Bell's inequalities. Lecture I

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March 7th, 2008.

Plan

- EPR: Quantum mechanics is incomplete
- Bell: Quantum mechanics violate inequalities satisfied by any local realistic theory
- Experiments and loopholes
- Simple proofs of Bell's theorem
- Bell inequalities and quantum information

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The first paper on "quantum nonlocality"

THE NEW YORK TIMES, SATURDAY, MAY 4, 1935.

EINSTEIN ATTACKS QUANTUM THEORY

Scientist and Two Colleagues Find It Is Not 'Complete' Even Though 'Correct.'

SEE FULLER ONE POSSIBLE

Believe a Whole Description of 'the Physical Reality' Can Be Provided Eventually.

PRINCETON, N. J., May 3.—Professor Albert Einstein will attack science's important theory of quantum mechanics, a theory of which he was a sort of grandfather. He concludes that while it is "correct" it is not "complete."

With two colleagues at the Institute for Advanced Study here, the noted scientist is about to report to the American Physical Society what is wrong with the theory of quantum mechanics, it has been learned exclusively by Science Service.

The quantum theory, with which science predicts with some success inter-atomic happenings, does not meet the requirements for a satisfactory physical theory, Professor Einstein will report in a joint paper with Dr. Boris Podolsky and Dr. N. Rosen.

In the quantum theory as now used, the latest Einstein paper will point out that where two physical quantities such as the position of a particle and its velocity interact, a knowledge of one quantity precludes knowledge about the other. This is the famous principle of uncertainty put forward by Professor Werner Heisenberg and incorporated in the quantum theory. This very fact, Professor Einstein feels, makes the quantum theory fail in the requirements necessary for a satisfactory physical theory.

Two Requirements Listed.

These two requirements are:

1. The theory should make possible a calculation of the facts of nature and predict results which can be accurately checked by experiment; the theory should be, in other words, correct.

2. Moreover, a satisfactory theory should, as a good image of the objective world, contain a counterpart for things found in the objective world; that is, it must be a complete theory.

Quantum theory, Professor Einstein and his colleagues will report, fulfills the correctness requirement but fails in the completeness requirement.

While proving that present quantum theory does not give a complete description of physical reality, Professor Einstein believes some later, still undeveloped, theory will make this possible. His conclusion is:

"While we have thus shown that the wave function [of quantum theory] does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible."

The development of quantum mechanics has proved very useful in exploring the atom. Six Nobel Prizes in physics, including one to Einstein, have been awarded for various phases of the researches leading up to quantum mechanics. The names of Planck, Bohr, de Broglie, Heisenberg, Dirac and Schroedinger, as well as Einstein, are linked with quantum mechanics.

The exact title of the Einstein-Podolsky-Rosen paper is: "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?"

Explanation by Podolsky.

In explaining the latest view of the physical world as revealed in their researches Dr. Podolsky, one of the authors, said:

"Physicists believe that there exist real material things independent of our minds and our theories. We construct theories and invent words (such as electron, positron, &c.) in an attempt to explain to ourselves what we know about our external world and to help us to obtain further knowledge of it. Before a theory can be considered to be satisfactory it must pass two very severe tests. First, the theory must enable us to calculate facts of nature, and these calculations must agree very accurately with observations and experiments. Second, we expect a satisfactory theory, as a good image of objective reality, to contain a counterpart for every element of the physical world. A theory satisfying the first requirement may be called a correct theory while, if it satisfies the second requirement, it may be called a complete theory.

"Hundreds of thousands of experiments and measurements have shown that, at least in cases when matter moves much slower than light, the theory of Planck, Einstein, Bohr, Heisenberg and Schroedinger known as quantum mechanics is a correct theory. Einstein, Podolsky and Rosen now discuss the question of the completeness of quantum mechanics. They arrive at the conclusion that quantum mechanics, in its present form, is not complete.

"In quantum mechanics the condition of any physical system, such as an electron, an atom, &c., is supposed to be completely described by a formula known as a 'wave function.' Suppose that we know the wave function for each of two physical systems, and that these two systems come together, interact, and again separate (as when two particles collide and move apart). Quantum mechanics, although giving us considerable information about such a process, does not enable us to calculate the wave function of each physical system after the separation. This fact is made use of in showing that the wave function does not give a complete description of physical reality. Since, however, description of physical systems by wave functions is an essential step of quantum mechanics, this means that quantum mechanics is not a complete theory."

Raises Point of Doubt.

Special to THE NEW YORK TIMES.

PRINCETON, N. J., May 3.—Asked to comment on the new ideas of Professor Einstein and his collaborators, Professor Edward U. Condon, mathematical physicist of Princeton University, said tonight:

"Of course, a great deal of the argument hinges on just what meaning is to be attached to the word 'reality' in connection with physics. They have certainly discussed an interesting point in connection with the theory. Dr. Einstein has never been satisfied with the statistical causality which in the new theories replaces the strict causality of the old physics.

"It is reported that when he first learned of the work of Schroedinger and Dirac, he said, 'Der lieber Gott wuerfelt nicht, [the good Lord does not throw dice]. For the last five years he has subjected the quantum mechanical theories to very searching criticism from this standpoint. But I am afraid that thus far the statistical theories have withstood criticism."

EPR: QM is “incomplete”

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

According to EPR, any satisfactory physical theory must be:

(1) Correct.

(2) “Complete”.

EPR's elements of reality

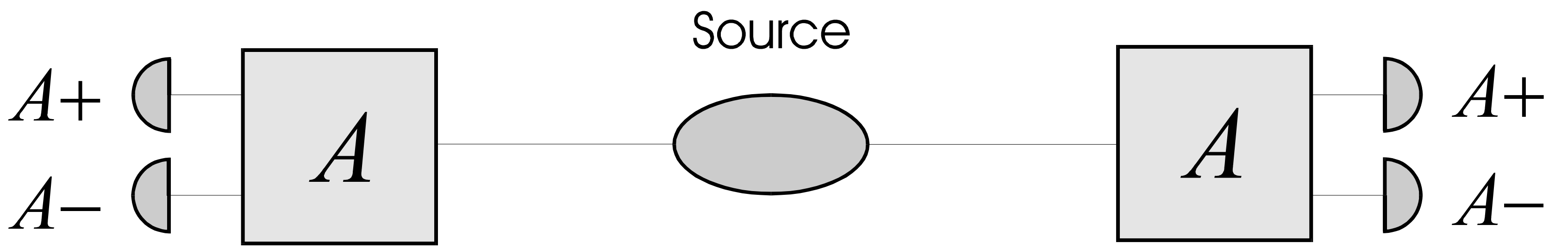
"If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."

EPR’s elements of reality

“Without in any way disturbing a system” = Spacelike separation.

“Predict with certainty” = Perfect correlations.

Bohm's version of EPR's argument



$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Bohm's version of EPR's argument

$$X_1 X_2 = -1$$

$$Y_1 Y_2 = -1$$

- X_2 and Y_2 are both "elements of reality".

- In QM, X_2 and Y_2 are incompatible observables (Heisenberg's uncertainty principle).

→ QM is incomplete (according to EPR).

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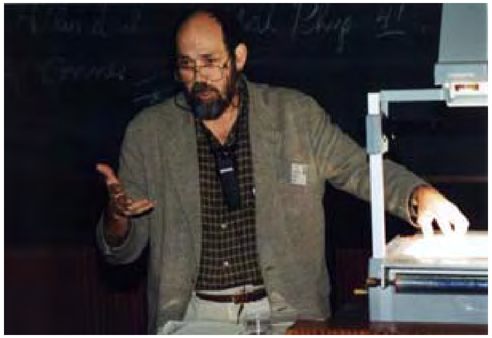
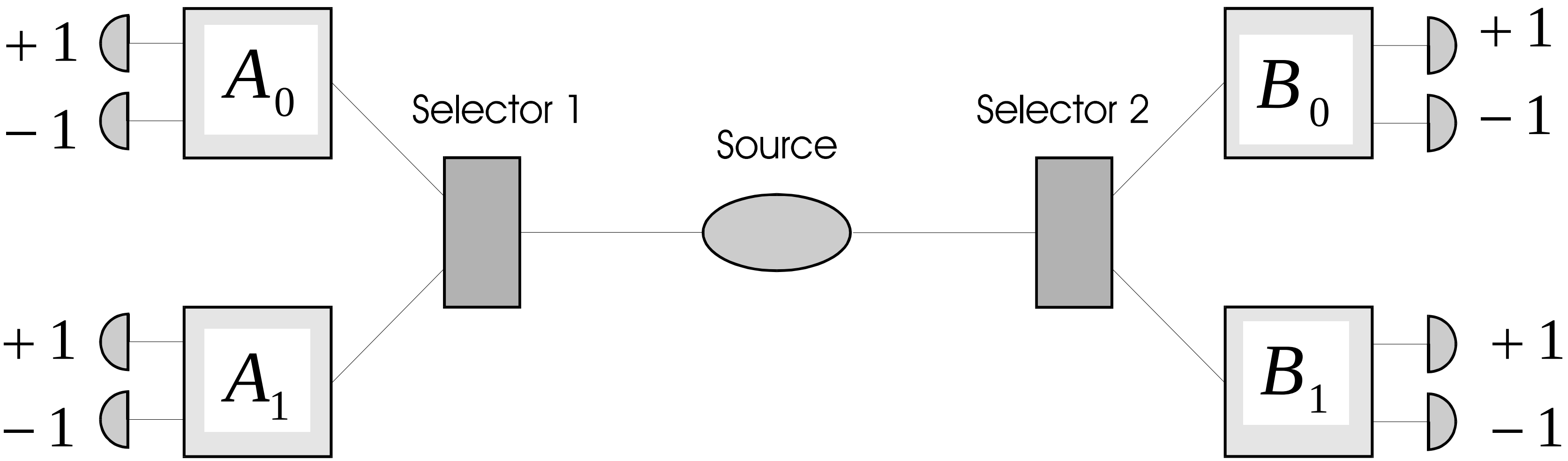
Bell’s theorem

It is impossible to complete QM with elements of reality because some predictions of QM cannot be reproduced with elements of reality.

The CHSH inequality





The CHSH inequality

$$A_0, A_1, B_0, B_1 \in \{-1, 1\}$$

$$(A_0 + A_1, A_0 - A_1) \in \{(\pm 2, 0), (0, \pm 2)\}$$

$$(A_0 + A_1)B_0 + (A_0 - A_1)B_1 \in \{-2, 2\}$$

$$-2 \leq \langle A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \rangle \leq 2$$

$$\left| \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \right| \leq 2$$

The CHSH inequality is violated

$$\beta_{\mathrm{QM}} = \left| \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \right|$$

$$= \left| -\cos\theta_{A_0 B_0} - \cos\theta_{A_0 B_1} - \cos\theta_{A_1 B_0} + \cos\theta_{A_1 B_1} \right|$$

$$\hat{A}_0 = \sigma_x$$

$$\hat{A}_1 = \sigma_y$$

$$\hat{B}_0 = (\sigma_y - \sigma_x) / \sqrt{2}$$

$$\hat{B}_1 = (\sigma_y + \sigma_x) / \sqrt{2}$$

$$\beta_{\mathrm{QM}} = 2\sqrt{2} > 2!!!$$

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Aspect’s experiments

VOLUME 47, NUMBER 7 PHYSICAL REVIEW LETTERS 17 AUGUST 1981

Experimental Tests of Realistic Local Theories via Bell’s Theorem

Alain Aspect, Philippe Grangier, and Gérard Roger
Institut d’Optique Théorique et Appliquée, Université Paris-Sud, F-91406 Orsay, France
(Received 30 March 1981)

We have measured the linear polarization correlation of the photons emitted in a radiative atomic cascade of calcium. A high-efficiency source provided an improved statistical accuracy and an ability to perform new tests. Our results, in excellent agreement with the quantum mechanical predictions, strongly violate the generalized Bell’s inequalities, and rule out the whole class of realistic local theories. No significant change in results was observed with source-polarizer separations of up to 6.5 m.

