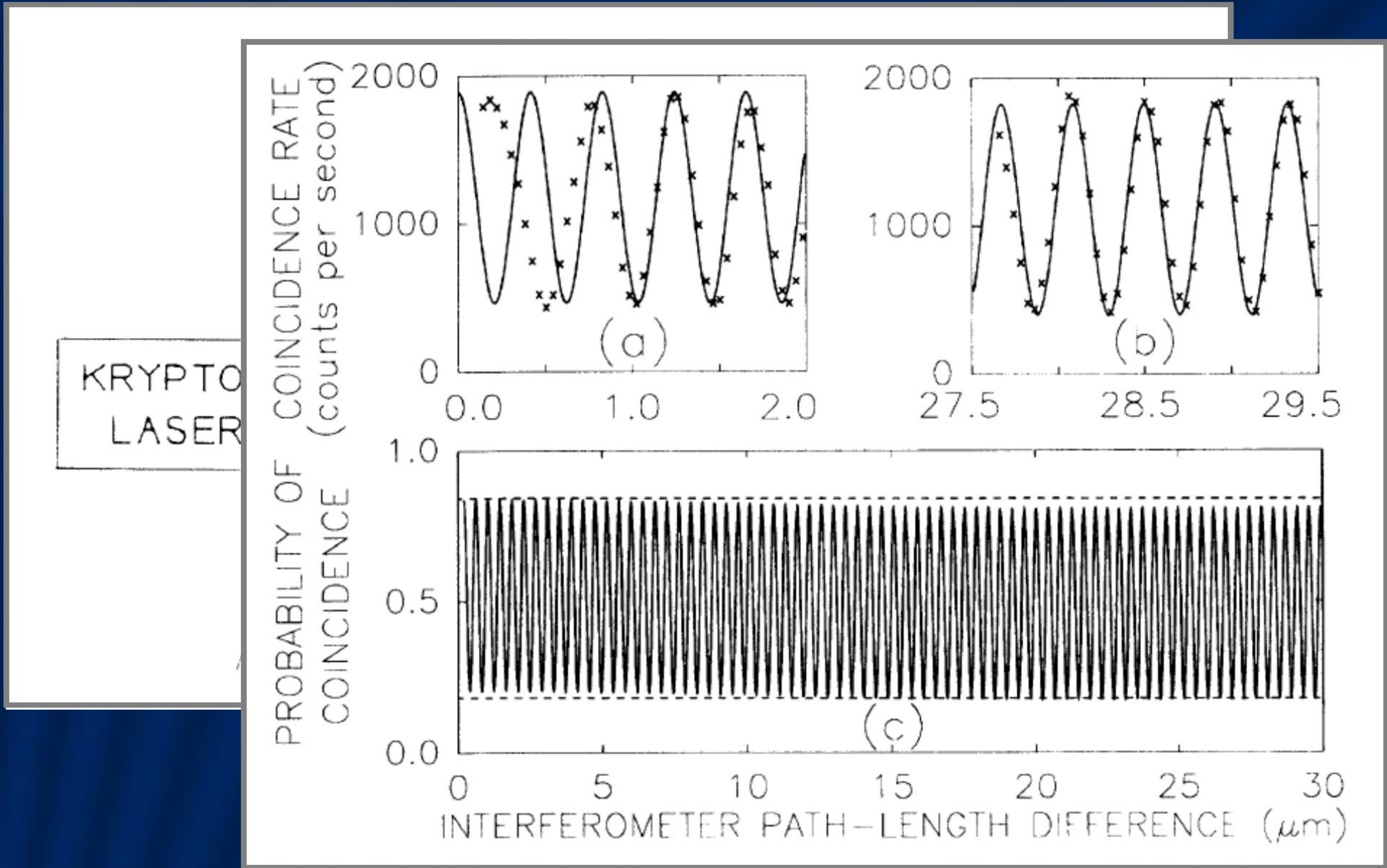
$$p_{11} = \tfrac{1}{2}(1 + \cos 2\phi)$$

$$p_{02} + p_{20} = \tfrac{1}{2}(1 - \cos 2\phi)$$

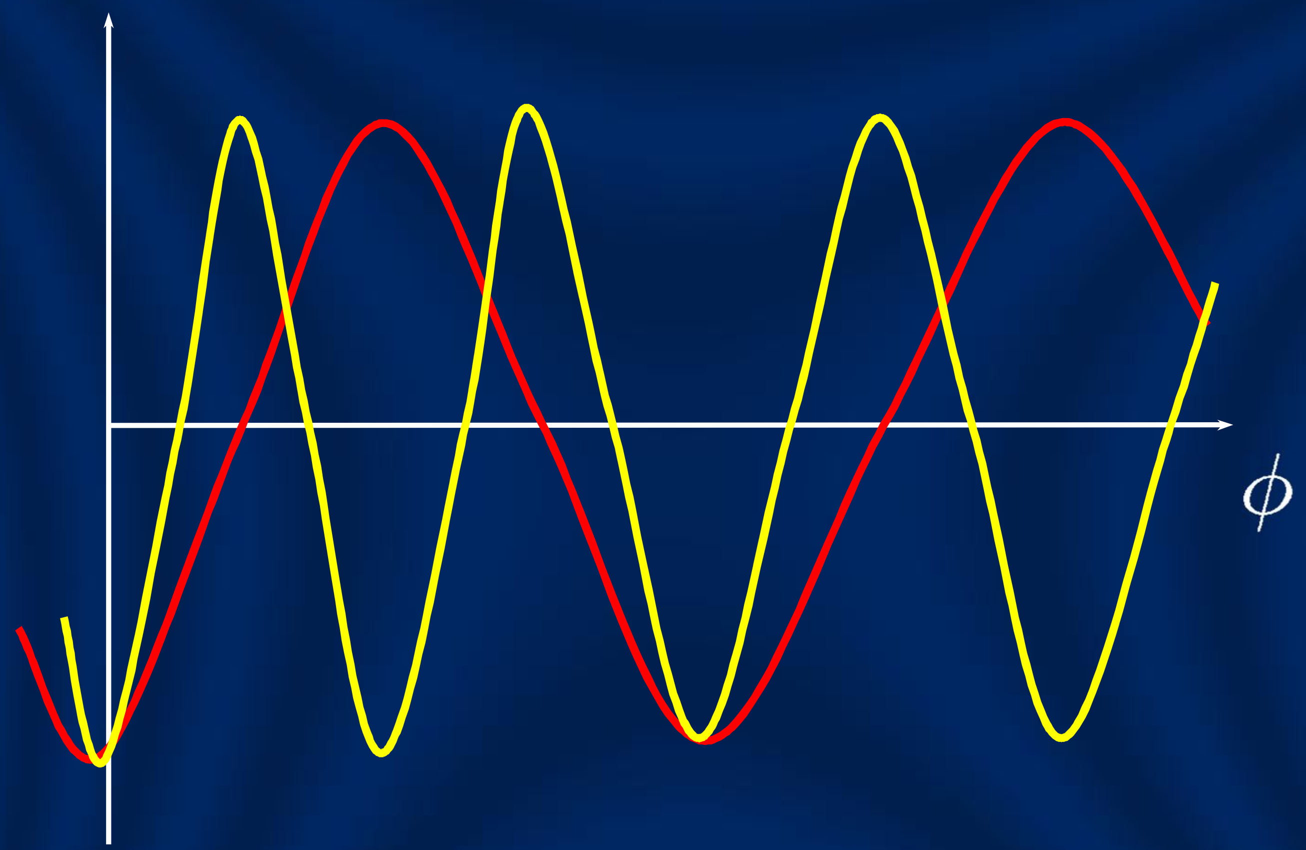
$$\frac{1}{\sqrt{2}}(\mathrm{e}^{2\mathrm{i}\phi}|20\rangle - |02\rangle)$$

Observation

J. G. Rarity *et al.,* Phys. Rev. Lett. **65**, 1348 (1990)



Fringe spacing



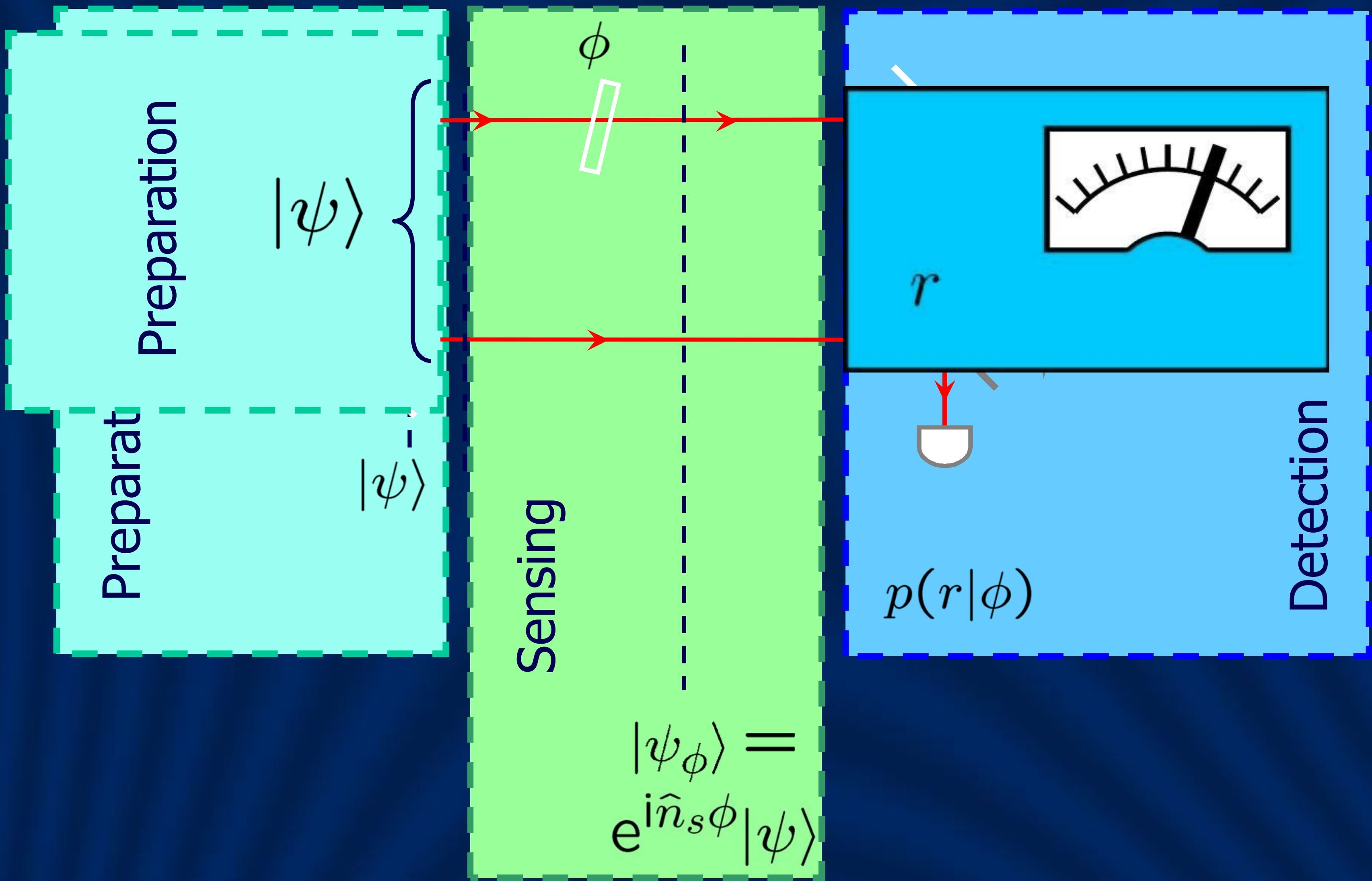
$$|\psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|20\rangle - |02\rangle)$$

For two copies $F = 1$ $F = 2$

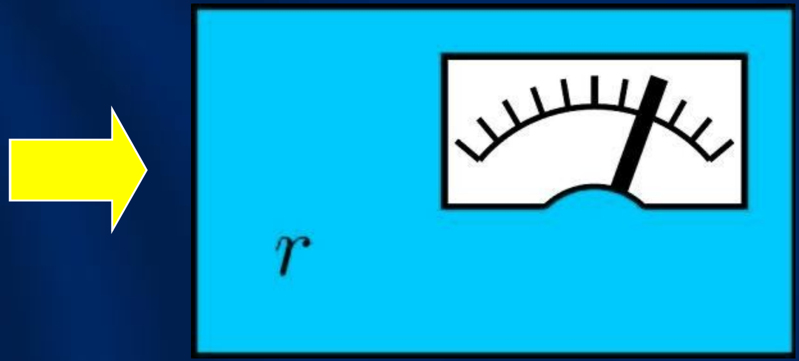
$F = 4$

General picture



Quantum measurement

$|\psi_\phi\rangle$



$\hat{M}_r \geq 0$

$\sum_r \hat{M}_r = \hat{\mathbb{1}}$

Result probability:

$$p(r|\phi) = \langle\psi_\phi|\hat{M}_r|\psi_\phi\rangle$$

Quantum Fisher information

Review: M. G. A. Paris, Int. J. Quant. Inf. **7**, 125 (2009)