VOLUME 49, NUMBER 2 PHYSICAL REVIEW LETTERS 12 JULY 1982

Experimental Realization of Einstein-Podolsky-Rosen-Bohm *Gedankenexperiment*: A New Violation of Bell’s Inequalities

Alain Aspect, Philippe Grangier, and Gérard Roger
Institut d’Optique Théorique et Appliquée, Laboratoire associé au Centre National de la Recherche Scientifique, Université Paris-Sud, F-91406 Orsay, France
(Received 30 December 1981)

The linear-polarization correlation of pairs of photons emitted in a radiative cascade of calcium has been measured. The new experimental scheme, using two-channel polarizers (i.e., optical analogs of Stern-Gerlach filters), is a straightforward transposition of Einstein-Podolsky-Rosen-Bohm *gedankenexperiment*. The present results, in excellent agreement with the quantum mechanical predictions, lead to the greatest violation of generalized Bell’s inequalities ever achieved.

VOLUME 49, NUMBER 25 PHYSICAL REVIEW LETTERS 20 DECEMBER 1982

Experimental Test of Bell’s Inequalities Using Time-Varying Analyzers

Alain Aspect, Jean Dalibard,[a] and Gérard Roger
Institut d’Optique Théorique et Appliquée, F-91406 Orsay Cédex, France
(Received 27 September 1982)

Correlations of linear polarizations of pairs of photons have been measured with time-varying analyzers. The analyzer in each leg of the apparatus is an acousto-optical switch followed by two linear polarizers. The switches operate at incommensurate frequencies near 50 MHz. Each analyzer amounts to a polarizer which jumps between two orientations in a time short compared with the photon transit time. The results are in good agreement with quantum mechanical predictions but violate Bell’s inequalities by 5 standard deviations.

Loophole-free Bell experiments

So far, the results of any performed Bell experiment admit an interpretation in terms of local realistic theories.

A loophole-free experiment would require:

• Spacelike separation between Alice's measurement choice and Bob's measurement in order to exclude the possibility that Alice's measurement choice influences the result of Bob's measurement (locality loophole).

• Sufficiently large number of detections of the prepared particles in order to exclude the possibility that the nondetections correspond to local hidden-variable instructions (detection loophole).

Photons, ions… the good news

- Photons are the best candidates for closing the locality loophole. For instance, one can do a Bell experiment with pairs of polarization-entangled photons separated d = 400 m, which is not subject to the locality loophole (Innsbruck 98).

- Ions are the best candidates for closing the detection loophole. For instance, one can do a Bell experiment with pairs of trapped ions with a detection efficiency $\eta = 1$ (Boulder 01, Maryland 08).

Photons, ions… the bad news

- Photo-detection efficiency (η = 0.05-0.33) is not high enough to close the detection loophole (η > 0.83 is required for the CHSH inequality).

- Separation between trapped ions (*d* = 1 m in the Maryland 08 experiment) is not enough to close the locality loophole (*d* > 15 km is required for the Maryland 08 experiment).

Problem

44 years after Bell's original paper we do not have a loophole-free Bell experiment!

Proposals for loophole-free experiments

- Eberhard-Kwiat: Bell inequalities for non-maximally entangled states (assuming photodetectors with $\eta > 0.67$ efficiency).
- Fry: spin measurements of atoms using a polarized pulse of laser light.
- Grangier: homodyne measurements.
- Simon-Weinfurter: entanglement swapping between two atom-photon pairs.
- Proposals for excluding some specific classes of local hidden-variables: Santos, Zukowski.
- Bipartite Bell inequalities exhibiting exponentially-growing-with-size nonlocality + two-photon hyperentanglement.

Entanglement, nonlocality and Bell's inequalities

- A state is (Popescu's) nonlocal if it cannot be prepared by local interactions and classical communication.
- A state violates a specific Bell inequality if the results of the experiment cannot be reproduced by any possible model with local properties and no communication.
- I prefer to call (Popescu's) nonlocality just "entanglement".
- Entanglement is a physical resource.
- (My) nonlocality is the price realistic theories have to pay to reproduce quantum mechanics.
- My point: "Measures of (Popescu's) nonlocality" might be not appropriate "measures of how conclusive a Bell experiment is".

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Hardy's nonlocality proof

Any pure two-qubit entangled, but not maximally entangled, state can be written as

$$|\eta\rangle = c_{++}|A+B+\rangle + c_{+-}|A+B-\rangle + c_{-+}|A-B+\rangle + c_{--}|A-B-\rangle$$

$$|\eta\rangle = d_{++}|A+b+\rangle + d_{-+}|A-b+\rangle + d_{--}|A-b-\rangle$$

$$|\eta\rangle = f_{++}|a+B+\rangle + f_{+-}|a+B-\rangle + f_{--}|a-B-\rangle$$

$$|\eta\rangle = g_{+-}|a+b-\rangle + g_{-+}|a-b+\rangle + g_{--}|a-b-\rangle$$

Hardy's nonlocality proof

Therefore,

$$P_\eta\left(A=+1,\, B=+1\right)=\left|c_{++}\right|^2$$

$$P_\eta\left(b=+1\middle|\, A=+1\right)=1$$

$$P_\eta\left(a=+1\middle|\, B=+1\right)=1$$

$$P_\eta\left(a=+1,\, b=+1\right)=0$$

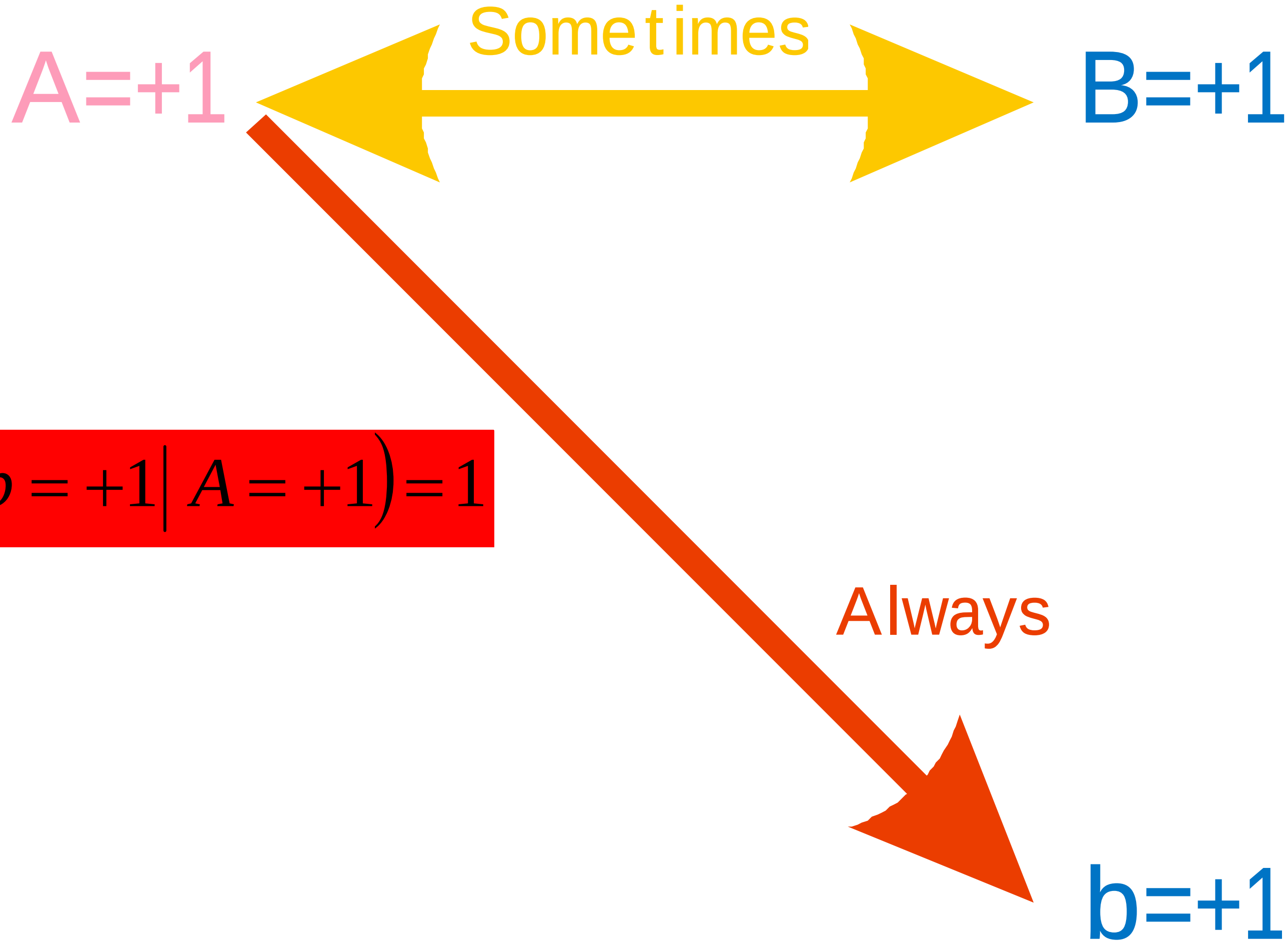
L. Hardy, PRL **71**, 1665 (1993).

Hardy's nonlocality proof

A=+1 ⟵ Sometimes ⟶ B=+1

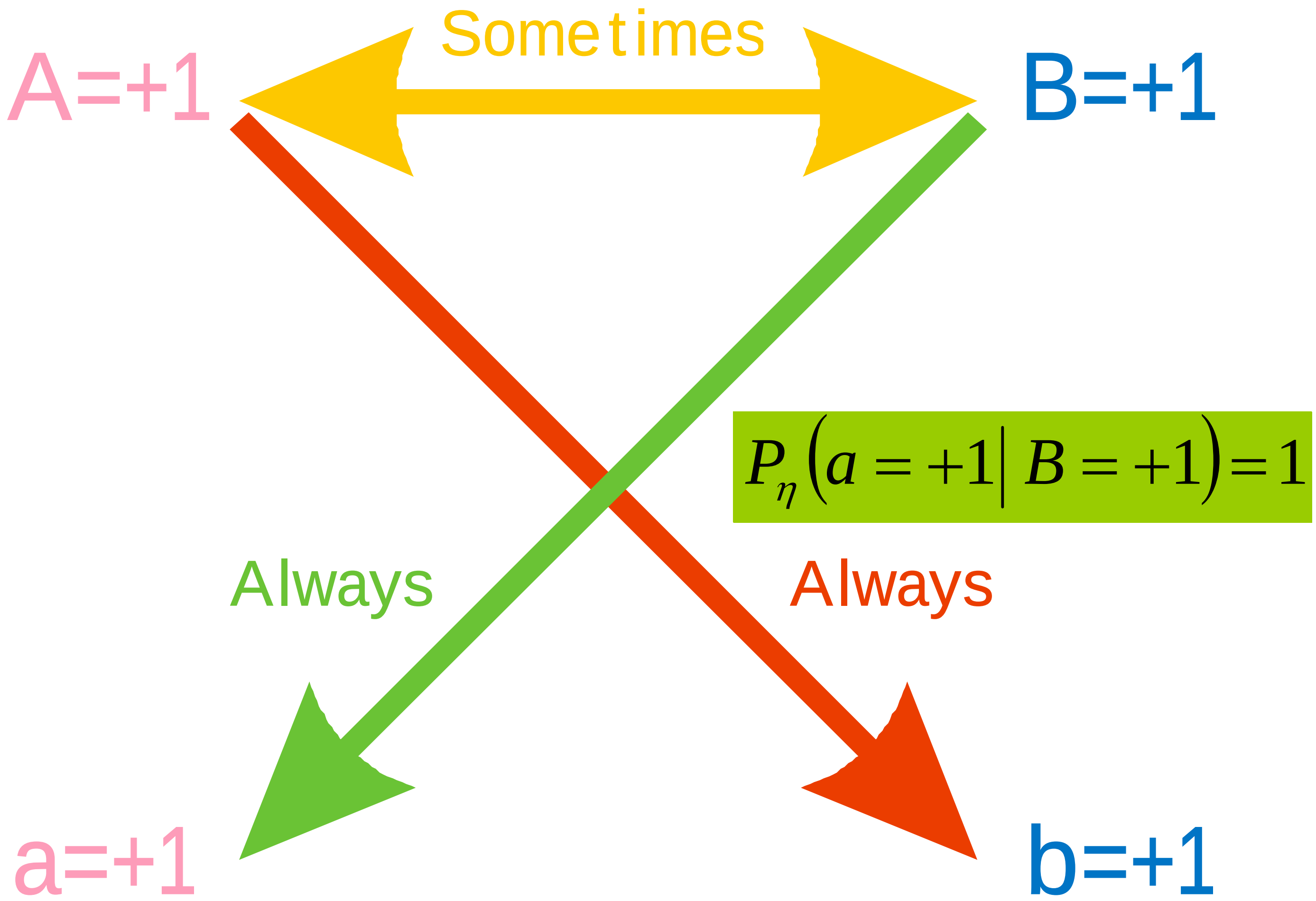
$$P_\eta(A=+1,\ B=+1)=\left|c_{++}\right|^2$$