For *any* measurement $\{\widehat{M}_r\}$

$$\mathsf{F}(\phi) \leq \mathsf{F}_Q(\phi) := 4\Big(\langle\partial_\phi\psi|\partial_\phi\psi\rangle - |\langle\psi_\phi|\partial_\phi\psi\rangle|^2\Big)$$

where

$$|\partial_\phi\psi\rangle = \frac{\partial}{\partial\phi}|\psi_\phi\rangle$$

Quantum Fisher information characterizes "local distinguishability" between states

$|\psi_\phi\rangle$ and $|\psi_{\phi+\delta\phi}\rangle \approx |\psi_\phi\rangle + \delta\phi|\partial_\phi\psi\rangle$

Symmetric logarthmic derivative

Implicit definition

$$\frac{\partial}{\partial \phi}\hat{\varrho}_\phi = \frac{1}{2}(\hat{L}_\phi \hat{\varrho}_\phi + \hat{\varrho}_\phi \hat{L}_\phi)$$

Explicit expression for $\hat{\varrho}_\phi = |\psi_\phi\rangle\langle\psi_\phi|$:

$$\hat{L}_\phi = 2\Big[\Big(\hat{I} - |\psi_\phi\rangle\langle\psi_\phi|\Big)|\partial_\phi\psi\rangle\langle\psi_\phi| \\ +|\psi_\phi\rangle\langle\partial_\phi\psi|\Big(\hat{I} - |\psi_\phi\rangle\langle\psi_\phi|\Big)\Big]$$

Upper bound:

$$\frac{1}{p(r|\phi)}\left(\frac{\partial}{\partial\phi}p(r|\phi)\right)^2 \leq \left|\frac{\mathrm{Tr}[\hat{\varrho}_\phi \hat{M}_r \hat{L}_\phi]}{\sqrt{\mathrm{Tr}(\hat{M}_r \hat{\varrho}_\phi)}}\right|^2$$

Schwarz inequality

$$\left|\mathrm{Tr}(\hat{A}^\dagger \hat{B})\right|^2 \leq \mathrm{Tr}(\hat{A}^\dagger \hat{A})\mathrm{Tr}(\hat{B}^\dagger \hat{B})$$

Take:

$$\hat{A}_r^\dagger = \frac{\sqrt{\hat{\varrho}_\phi}\sqrt{\hat{M}_r}}{\sqrt{\mathrm{Tr}(\hat{M}_r \hat{\varrho}_\phi)}}, \qquad \hat{B}_r = \sqrt{\hat{M}_r}\hat{L}_\phi\sqrt{\hat{\varrho}_\phi}$$

Then:

$$\begin{aligned} \mathrm{F}(\phi) &\leq \sum_r |\mathrm{Tr}(\hat{A}_r^\dagger \hat{B}_r)|^2 \\ &\leq \mathrm{Tr}\left(\sum_r \hat{A}_r^\dagger \hat{A}_r\right) \mathrm{Tr}\left(\sum_r \hat{B}_r^\dagger \hat{B}_r\right) \\ &= \mathrm{Tr}(\hat{\varrho}_\phi \hat{L}_\phi^2) = \mathrm{F}_Q(\phi) \end{aligned}$$

Phase measurement

Transformation of the input state by a phase shifter:

$$|\psi_\phi\rangle = \mathrm{e}^{\mathrm{i}\widehat{n}_s\phi}|\psi\rangle$$

where $\widehat{n}_s$ is operator of the number of photons sent through the phase shifter.

Quantum Fisher information

$$\mathrm{F}_Q(\phi) = 4(\Delta n_s)^2$$

is proportional to the *variance* of the photon number in the sensing arm!

Interferometric Cramér-Rao bound

"Heisenberg" uncertainty relation:

$$\Delta\phi\Delta n_s \geq \frac{1}{2}$$

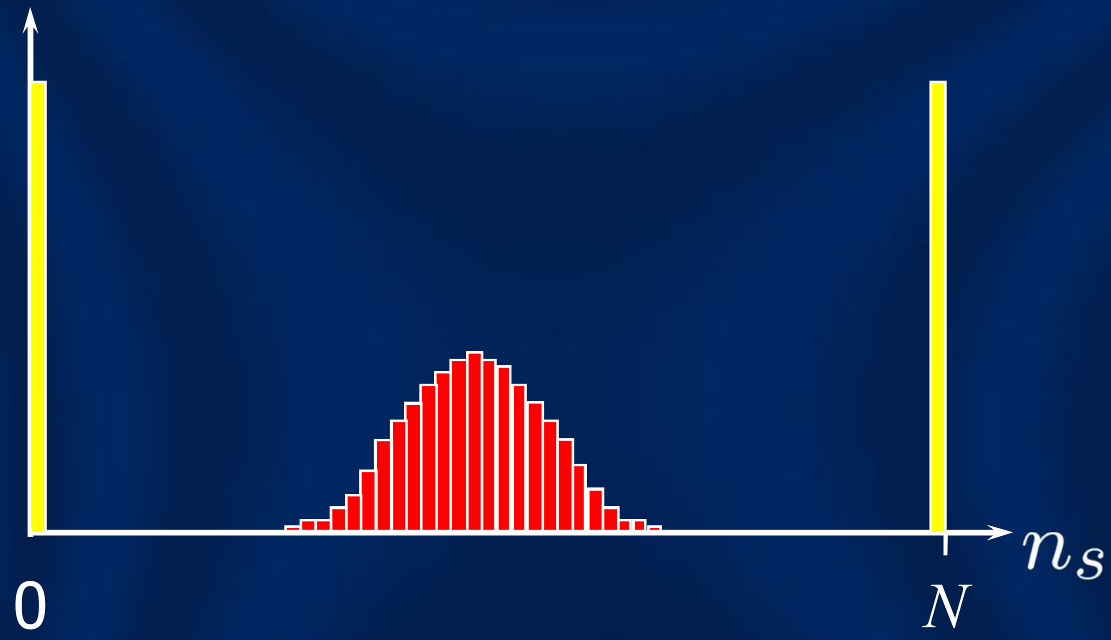
Δn_s – photon number uncertainty in the sensing arm

$\Delta\phi$ – precision of phase estimation

Task: maximize Δn_s for a fixed total number of photons N.

Optimal precision

For the total number of N photons:



N photons sent to a 50/50 beam splitter yield the shot-noise limit:

$$\Delta\phi = \frac{1}{\sqrt{N}}$$

Maximum possible Δn_s defines the *Heisenberg limit*:

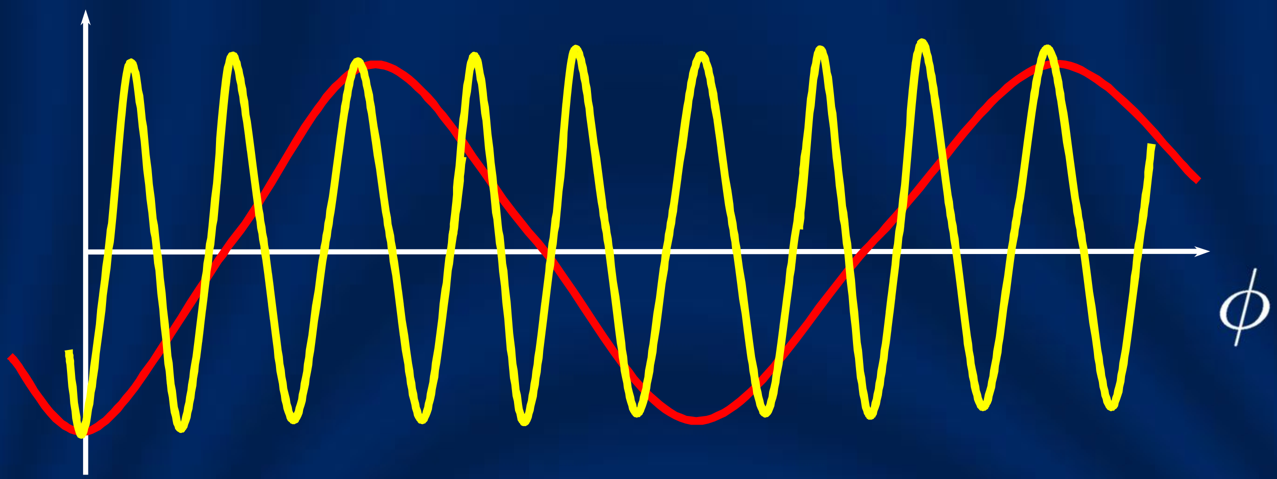
$$\Delta\phi = \frac{1}{N}$$

NOON state

The optimal N photon state:

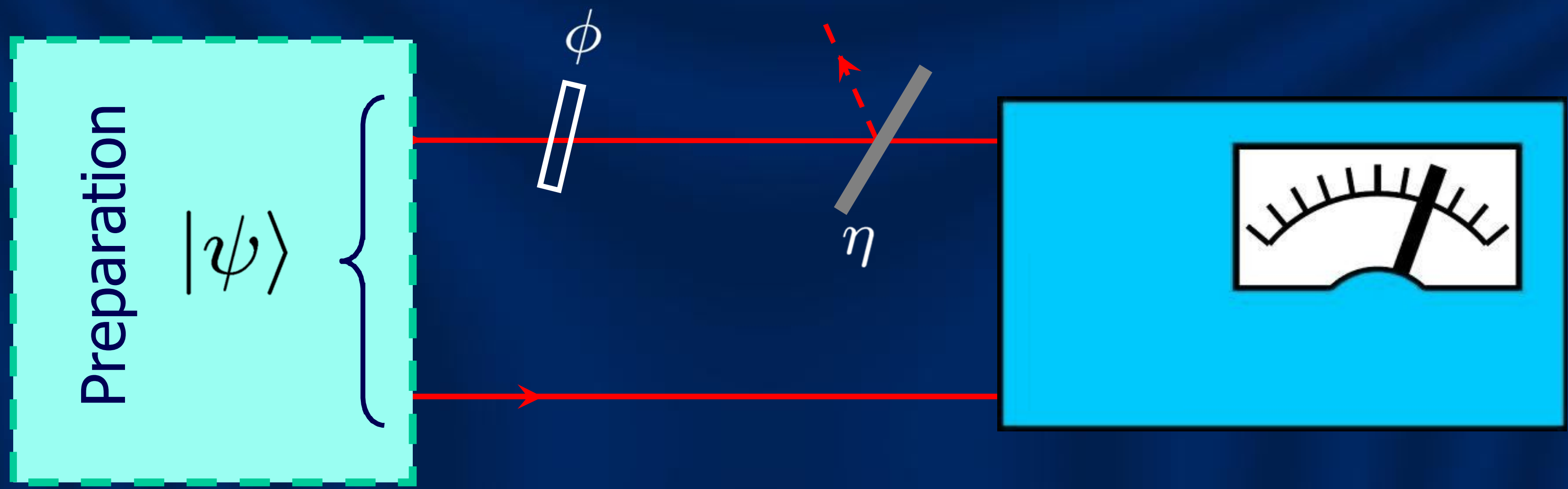
$$|\Psi_N\rangle = \frac{1}{\sqrt{2}}(|N0\rangle - |0N\rangle)$$

$$\longrightarrow \frac{1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}N\phi}|N0\rangle - |0N\rangle)$$



For a review: V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004)

Losses



If a photon is lost: $|\Psi_N\rangle \rightarrow \mathrm{e}^{\mathrm{i}N\phi}|N-1,0\rangle$

When no photons are lost:

$$|\Psi_N\rangle \rightarrow \frac{1}{\sqrt{2}}(\sqrt{\eta}^N \mathrm{e}^{\mathrm{i}N\phi}|N0\rangle - |0N\rangle)$$

M.A. Rubin and S. Kaushik, Phys. Rev. A **75**, 053805 (2007)
G. Gilbert, M. Hamrick, Y.S. Weinstein, J. Opt. Soc. Am. B **25**, 1336 (2008)

Two-photon case

$$|\psi\rangle = \alpha|20\rangle + \beta|11\rangle + \gamma|02\rangle$$

No photon lost:

$$|\psi_0\rangle = \eta\alpha|20\rangle + \sqrt{\eta}\beta|11\rangle + \gamma|02\rangle$$

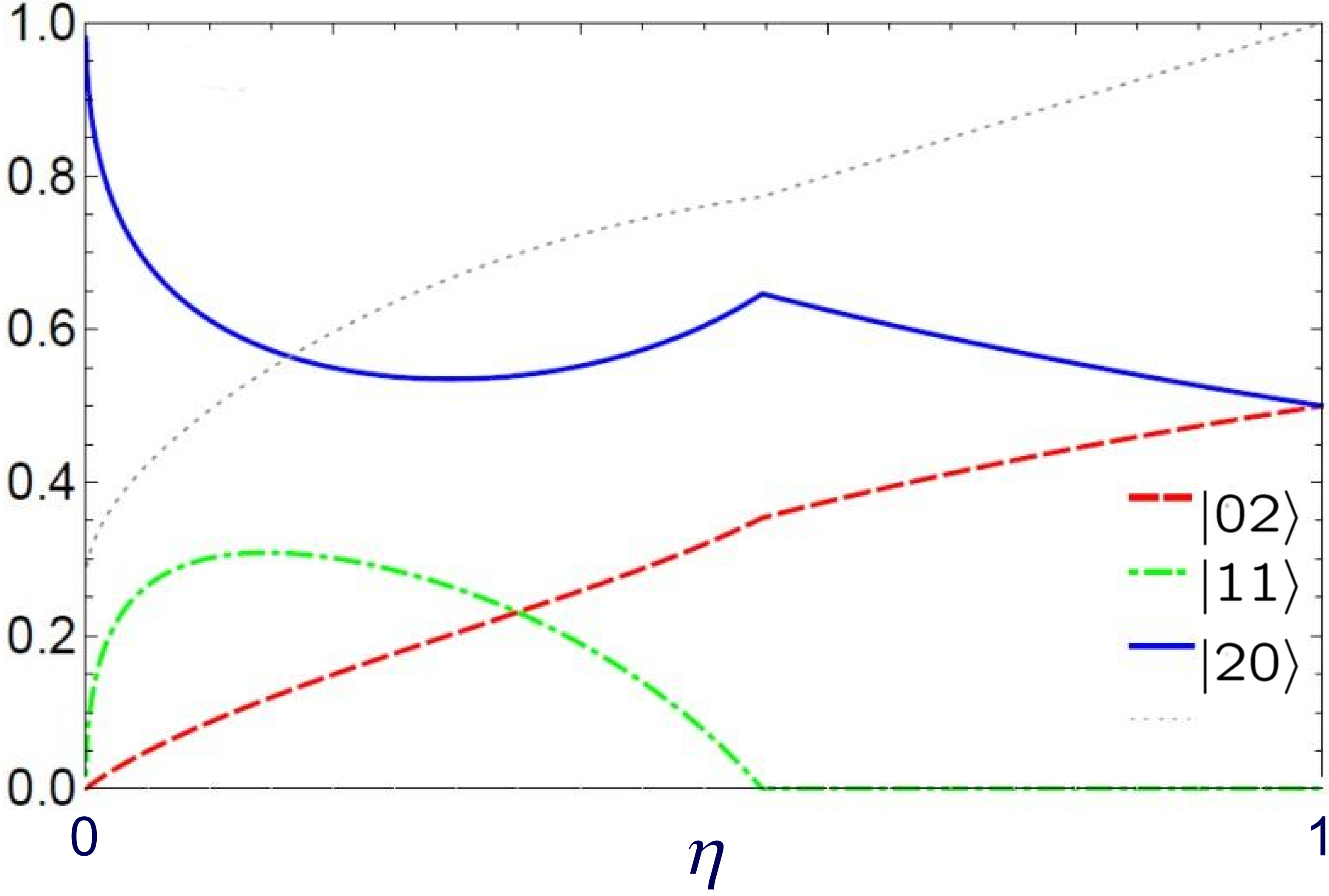
One photon lost:

$$|\psi_1\rangle = \sqrt{2\eta(1-\eta)}\alpha|10\rangle + \sqrt{1-\eta}\beta|01\rangle$$

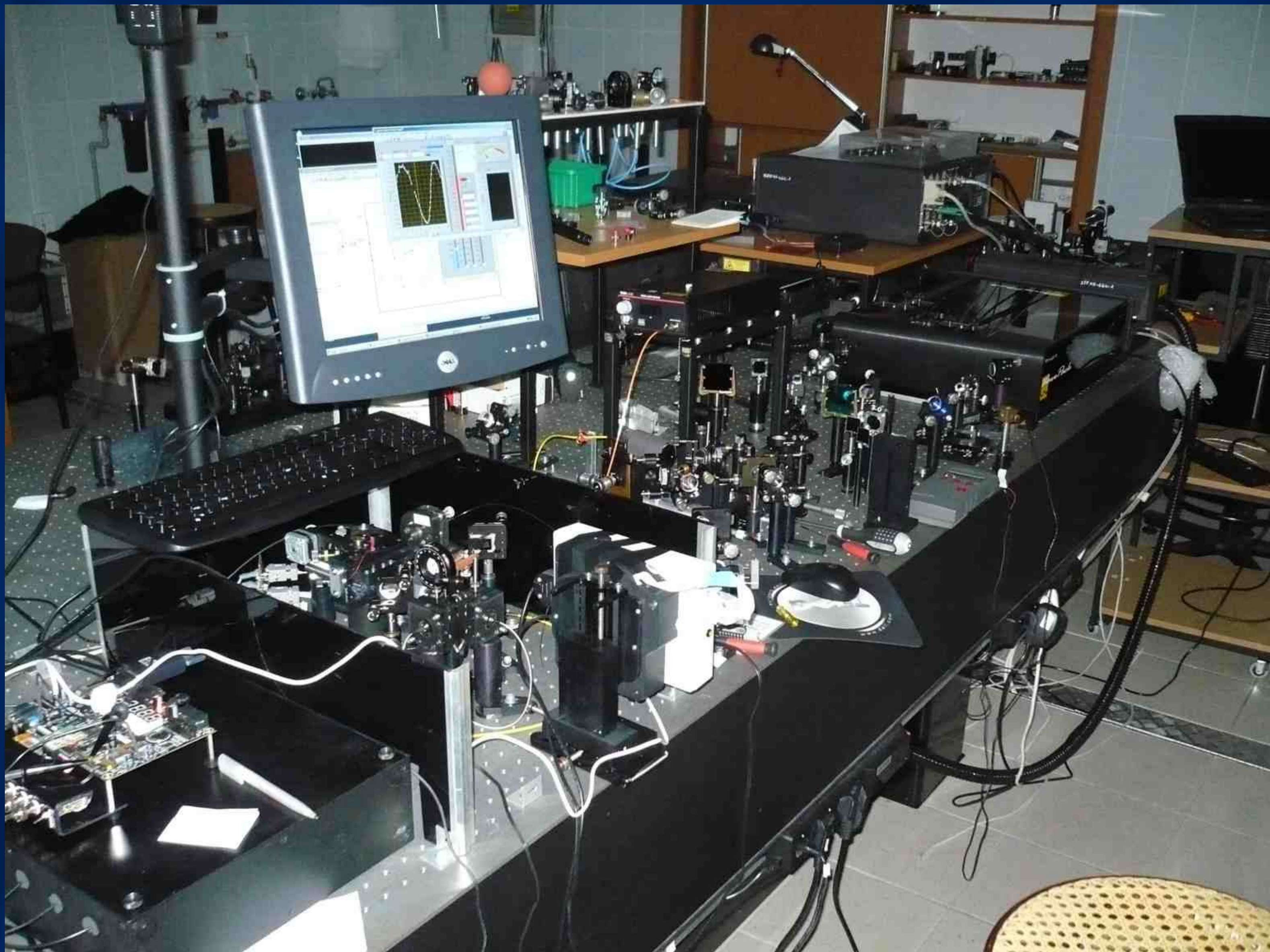
Two photons lost:

$$|\psi_2\rangle = (1-\eta)\alpha|00\rangle$$

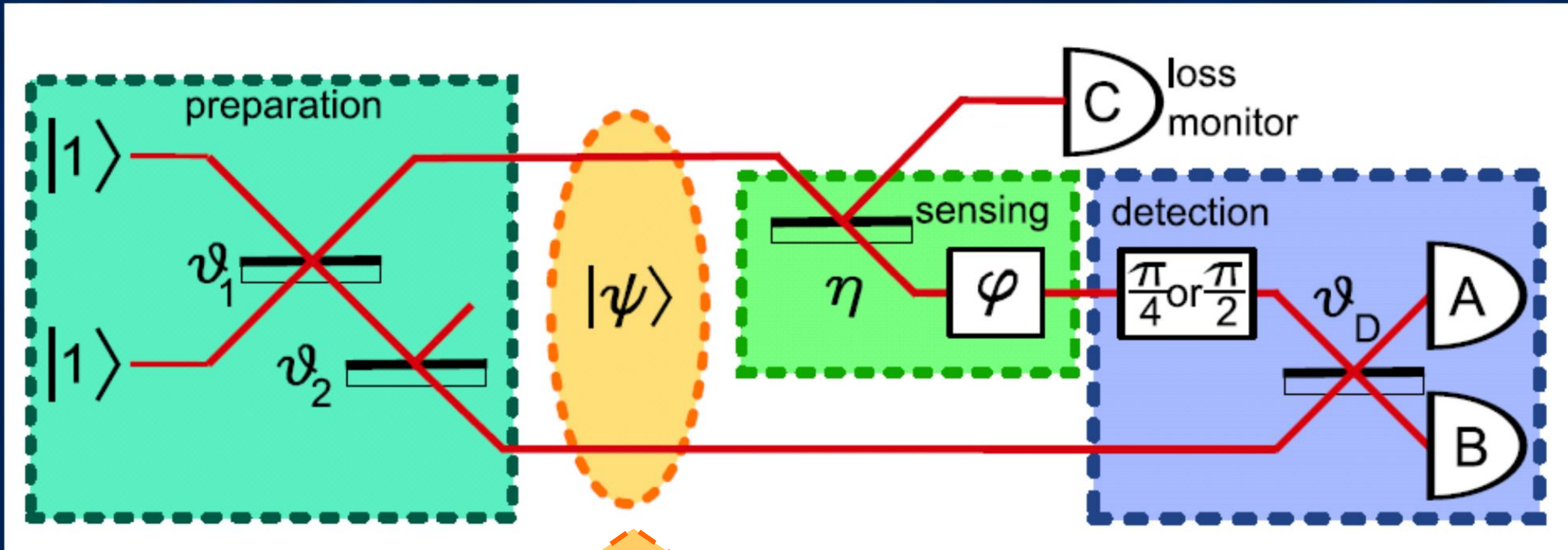
Weights



Lab

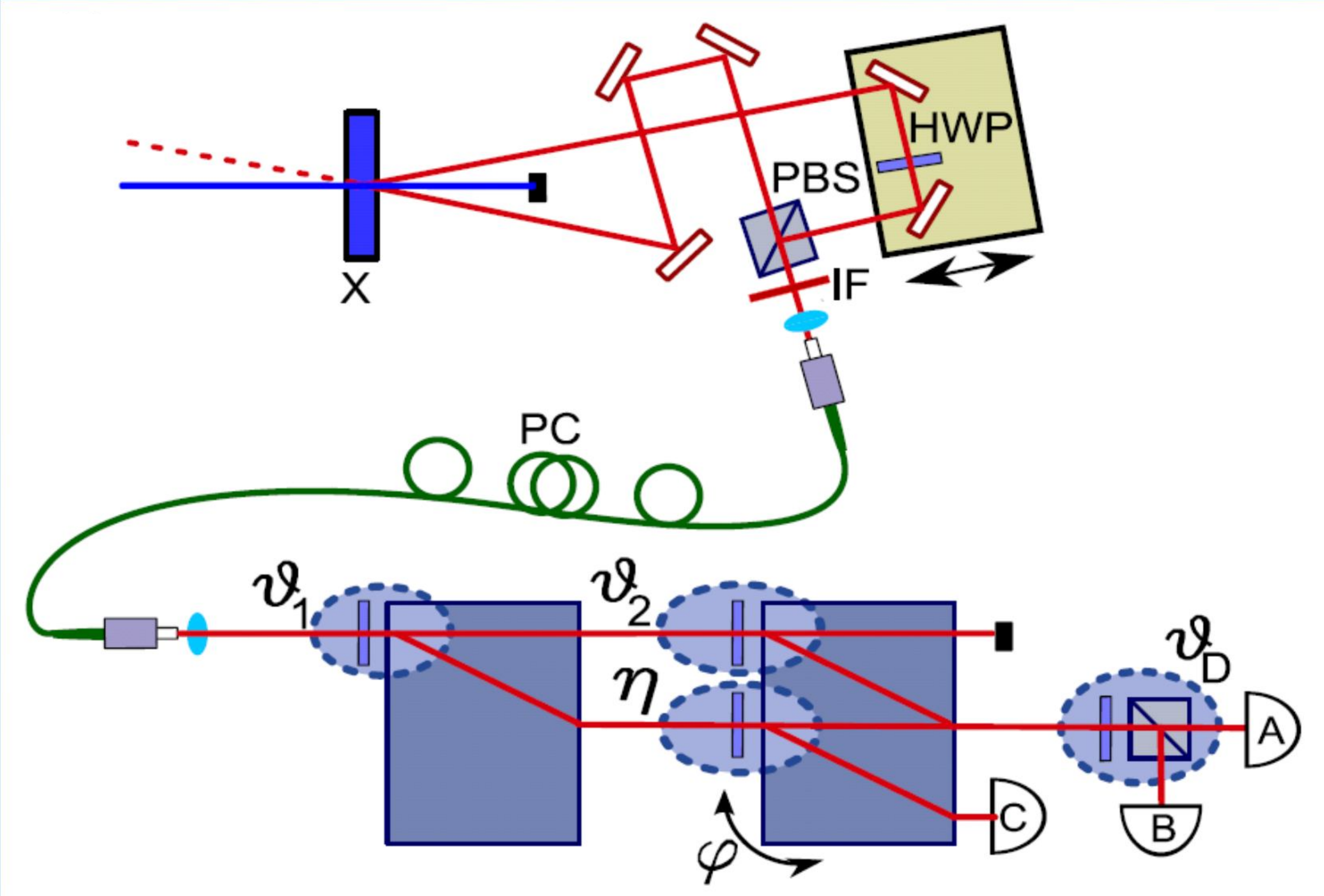


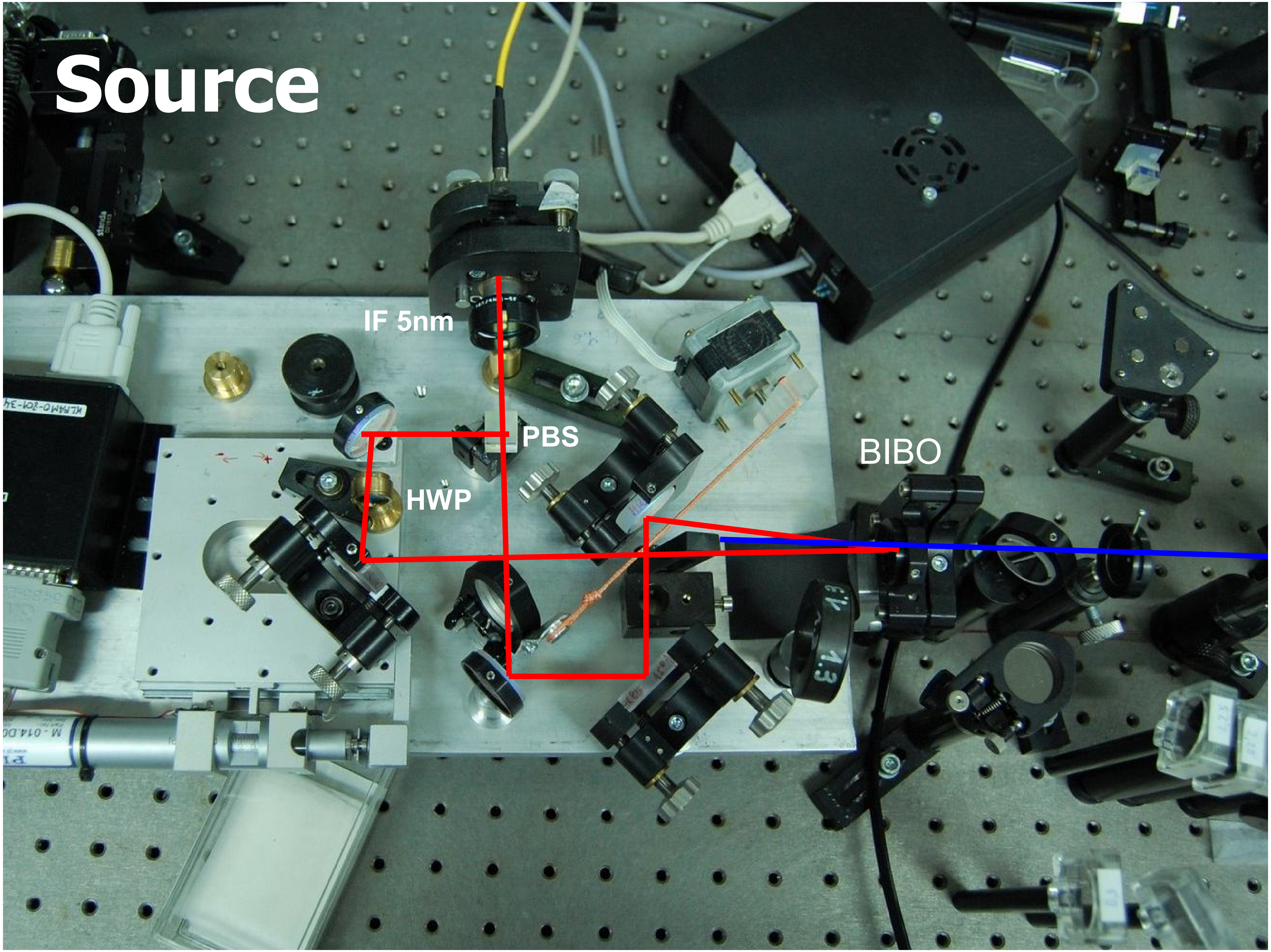
Schematic

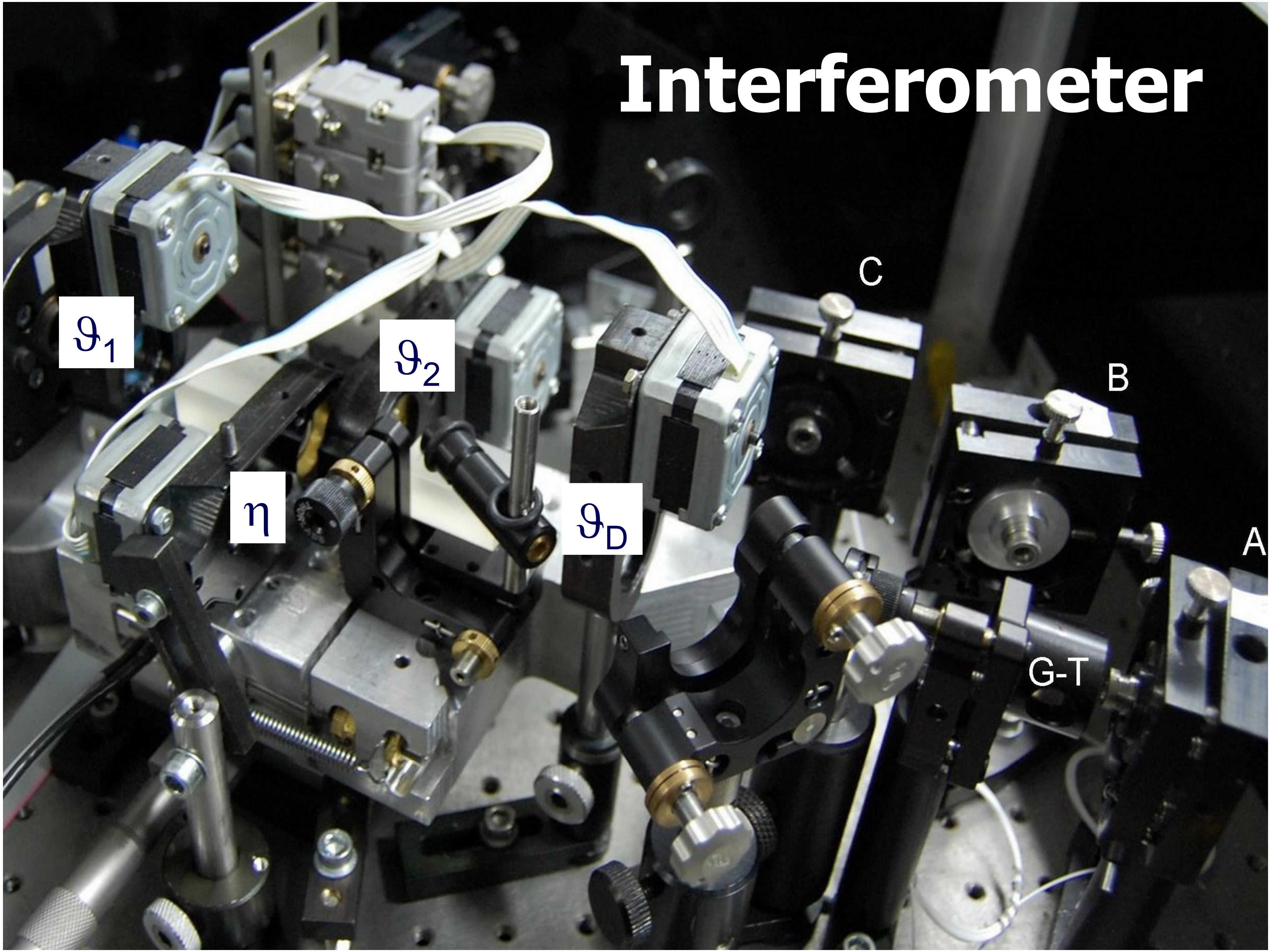


$$\sqrt{2\vartheta_1(1-\vartheta_1)}|20\rangle+\sqrt{\vartheta_2(2\vartheta_1-1)}|11\rangle-\vartheta_2\sqrt{2\vartheta_1(1-\vartheta_1)}|02\rangle$$

Realization




Source
IF 5nm
PBS
HWP
BIBO


Interferometer
C
B
A
ϑ_1
ϑ_2
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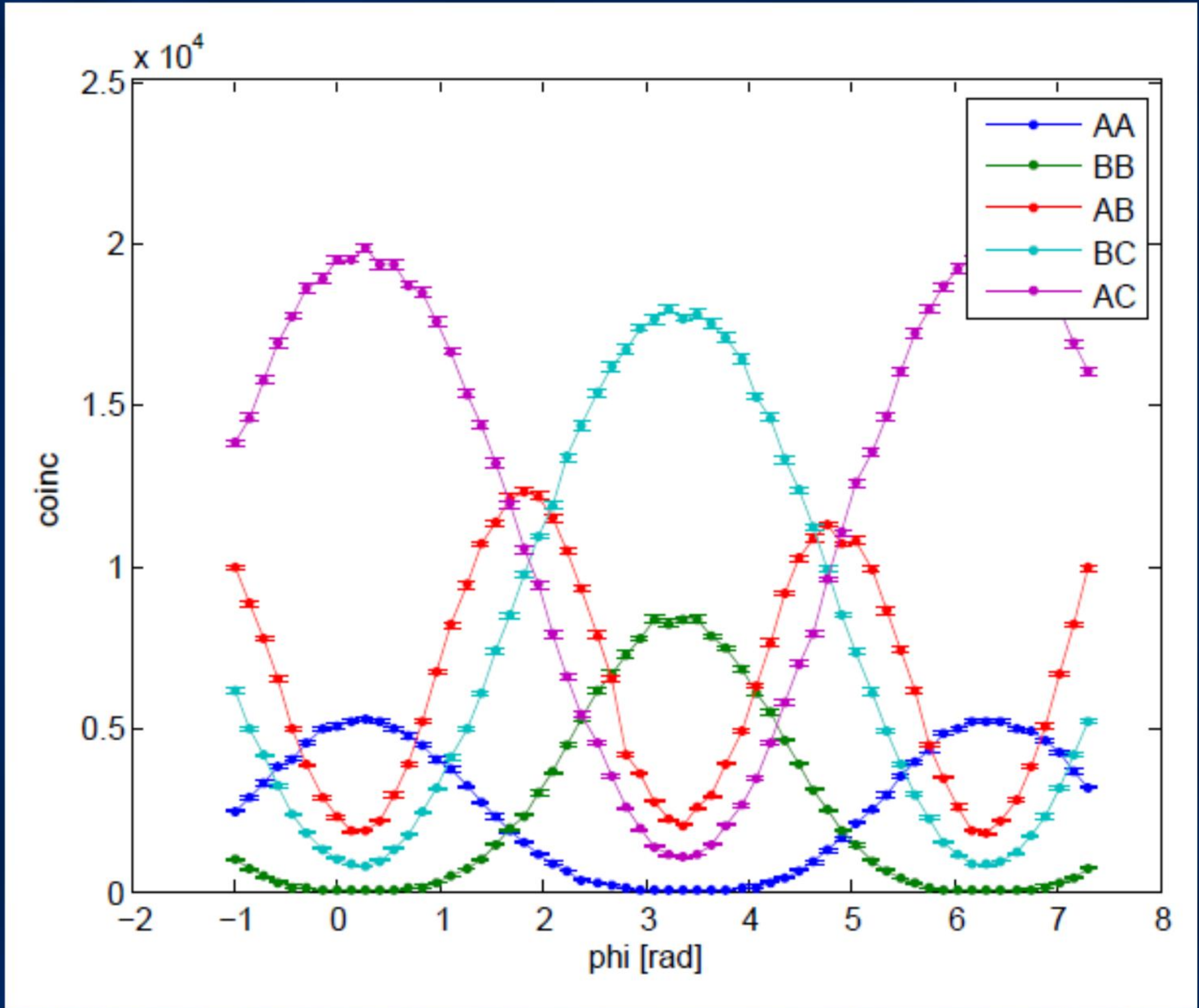
ϑ_D
G-T