Hardy’s nonlocality proof

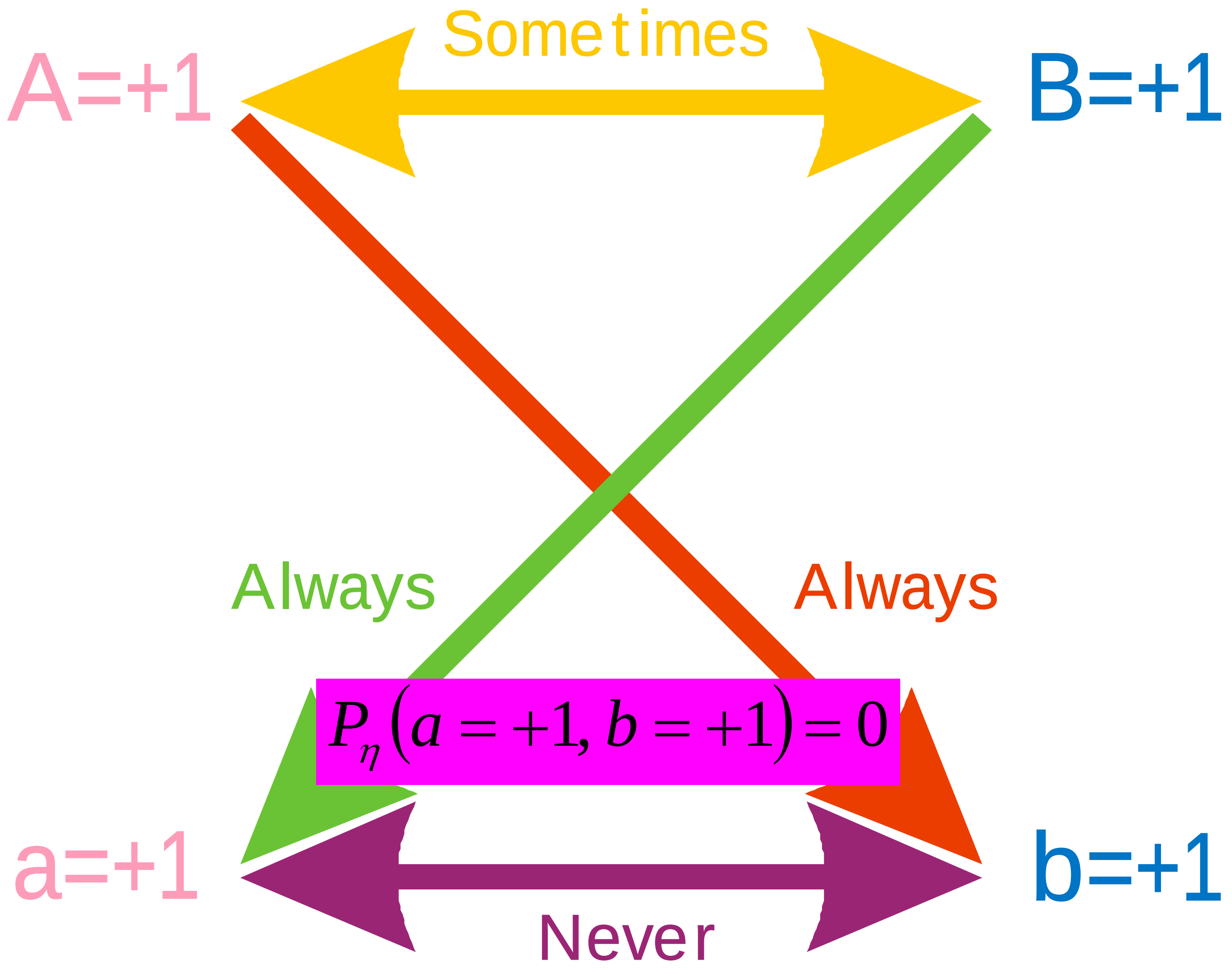


$$P_{\eta}\left(b=+1 \,\middle|\, A=+1\right)=1$$

Hardy’s nonlocality proof



Hardy's nonlocality proof



Rotationally invariant proof

Prepare the 8-qubit state

$$|\eta\rangle = \frac{1}{\sqrt{7}}\left(|\phi_0\phi_0\rangle + \sqrt{3}|\phi_0\phi_1\rangle + \sqrt{3}|\phi_1\phi_0\rangle\right)$$

where

$$|\phi_0\rangle = |\psi^-\rangle \otimes |\psi^-\rangle = \frac{1}{2}\left(|0101\rangle - |0110\rangle - |1001\rangle + |1010\rangle\right)$$

$$|\phi_1\rangle = \frac{1}{2\sqrt{3}}\left(2|0011\rangle - |0101\rangle - |0110\rangle - |1001\rangle - |1010\rangle + 2|1100\rangle\right)$$

AC, PRL **91**, 230403 (2003).

Rotationally invariant proof

The local (4-qubit) observables are:

$$F = -|\phi_0\rangle\langle\phi_0| + |\phi_1\rangle\langle\phi_1|$$

$$G = -|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|$$

where

$$|\psi_j\rangle = P_{23}|\phi_j\rangle$$

Properties:

$$P_\eta\left(R_A F = 1, R_B F = 1\right) = 0$$

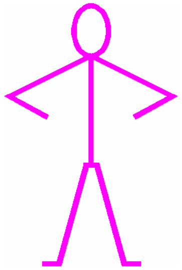
$$P_\eta\left(R_A F = 1 \,|\, R'_B G = 1\right) = 1$$

$$P_\eta\left(R_B F = 1 \,|\, R'_A G = 1\right) = 1$$

$$P_\eta\left(R'_A G = 1, R'_B G = 1\right) = \frac{9}{112}$$

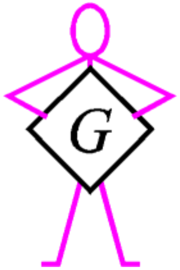
Rotationally invariant proof

This is Alice



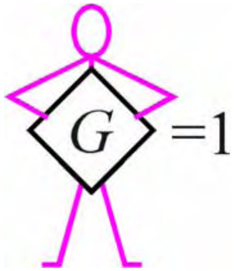
Rotationally invariant proof

Let us suppose that she measures *G*...



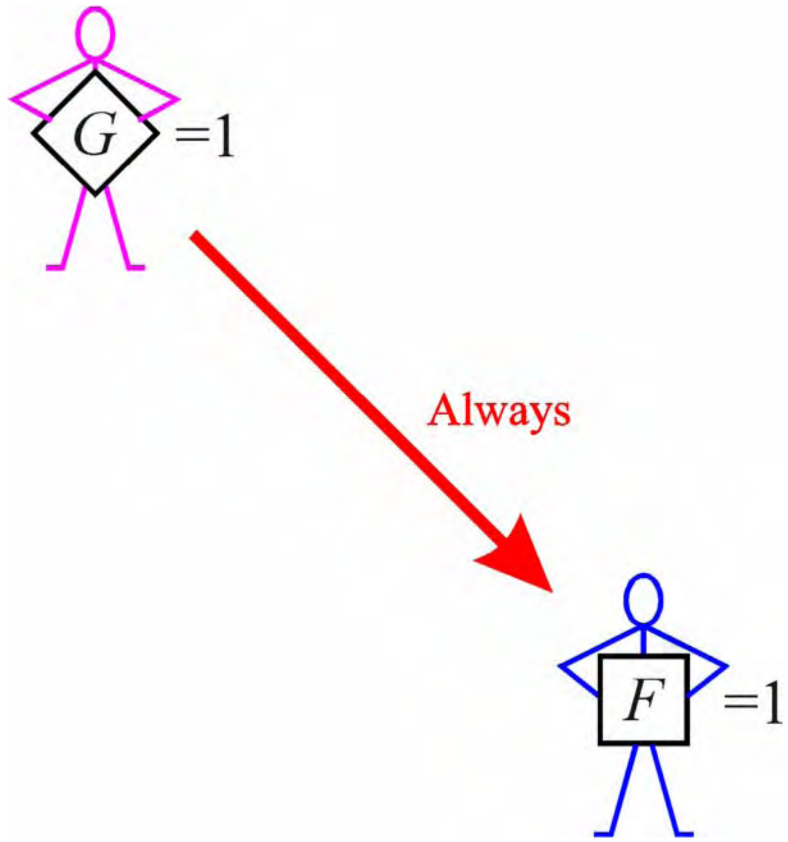
Rotationally invariant proof

...and obtains the result 1



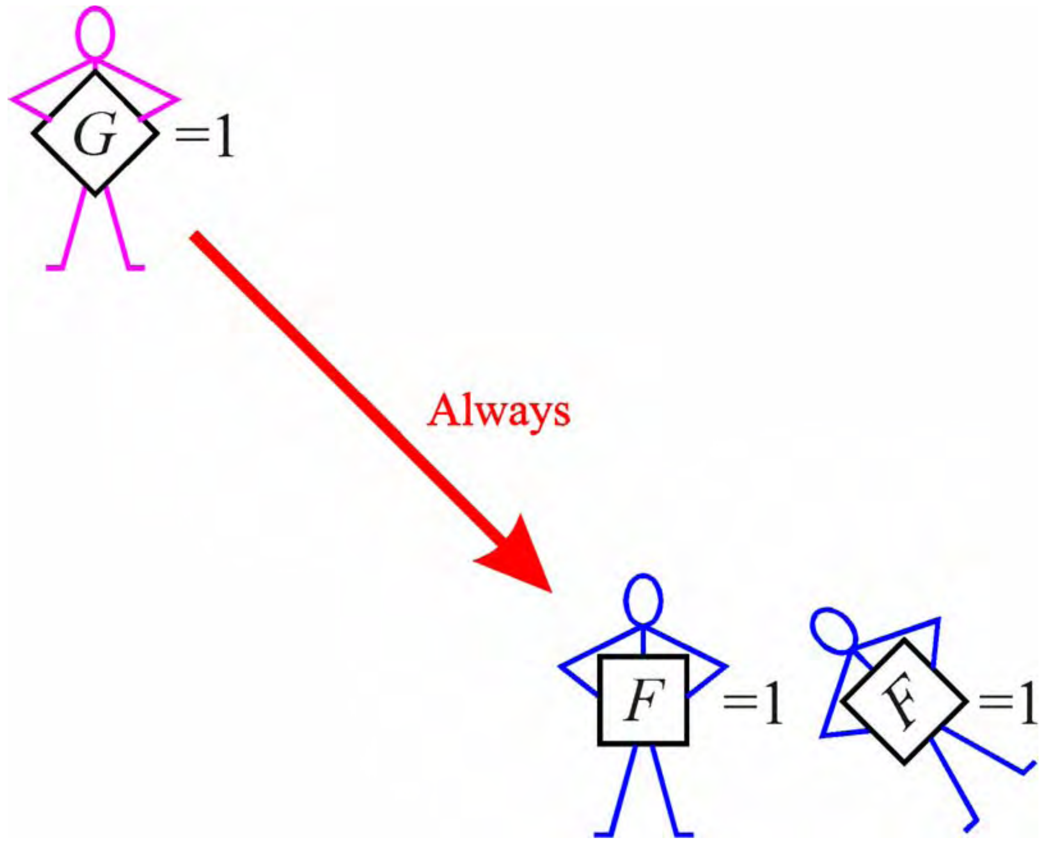
Rotationally invariant proof

Then, if Bob (whose measurement is spacelike separated from Alice's decision) measures *F*, he always obtains 1...



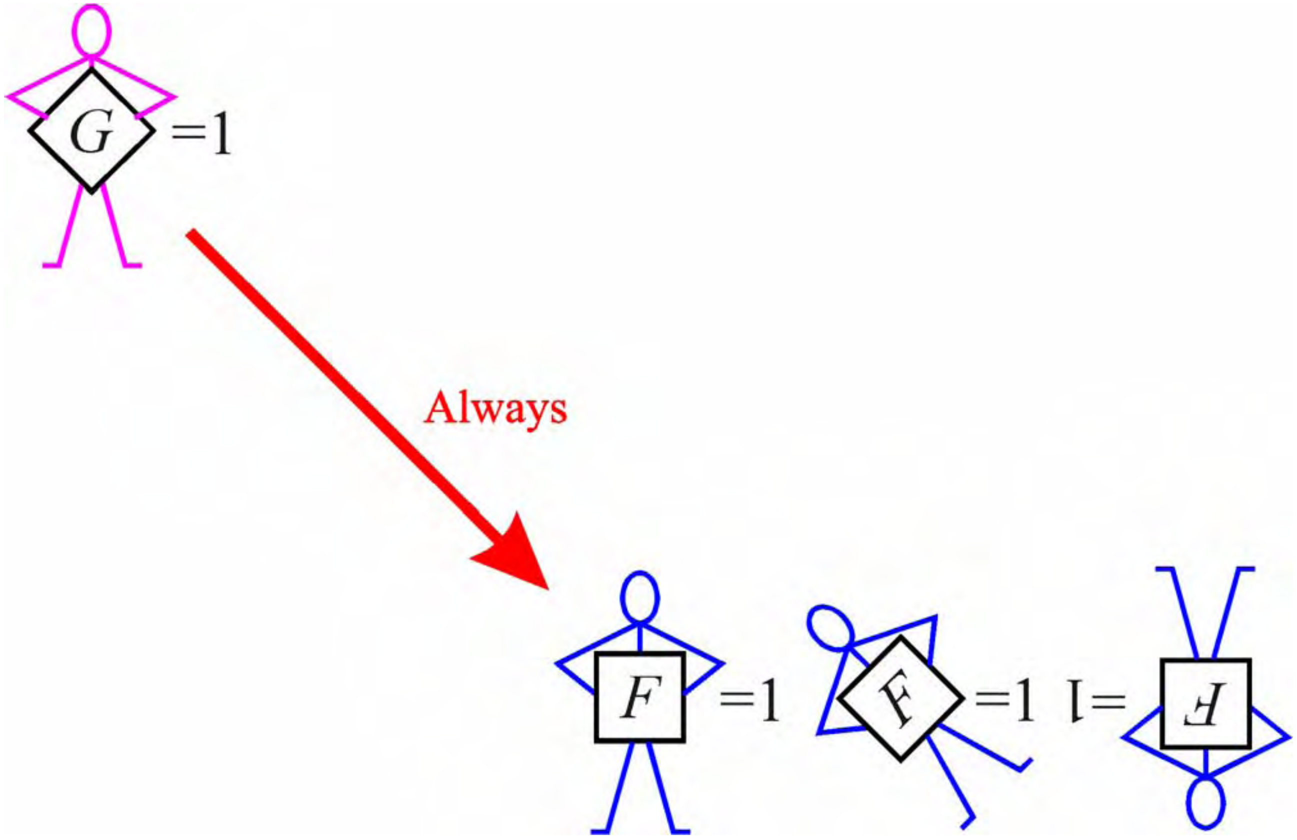
Rotationally invariant proof

...even if Bob rotates his apparatus



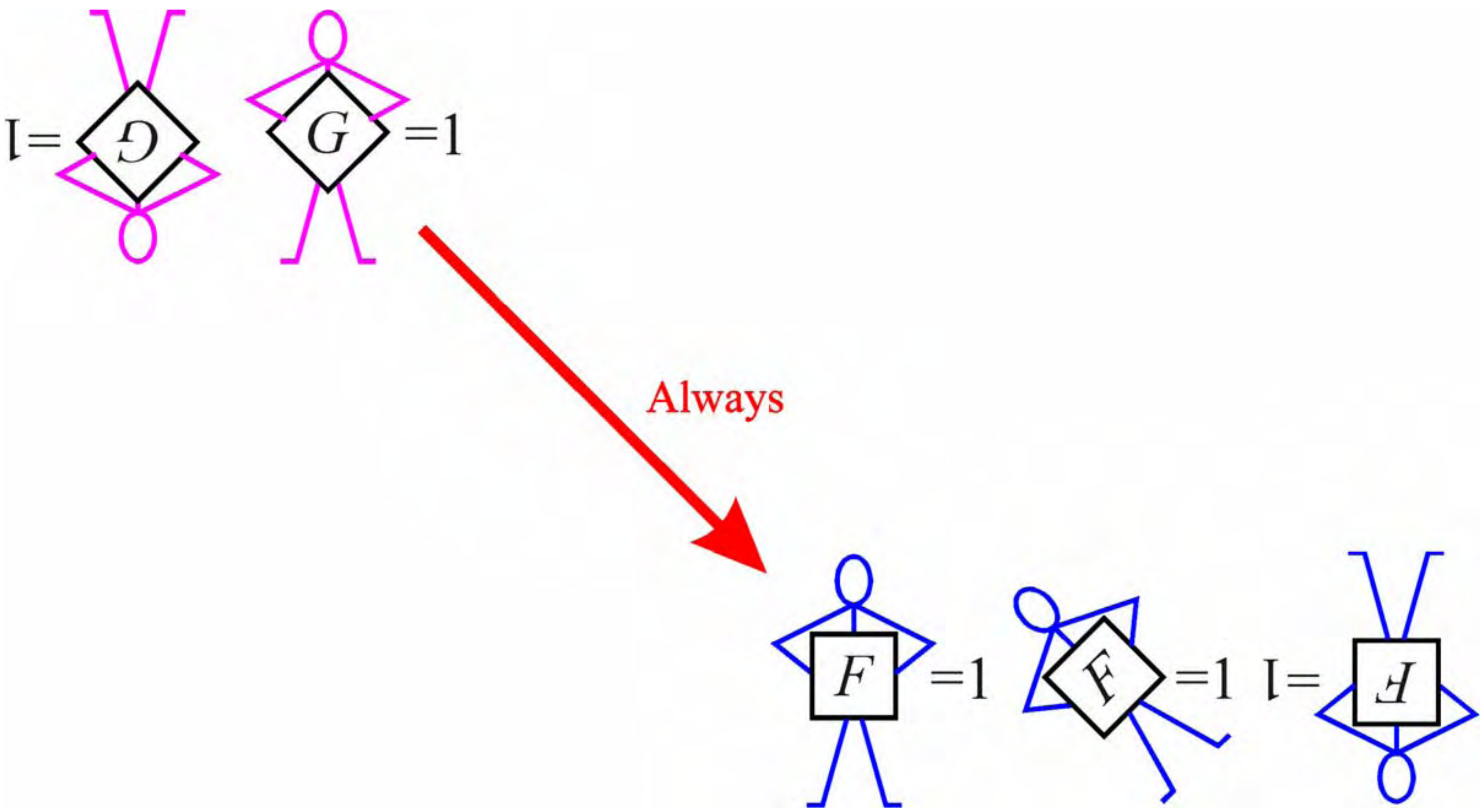
Rotationally invariant proof

He always obtains 1!



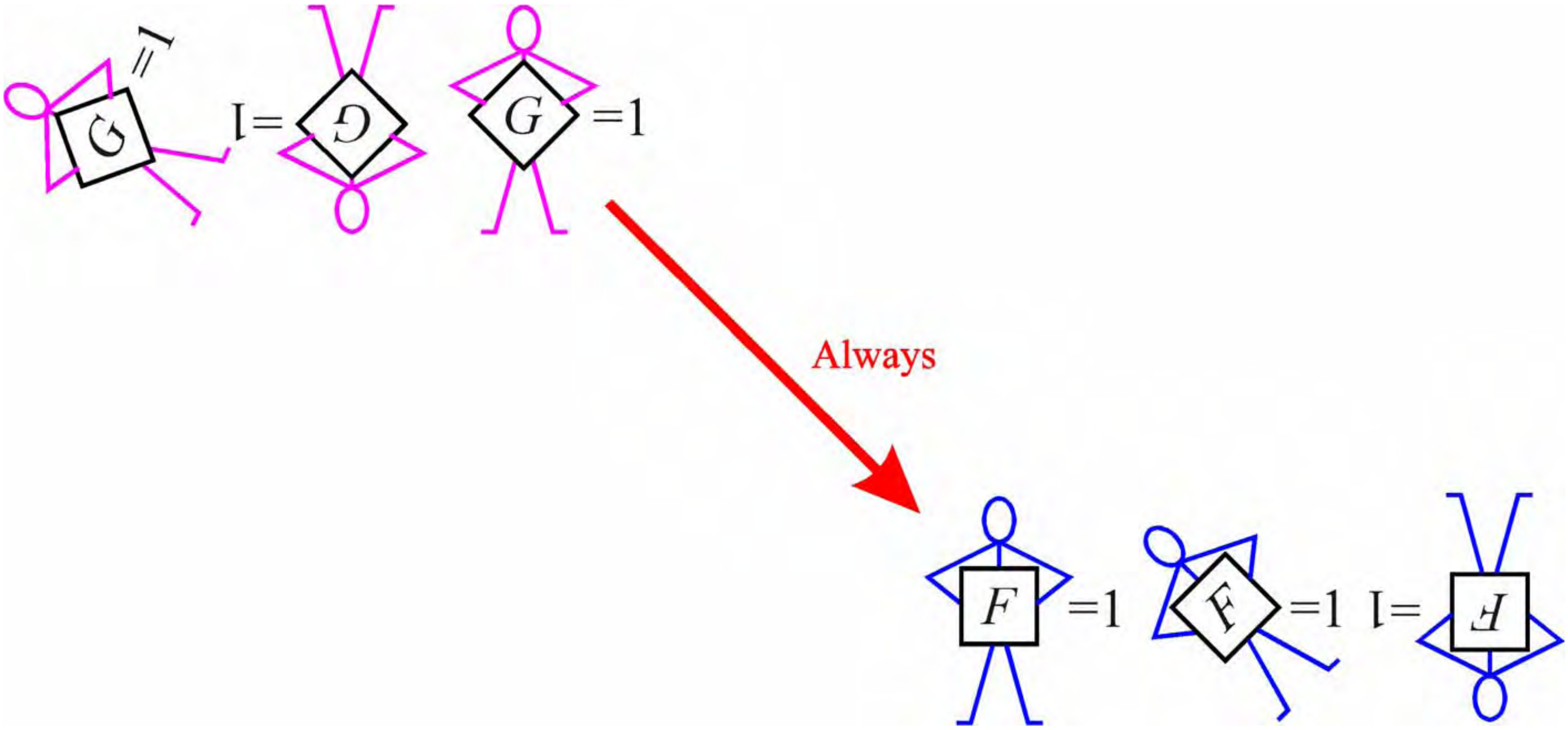
Rotationally invariant proof

Even if Alice has rotated her apparatus!