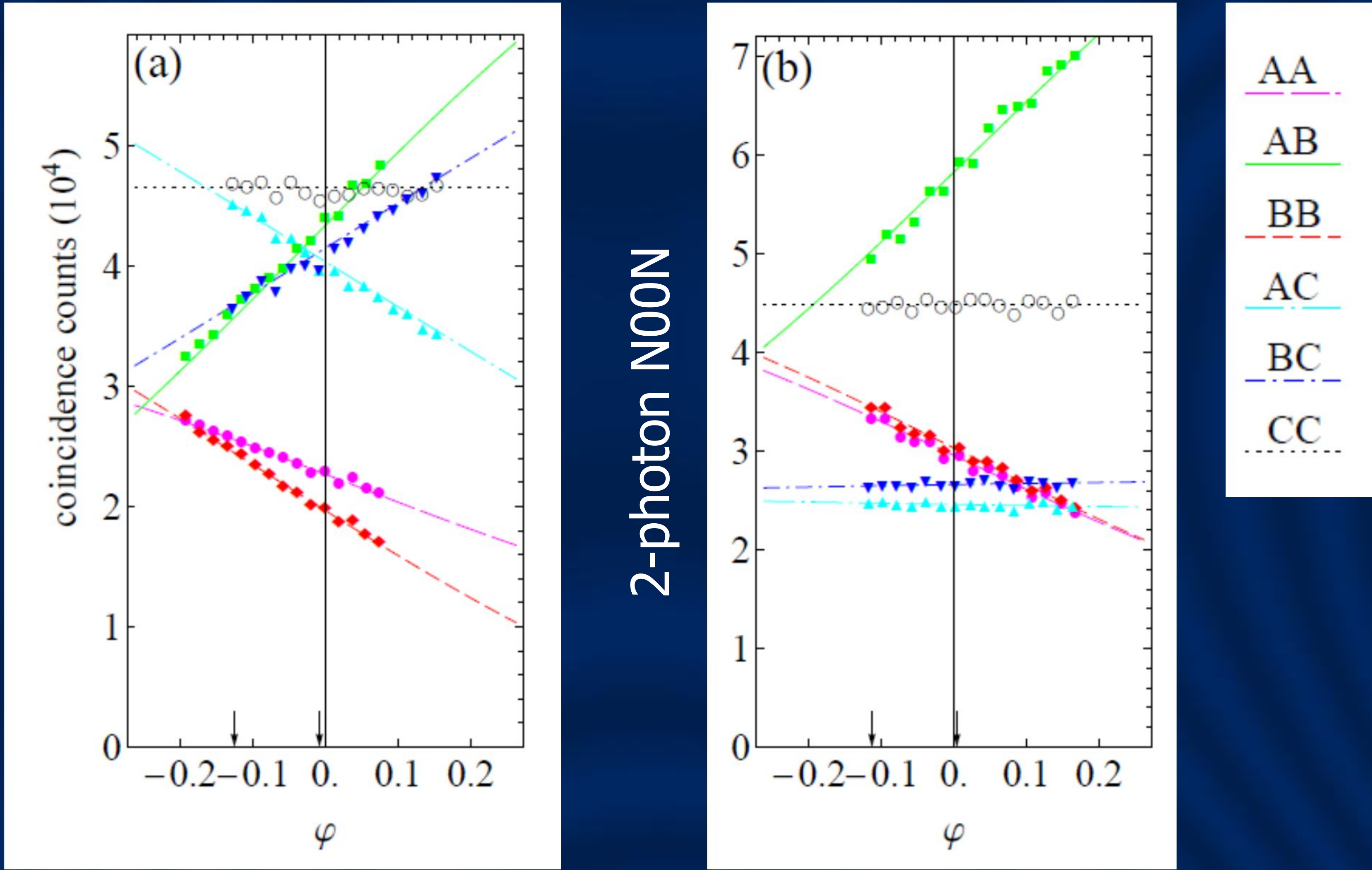
Fringes



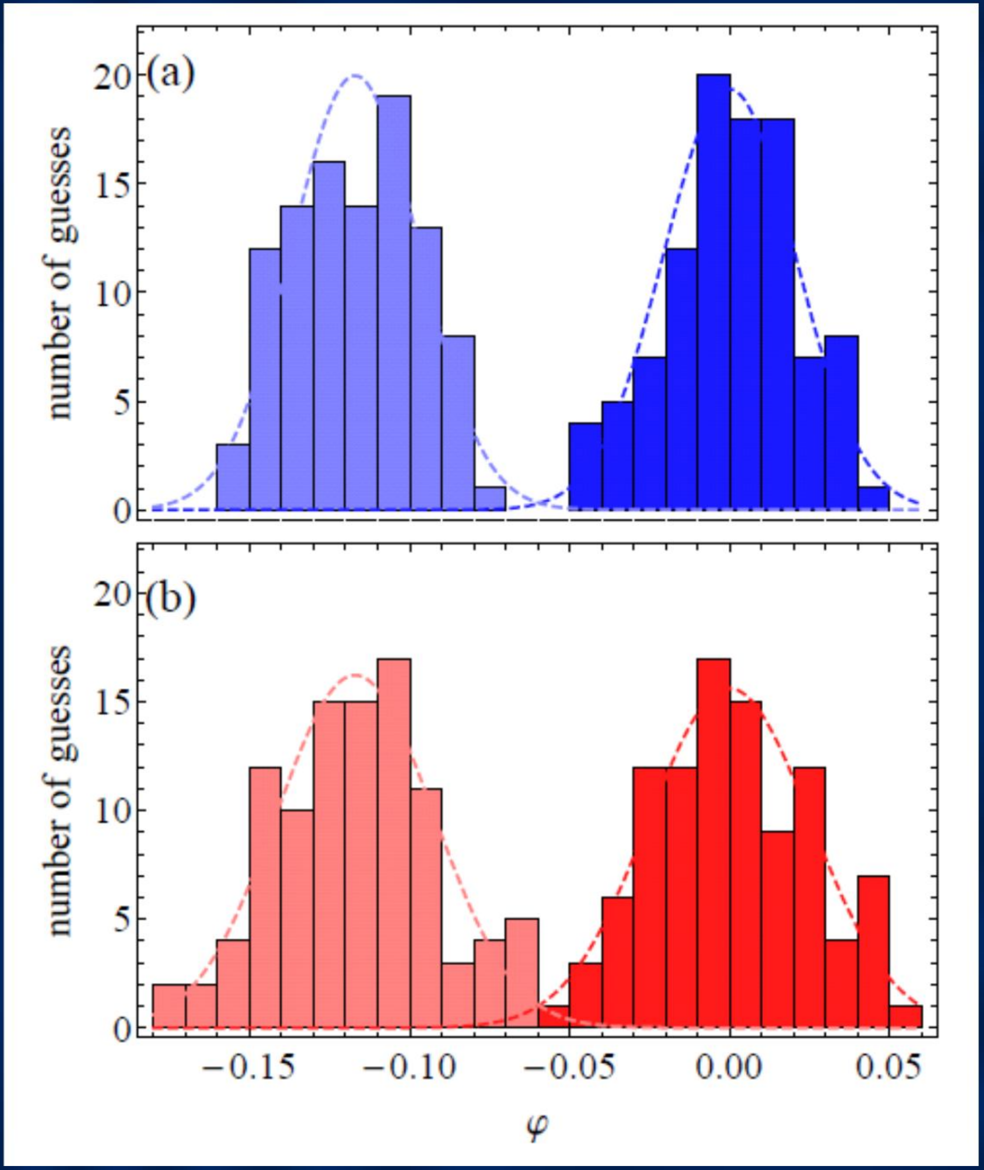
Measurement results

Optimal

2-photon N00N



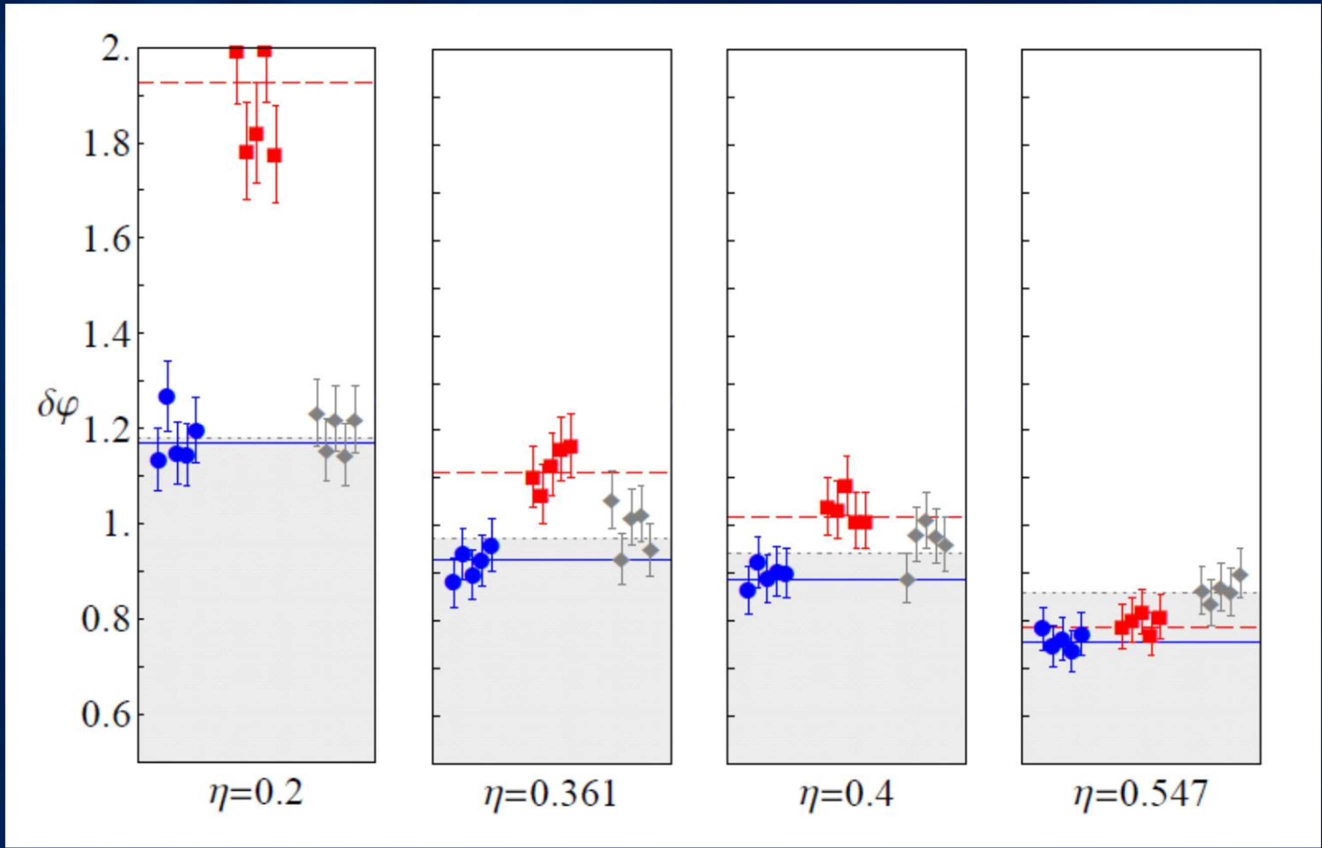
Reconstructed phase



Optimal

2-photon
NOON

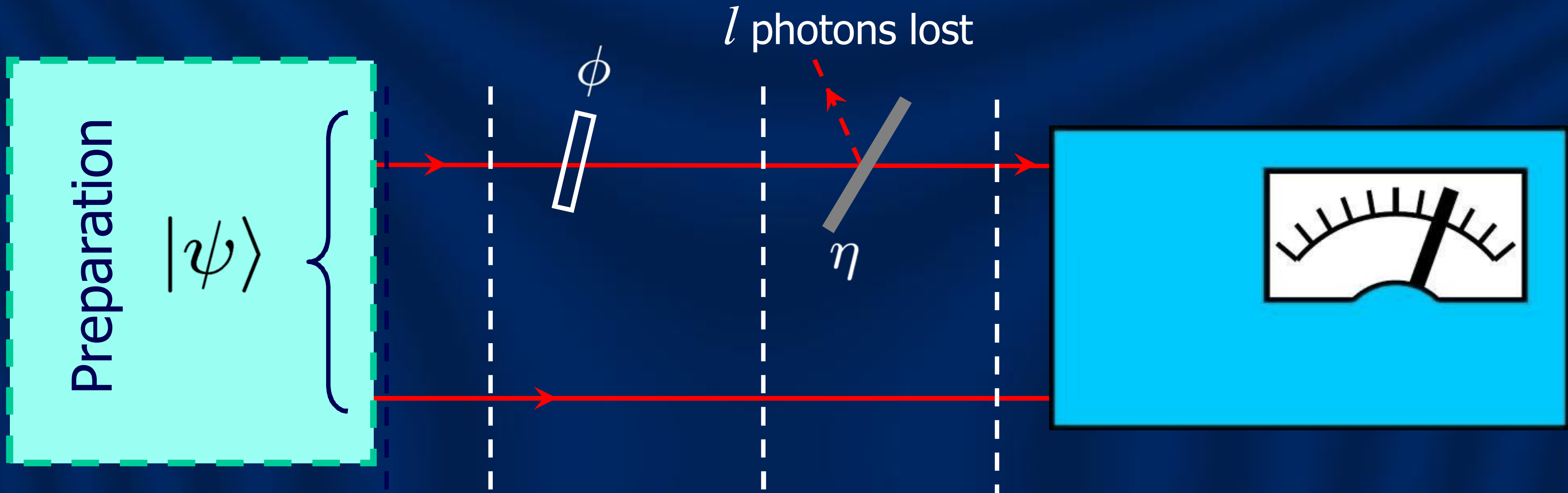
Precision



- Optimal
- 2-NOON
- Shot noise

M. Kacprowicz, R. Demkowicz-Dobrzański, W. Wasilewski, K. Banaszek, and I. A. Walmsley, Nature Photonics **4**, 357 (2010)