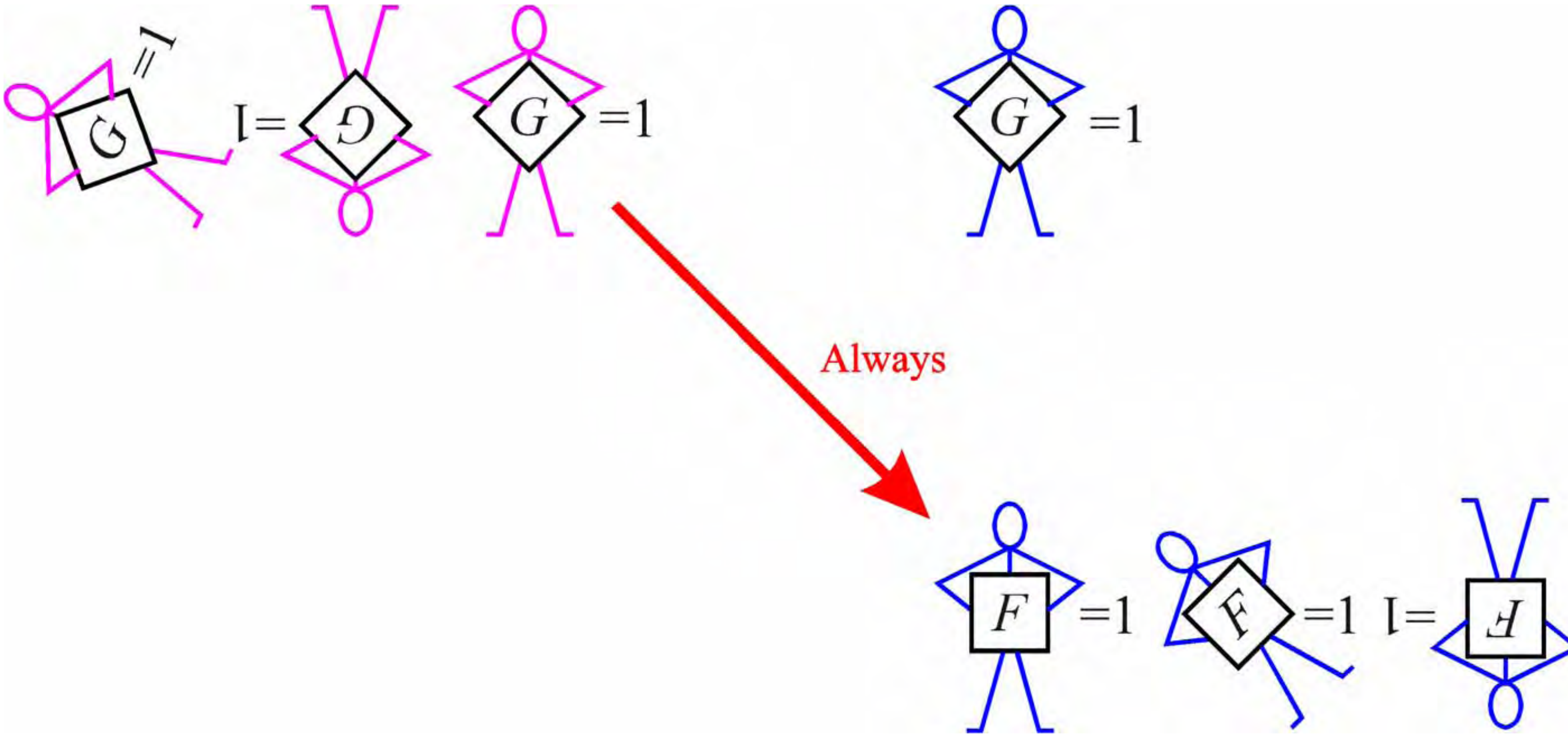
Rotationally invariant proof

In any way!



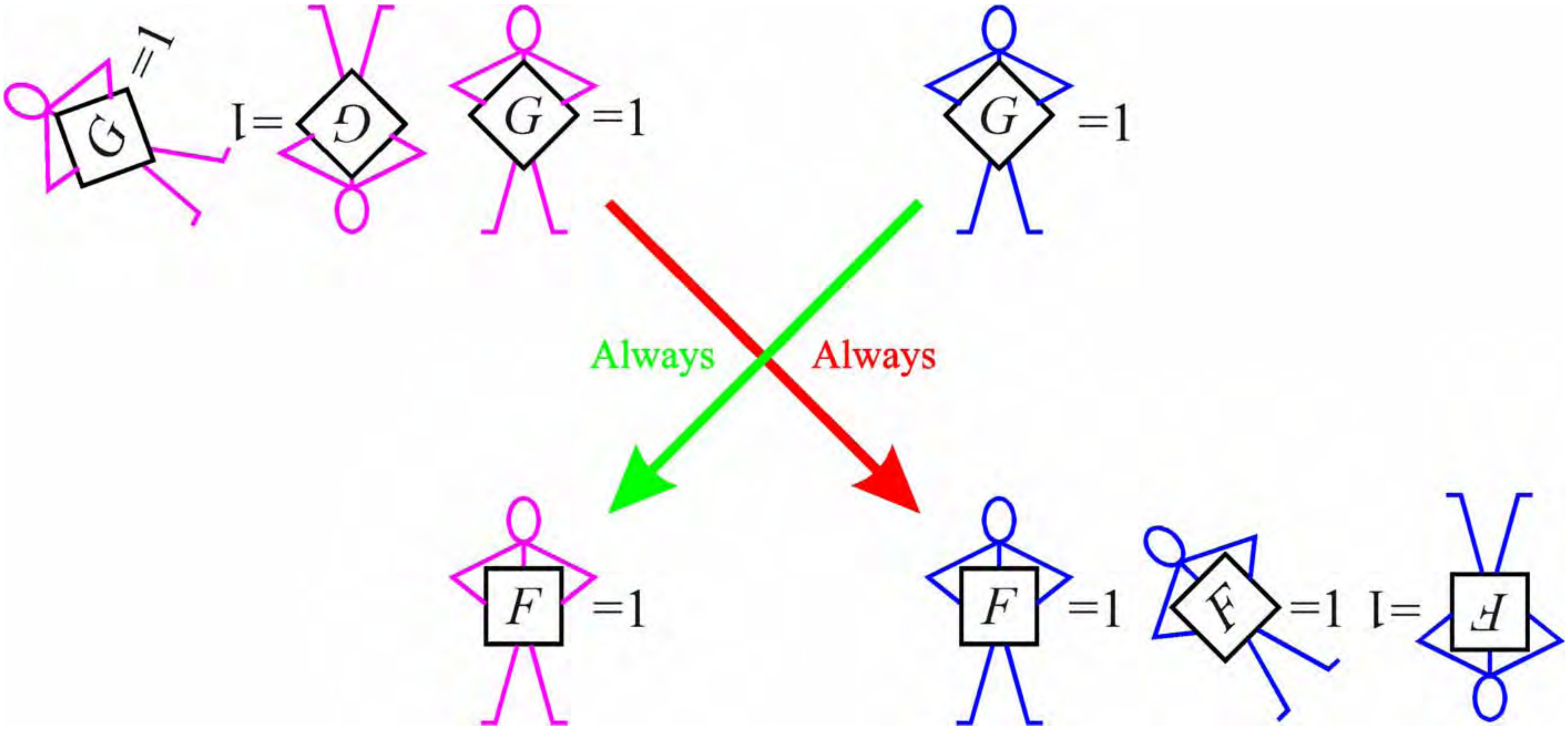
Rotationally invariant proof

Analogously, if Bob measures *G* and obtains 1...



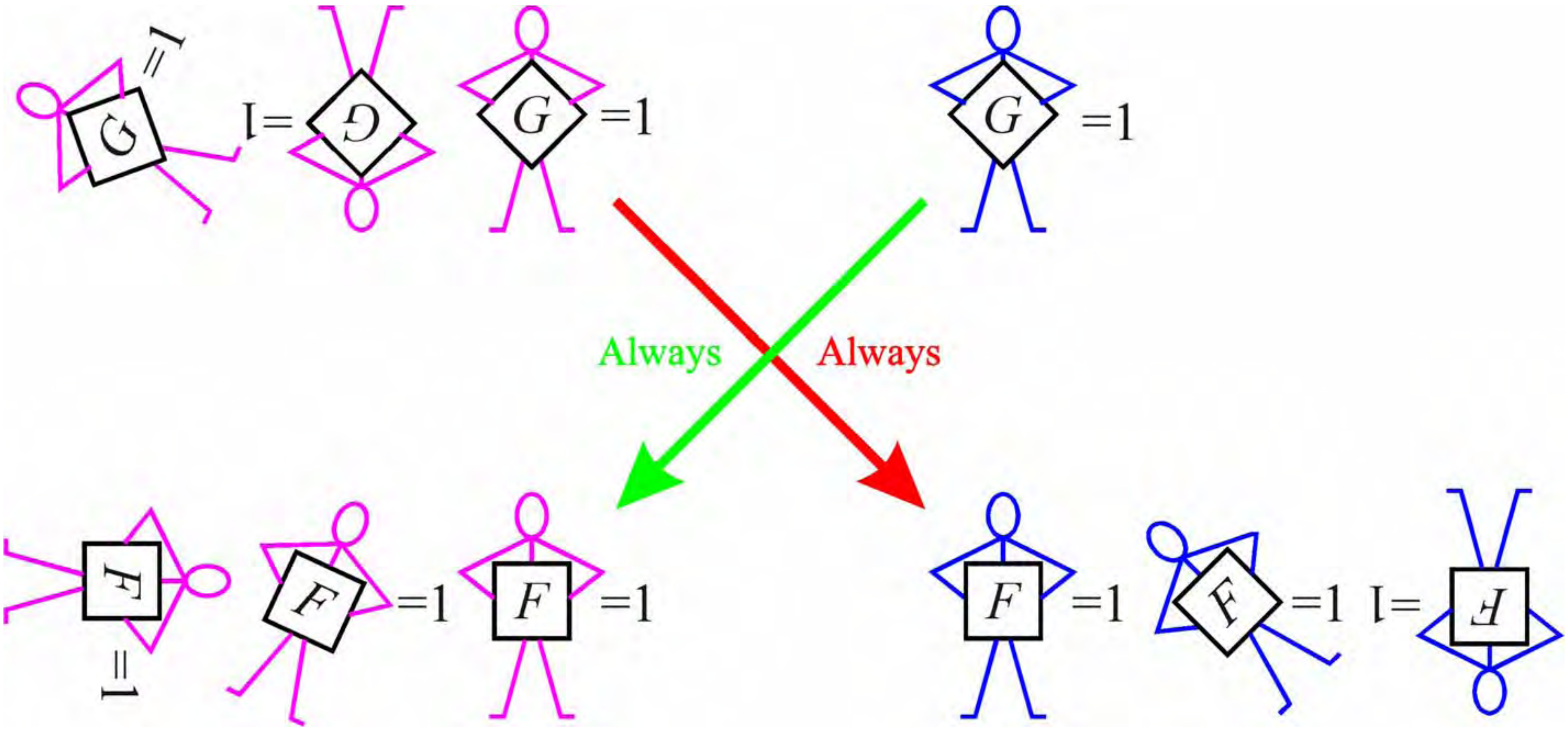
Rotationally invariant proof

...then he can predict that, if Alice measures *F*, she always obtains 1



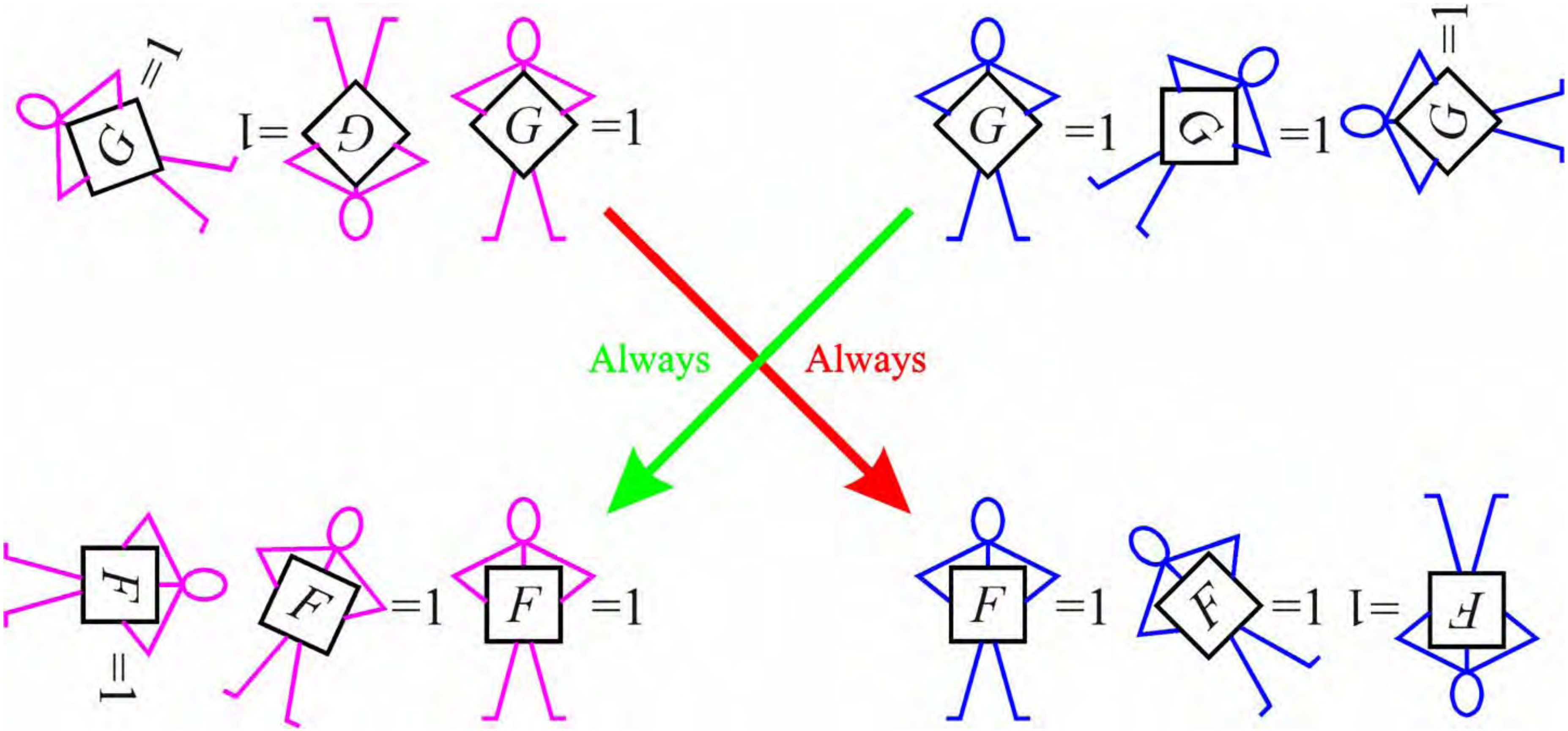
Rotationally invariant proof

Even if Alice rotates her apparatus!



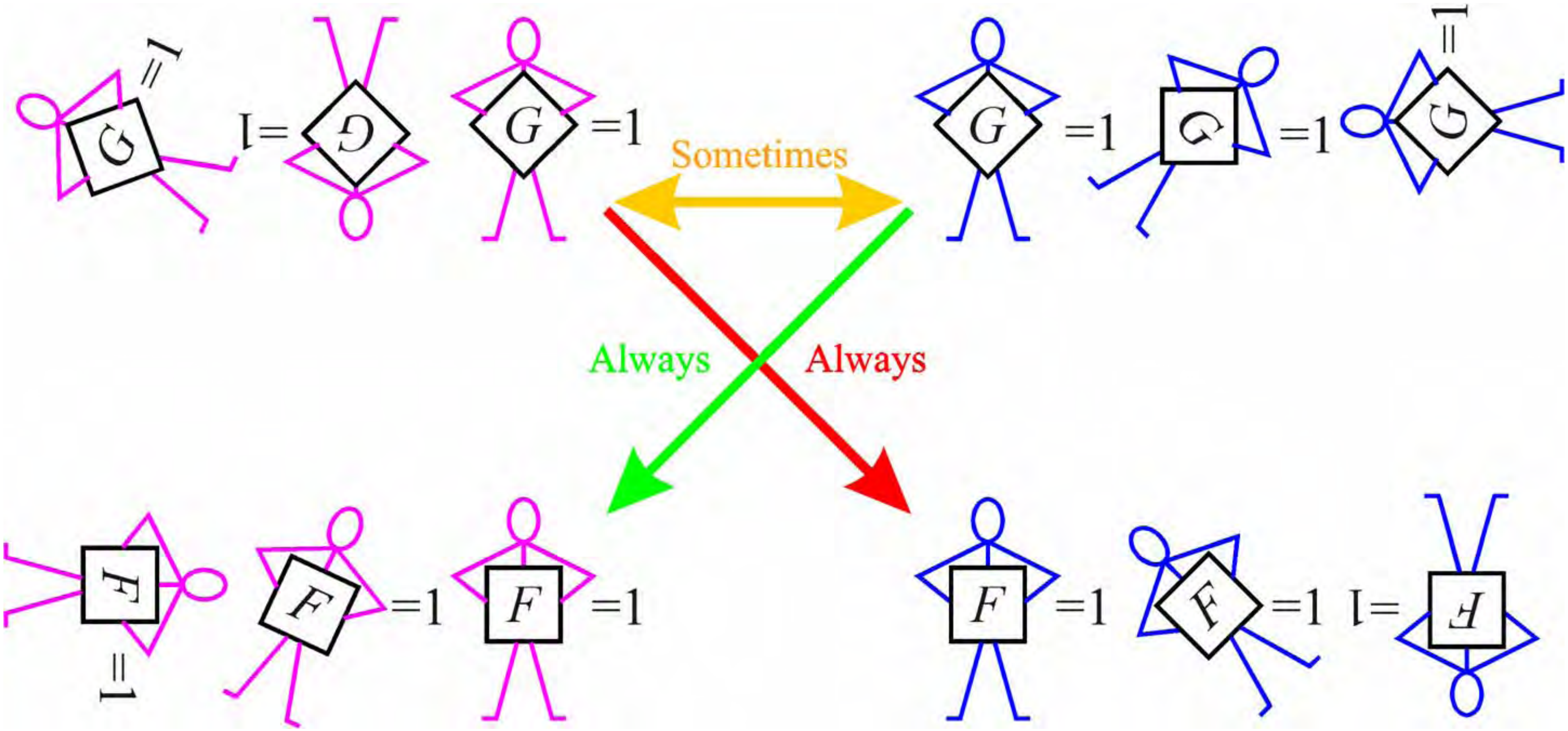
Rotationally invariant proof

...or Bob!



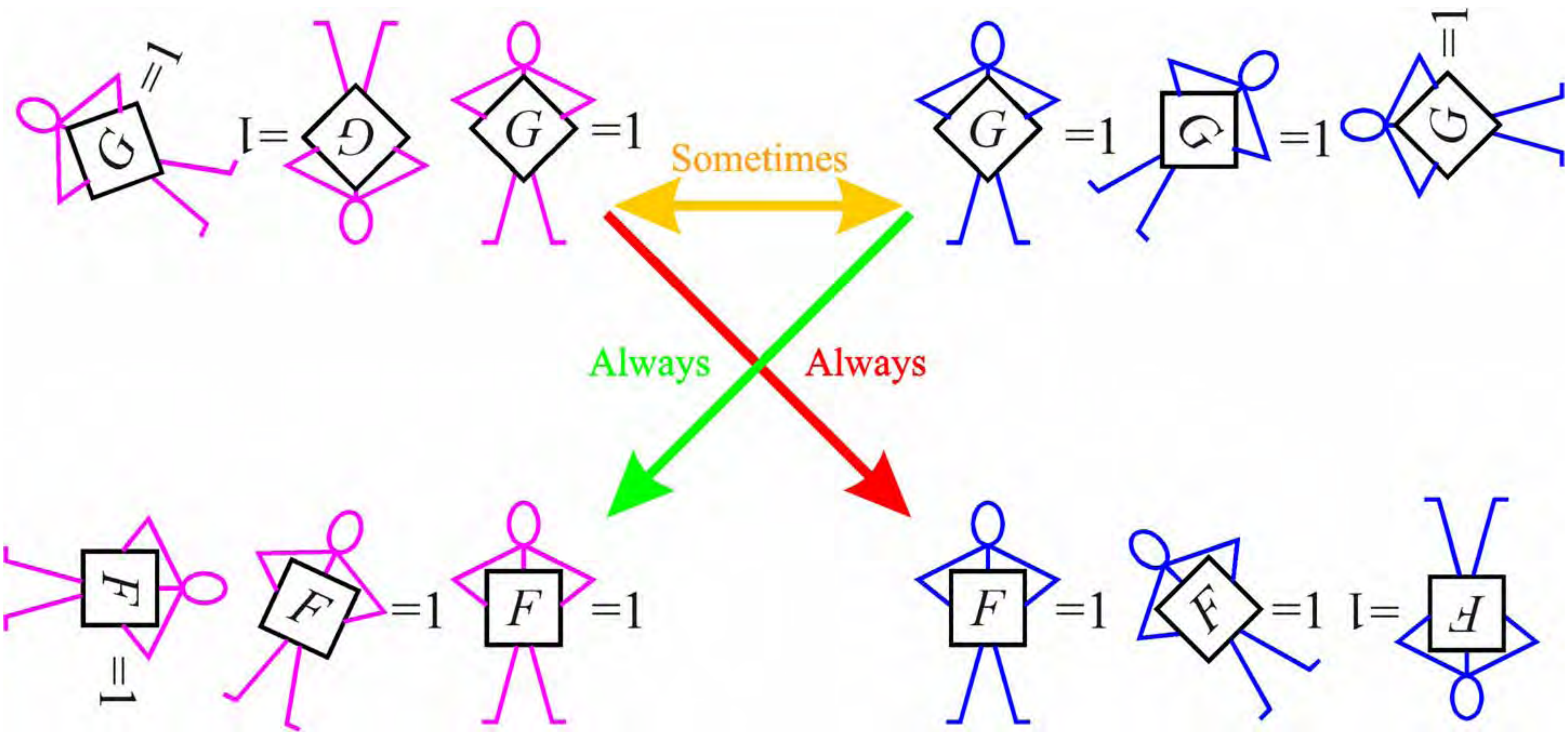
Rotationally invariant proof

If Alice and Bob measure *G*, sometimes (in 8% of the cases) they both obtain 1...