General approach: one-arm losses



$$|\psi\rangle = \sum_{k=1}^{N} \alpha_k |k, N-k\rangle$$

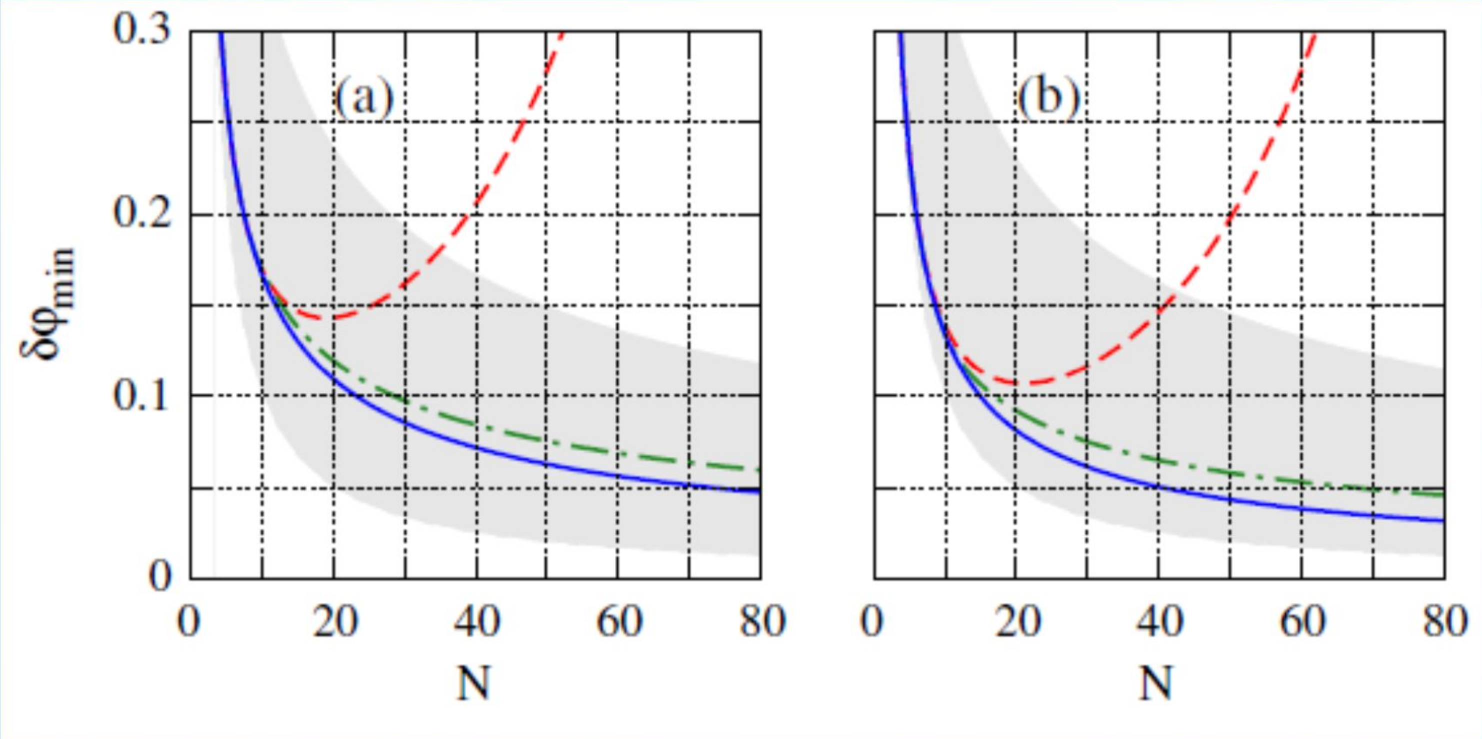
$$|\psi(\phi)\rangle = \sum_{k=1}^{N} \mathrm{e}^{\mathrm{i}k\phi} \alpha_k |k, N-k\rangle$$

$$|\psi_l(\phi)\rangle = \sum_{k=1}^{N} \mathrm{e}^{\mathrm{i}k\phi} \alpha_k \sqrt{B_l^k(\eta)} |k-l, N-k\rangle$$

Precision

$\eta = 90\%$

One-arm losses Two-arm losses



Optimal

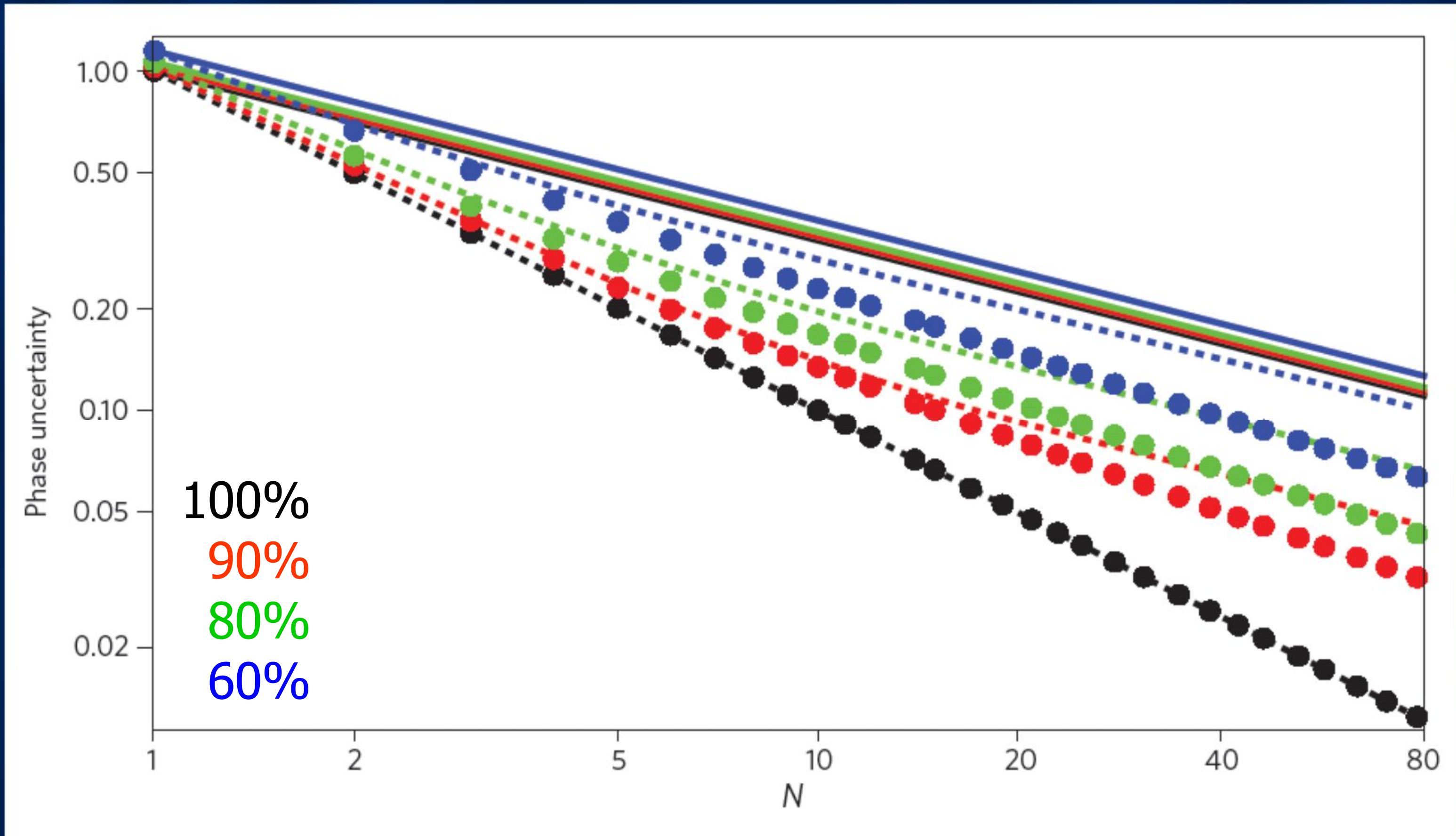
Chopped n00n

N00N state

U. Dorner, R. Demkowicz-Dobrzański *et al.*, Phys. Rev. Lett. **102**, 040403 (2009)

R. Demkowicz-Dobrzański, U. Dorner *et al.*, Phys. Rev. A **80**, 013825 (2009)

Scaling



shot noise

quantum

multipass

K. Banaszek, R. Demkowicz-Dobrzański, and I. A. Walmsley,
Quantum states made to measure, Nature Photonics **3**, 673 (2009)

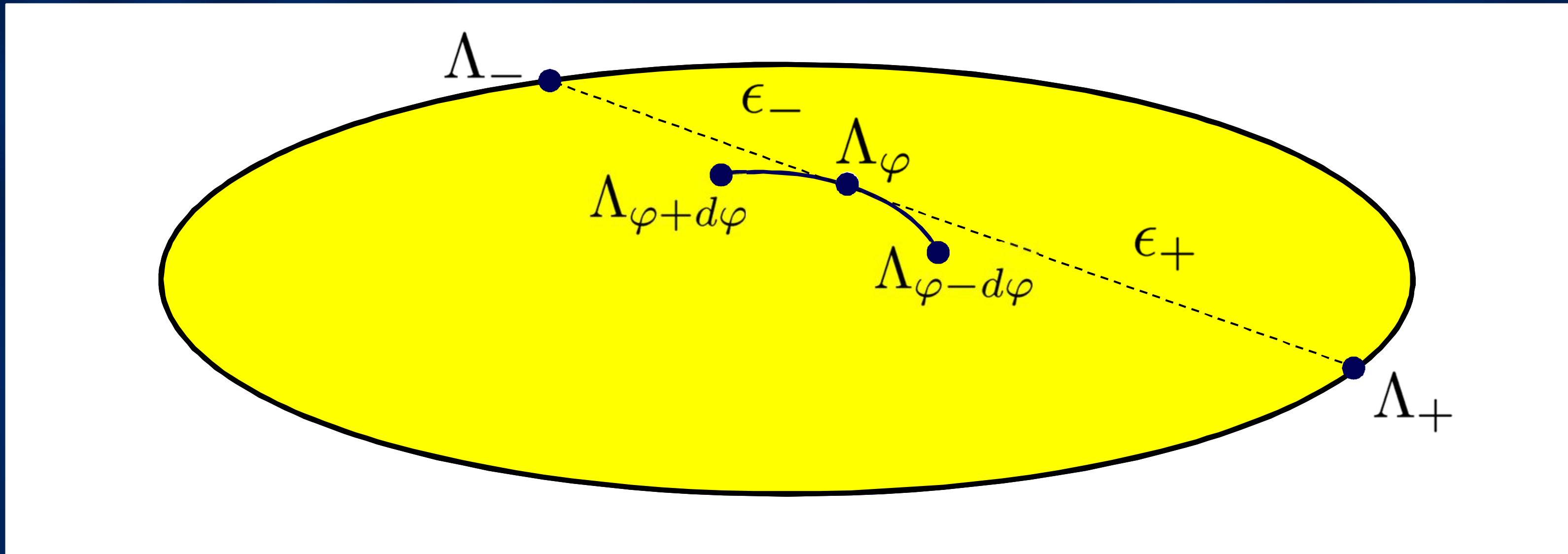
General picture

Actual value ϕ $\xrightarrow{\Lambda_\phi}$ $\hat{\varrho}_\phi = \Lambda_\phi(\hat{\varrho}_{\mathrm{ini}})$