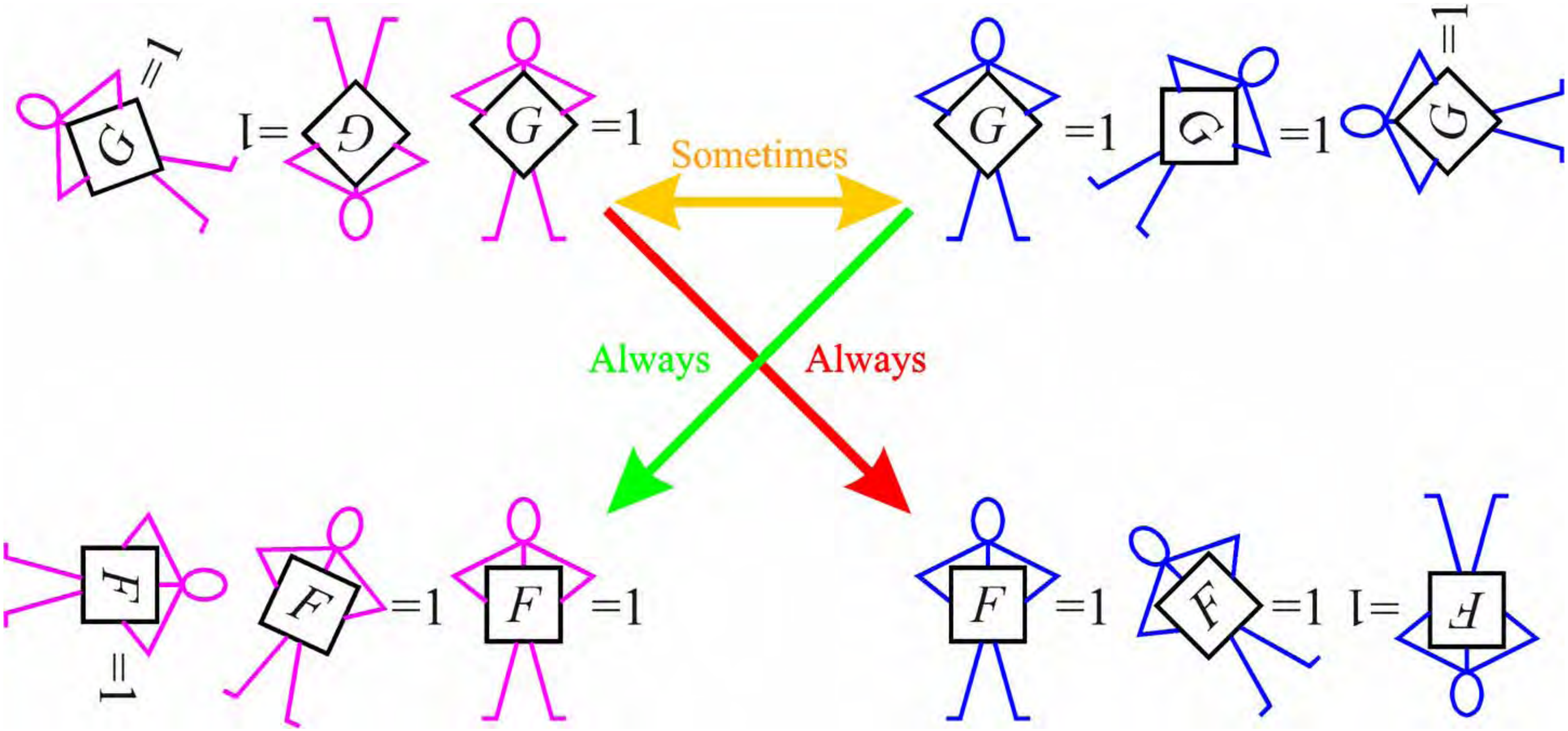
Rotationally invariant proof

In those cases, what if, instead of measuring *G*, they had measured *F*?



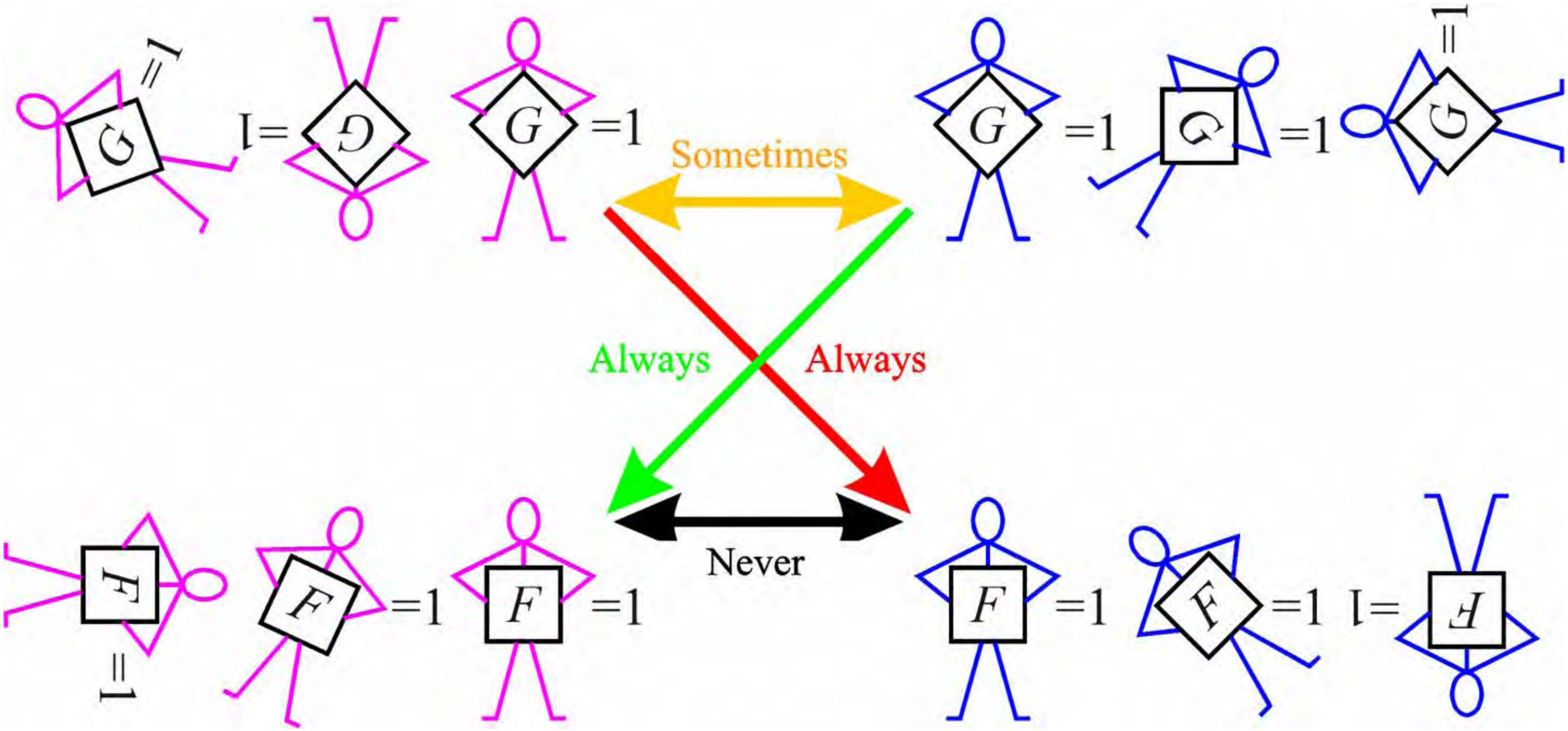
Rotationally invariant proof

If EPR's elements of reality do exist, then, at least in 8% of the cases, both of them would have obtained *F*=1