Quantum Cramér-Rao bound using SLD:

$$F(\phi) \leq \mathrm{Tr}(\hat{\varrho}_\phi \hat{L}_\phi^2), \quad \frac{\partial}{\partial\phi}\hat{\varrho}_\phi = \frac{1}{2}(\hat{L}_\phi\hat{\varrho}_\phi + \hat{\varrho}_\phi\hat{L}_\phi)$$

Completely positive maps



$$\Lambda_\phi = p_+(\phi)\Lambda_+ + p_-(\phi)\Lambda_- + \mathcal{O}((d\phi)^2)$$

Let $\Lambda_\pm$ be extremal and “distances” $\epsilon_\pm$ be defined through

$$\Lambda_\pm = \Lambda_\phi \pm \epsilon_\pm \partial_\phi \Lambda_\phi$$

K. Matsumoto, arXiv:1006.0300 (2010)

Classical simulation

Actual value ϕ

$p_{\pm}(\phi)$

Λ_ϕ

$\hat{\varrho}_\phi = \Lambda_\phi(\hat{\varrho}_{\mathrm{ini}})$

$p(r|\phi)$

$$\Delta\phi \geq \sqrt{\frac{\epsilon_+ \epsilon_-}{N}}$$

R. Demkowicz-Dobrzański, J. Kołodyński, and M. Guţă, Nature Comm. **3**, 1063 (2012)

Specific channels

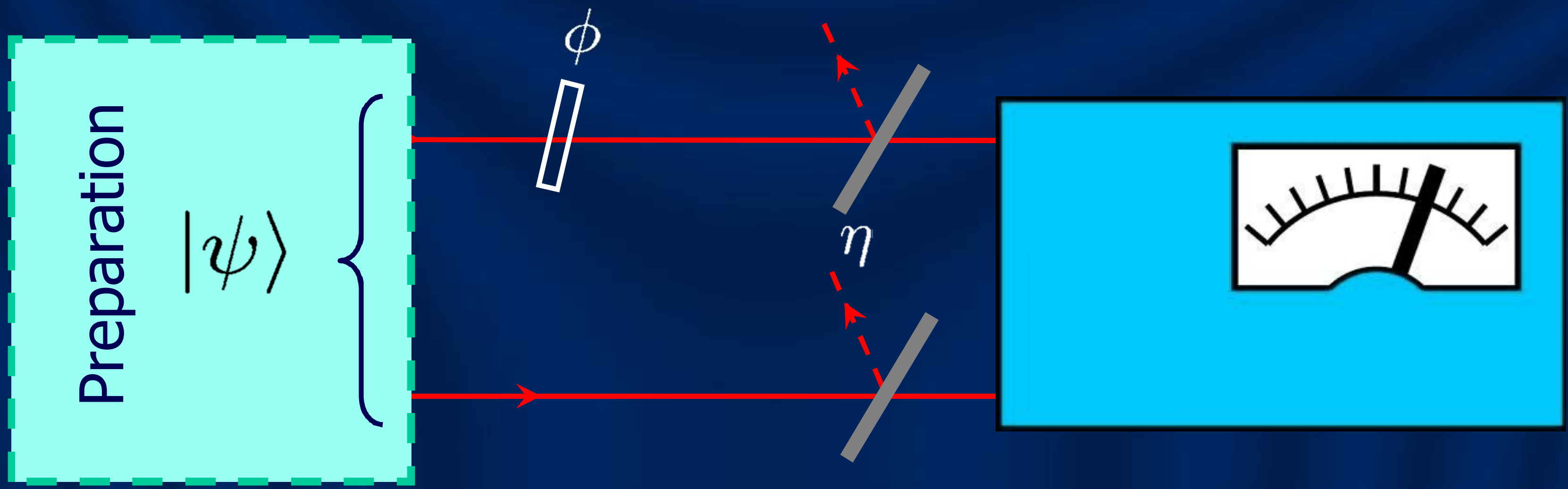
Table 1 | Precision bounds of the most relevant models in quantum-enhanced metrology.

Channel considered	Classical simulation	Channel extension
Depolarisation	$\sqrt{(1-\eta)(1+3\eta)/4\eta^2}$	$\sqrt{(1-\eta)(1+2\eta)/2\eta^2}$
Dephasing	$\sqrt{1-\eta^2/\eta}$	$\sqrt{1-\eta^2/\eta}$
Spontaneous emission	NA	$(1/2)\sqrt{1-\eta/\eta}$
Lossy interferometer	NA	$\sqrt{1-\eta/\eta}$

NA, not available.
The bounds are derived using the two methods discussed in the paper. All the bounds are of the form $\Delta\varphi_N \geq (\mathrm{const}/\sqrt{N})$, where constant factors are given in the table. Classical simulation method does not provide bounds for spontaneous emission and lossy interferometer, as these channels are φ-extremal. For the dephasing model, it surprisingly yields an equally tight bound as the more powerful channel extension method.

R. Demkowicz-Dobrzański, J. Kołodyński, and M. Guţă, Nature Comm. **3**, 1063 (2012)

Asymptotic scaling



When N photons are used:

$$\mathsf{F} \leq \frac{\eta N}{1-\eta} \qquad \Delta\tilde{\phi} \geq \sqrt{\frac{1-\eta}{\eta N}}$$

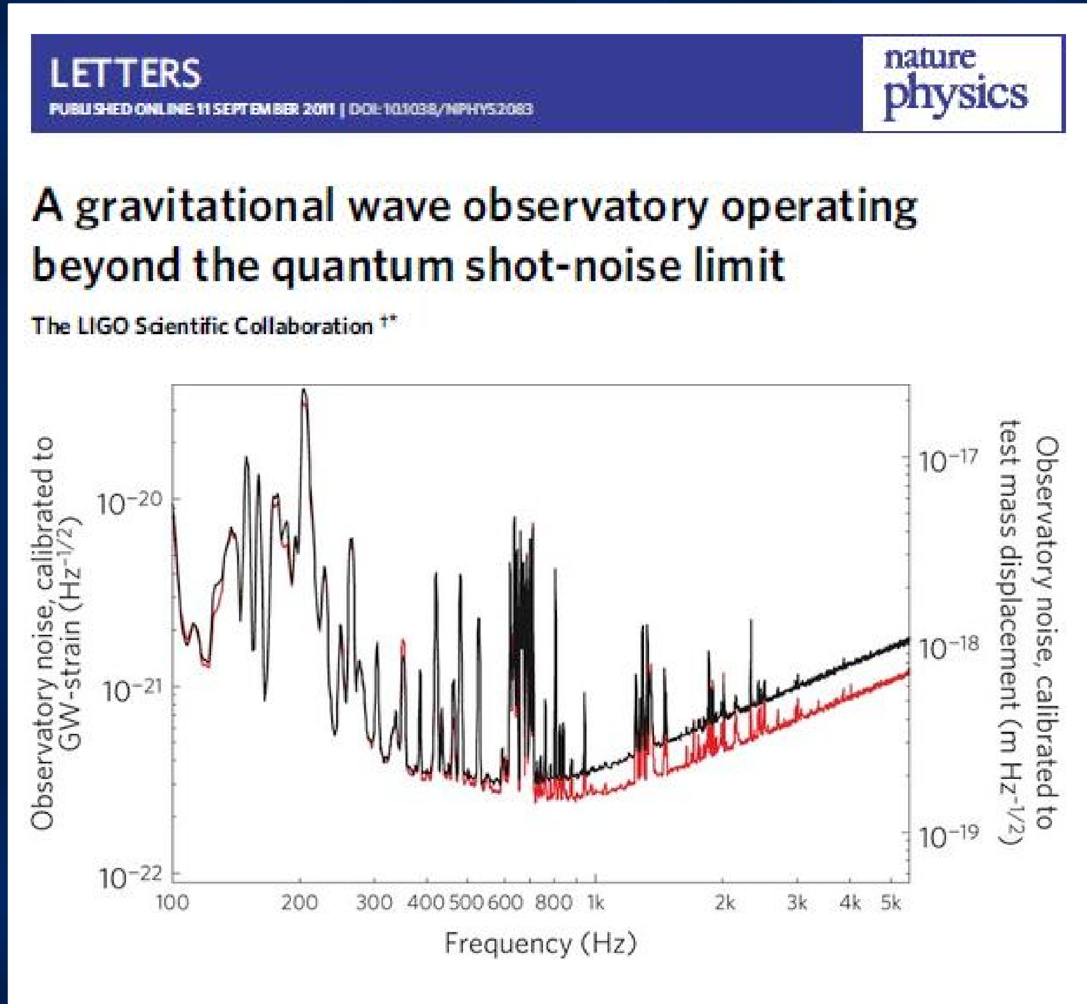
Theoretical toolbox:

J. Kołodyński and R. Demkowicz-Dobrzański, New J. Phys. **15**, 073043 (2013)

Gravitational wave detection

GEO600 Experiment

$$\frac{\Delta\tilde{\phi}_{\text{squeezed}}}{\Delta\tilde{\phi}_{\text{standard}}} \approx 0.66$$

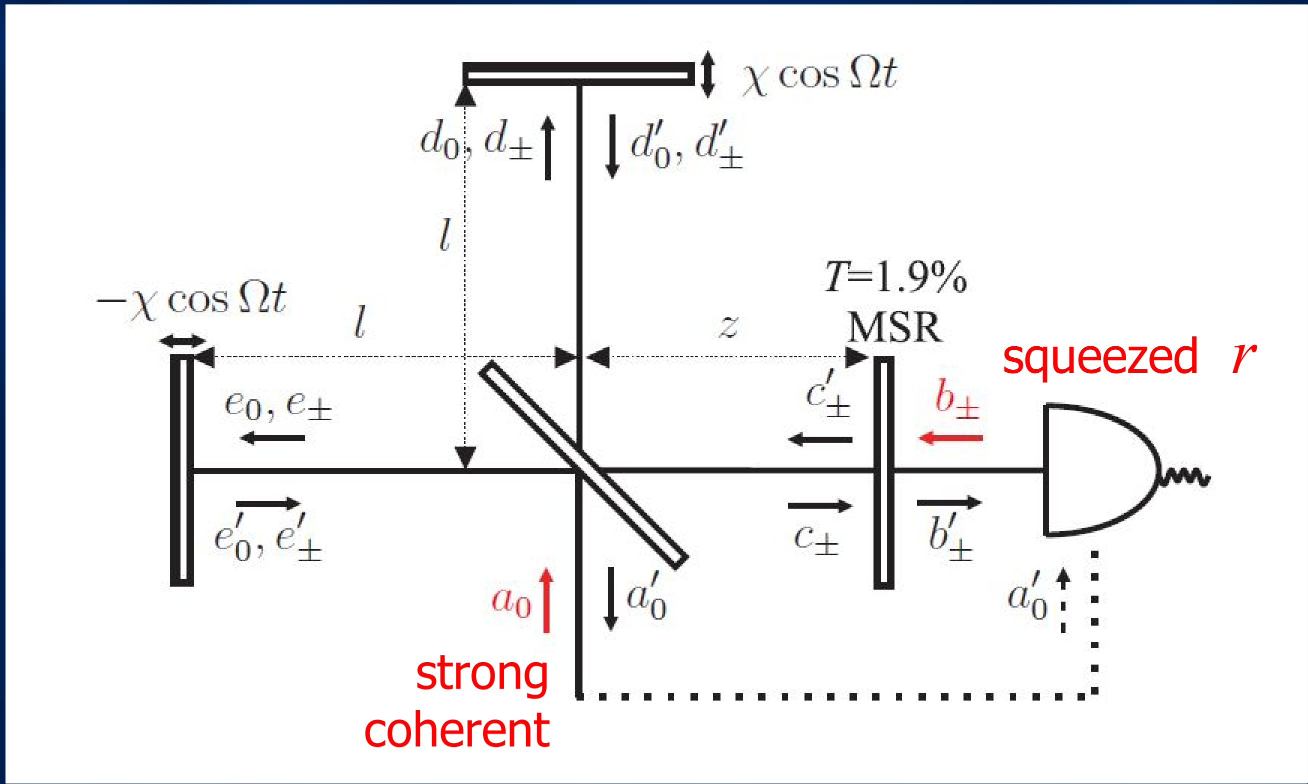

LETTERS
nature physics
PUBLISHED ONLINE: 11 SEPTEMBER 2011 | DOI: 10.1038/NPHYS2083

A gravitational wave observatory operating beyond the quantum shot-noise limit

The LIGO Scientific Collaboration †*

J. Abadie *et al.* (The LIGO Scientific Collaboration), Nature Phys. **7**, 962 (2011)

Model



When power is carried dominantly by the coherent field

$$\Delta\tilde{\phi} = \sqrt{\frac{1-\eta+\eta e^{-2r}}{\eta\langle N\rangle}}$$

Undefined photon number

When no external phase is used:

$$\hat{\varrho} = \bigoplus_{N=0}^{\infty} p_n \hat{\varrho}_N$$

Convexity of Fisher information:

$$\mathrm{F}(\hat{\varrho}) \leq \sum_{N=0}^{\infty} p_n \mathrm{F}(\hat{\varrho}_N)$$

Bound for the fixed photon number:

$$\leq \sum_{N=0}^{\infty} p_n \frac{\eta N}{1-\eta} = \frac{\eta \langle N \rangle}{1-\eta}$$

General limit:

$$\Delta\tilde{\phi} \geq \sqrt{\frac{1-\eta}{\eta\langle N\rangle}}$$

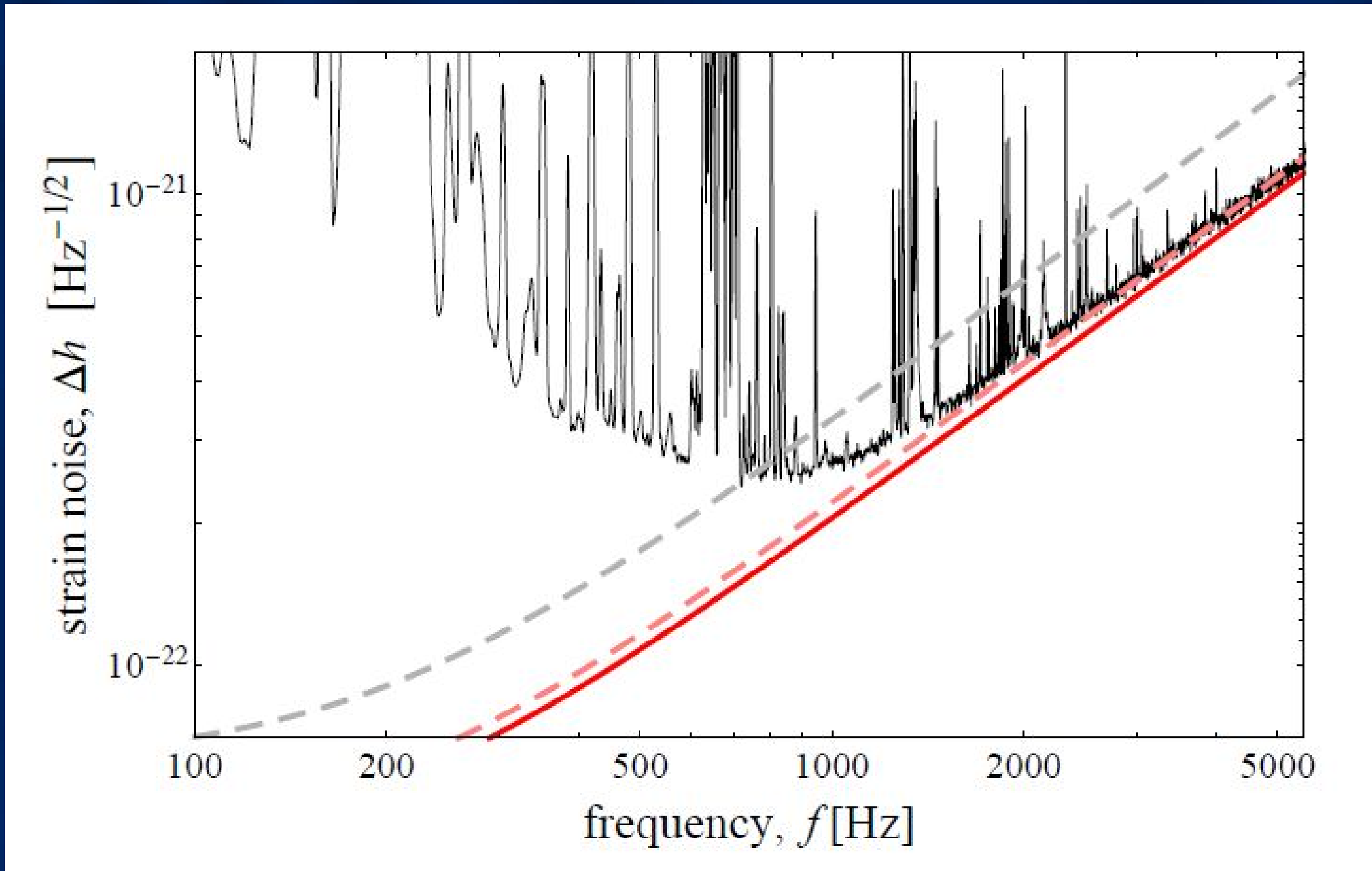
Squeezed scheme:

$$\Delta\tilde{\phi} = \sqrt{\frac{1-\eta+\eta \mathrm{e}^{-2r}}{\eta\langle N\rangle}}$$

Saturates if $\mathrm{e}^{-2r} \ll (1-\eta)/\eta$

Result

R. Demkowicz-Dobrzański, K. Banaszek, and R. Schnabel, Phys. Rev. A **88**, 041802(R) (2013)



Shot noise limit

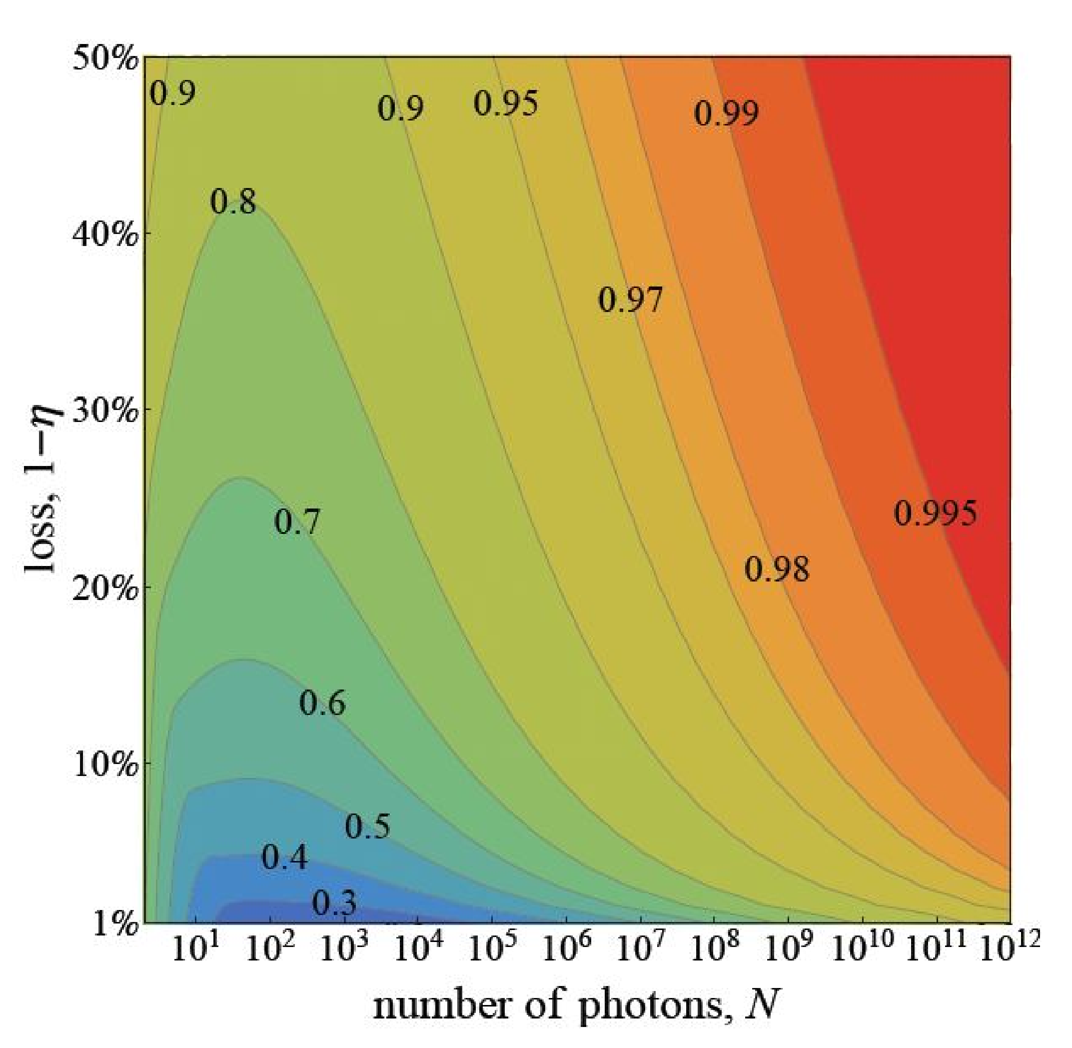
10dB squeezing (implemented)

16dB squeezing and ultimate bound

Assumed uniform transmission $\eta = 62\%$

Optimality of squezed states

R. Demkowicz-Dobrzański, K. Banaszek, and R. Schnabel,
Phys. Rev. A **88**, 041802(R) (2013)



$$\frac{\Delta\phi_{\text{optimal}}}{\Delta\phi_{\text{squeezed}}}$$

Outlook

Either...

- ideal single-photon sources
- deterministic state preparation
- quantum non-demolition measurements
- 100% efficient detectors

... or ...

- imperfection-tolerant schemes

... or ...

- a combination of the above

Resources:

- total amount of light used
- number of photons sent through the sample
- passes through the sample
- external phase reference

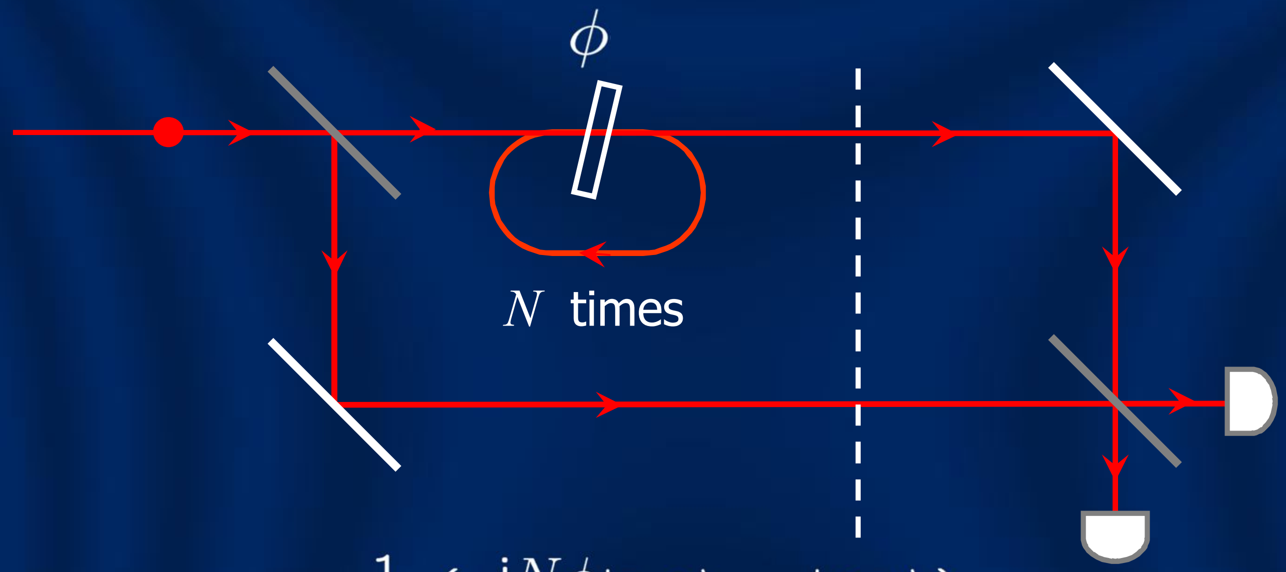
Performance:

- statistical uncertainty
- resolution
- ...?

K. Banaszek, R. Demkowicz-Dobrzański, and I. A. Walmsley, *Quantum states made to measure*, Nature Photonics **3**, 673 (2009)

Multipass strategy

Inspired by B. L. Higgins *et al.*, Nature **450**, 393 (2007)



$$\frac{1}{\sqrt{2}}(e^{iN\phi}|10\rangle - |01\rangle)$$

- The acquired phase exhibits Heisenberg-type scaling
- Sensitivity to losses is analogous as for N00N states!

R. Demkowicz-Dobrzański, Laser Phys. **20**, 1197 (2010)