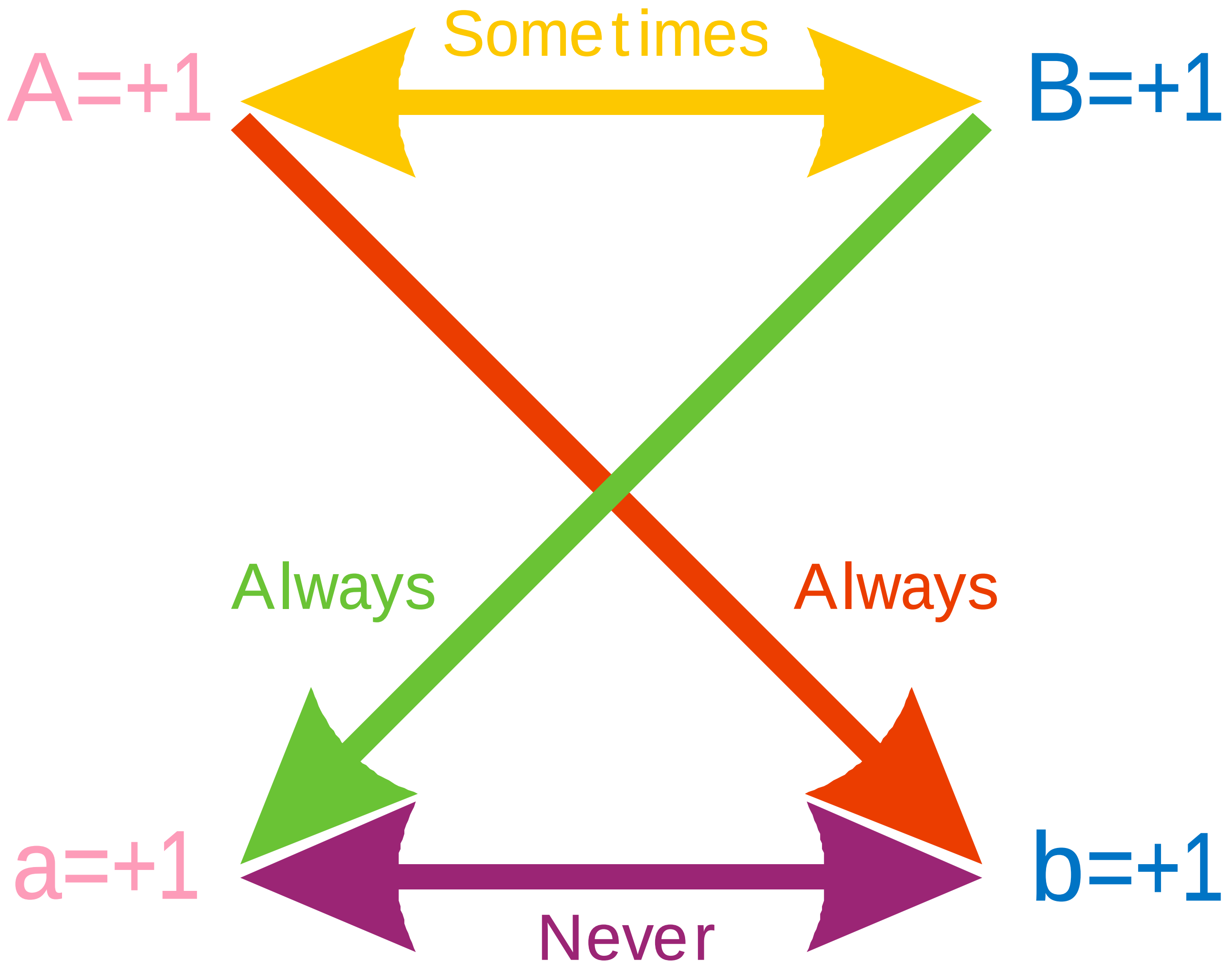


Rotationally invariant proof

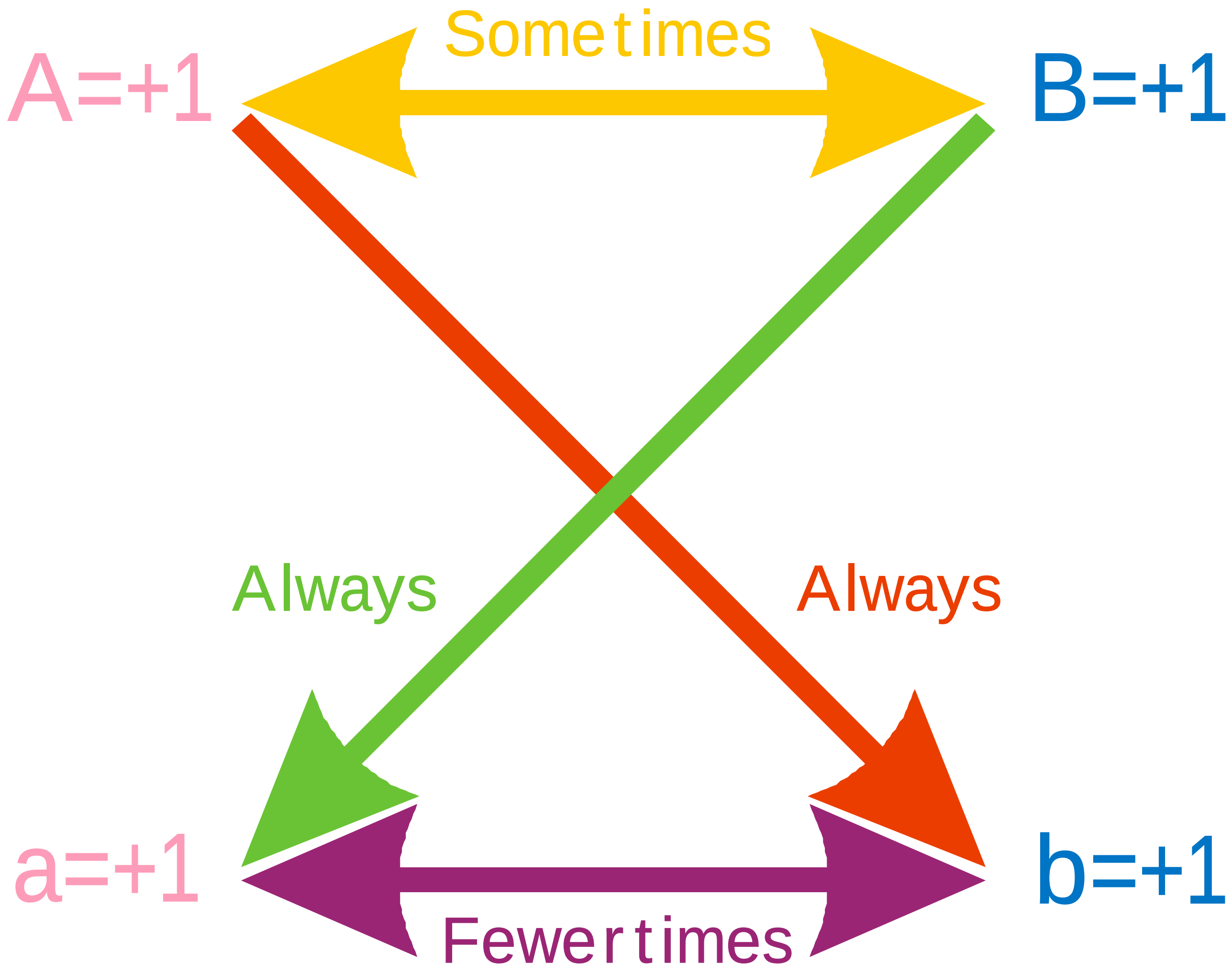
However, they NEVER both obtain 1!!!



Hardy's nonlocality proof



Extended Hardy's nonlocality proof



Hardy's is a particular case of the CHSH inequality

$$-1 \le P(A_0 = 1, B_0 = 1) - P(A_0 = 1, B_1 = -1) - P(A_1 = -1, B_0 = 1) - P(A_1 = -1, B_1 = -1) \le 0$$

$$\left| \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \right| \le 2$$

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Bell inequalities and quantum information

- Entanglement witnesses
- State analysis and discrimination
- Security of quantum key distribution
- Entanglement-assisted reduction of communication

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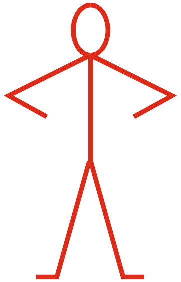
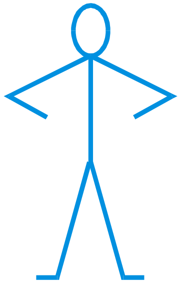
Entanglement-assisted reduction of communication

Rules





Entanglement-assisted reduction of communication

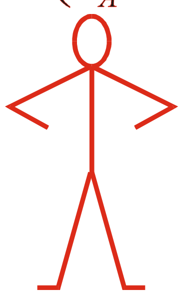
Rules

Entanglement-assisted reduction of communication

Rules

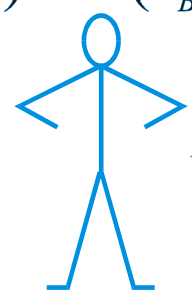
$P(x_A = 0) = P(x_A = 1) = 1/2$

$x_A \in \{0,1\}$



$P(x_B = 0) = P(x_B = 1) = 1/2$

$x_B \in \{0,1\}$

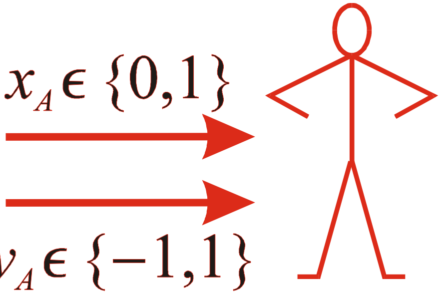


Entanglement-assisted reduction of communication

Rules

$P(x_A = 0) = P(x_A = 1) = 1/2$

$x_A \in \{0,1\}$

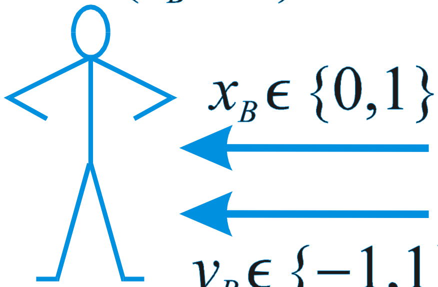


$y_A \in \{-1,1\}$

$P(y_A = -1) = P(y_A = 1) = 1/2$

$P(x_B = 0) = P(x_B = 1) = 1/2$

$x_B \in \{0,1\}$



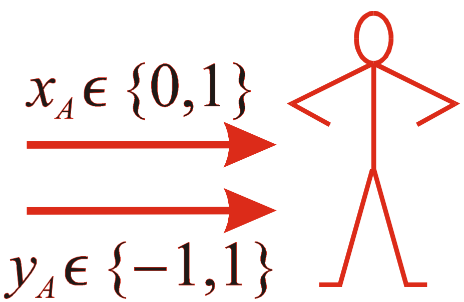
$y_B \in \{-1,1\}$

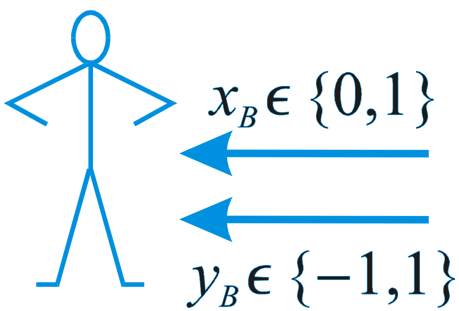
$P(y_A = -1) = P(y_A = 1) = 1/2$

Entanglement-assisted reduction of communication

Rules

$$f = y_A y_B (-1)^{xA\,xB}$$

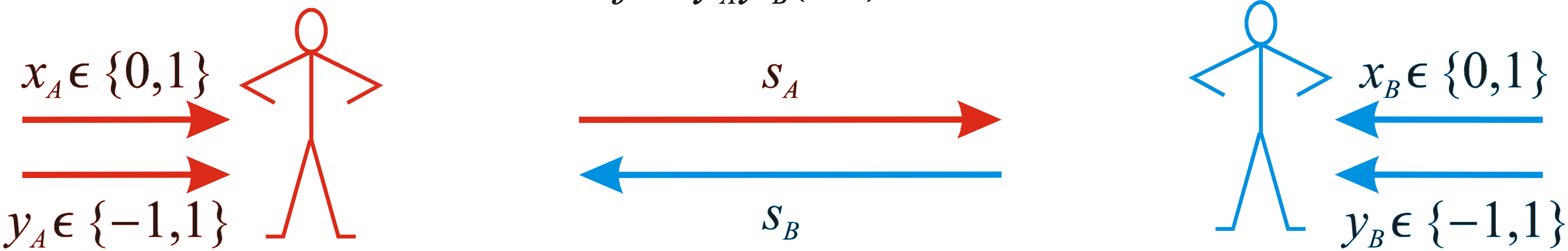




Entanglement-assisted reduction of communication

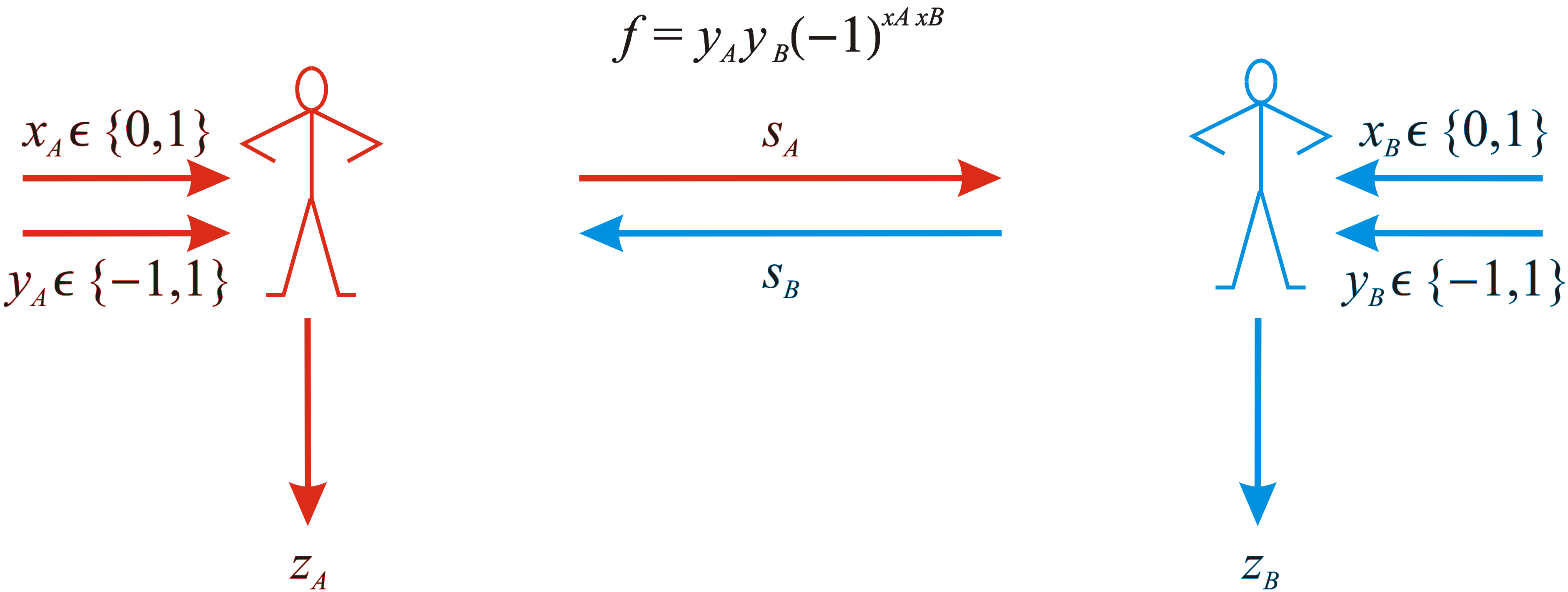
Rules

$$f = y_A y_B (-1)^{xA\,xB}$$



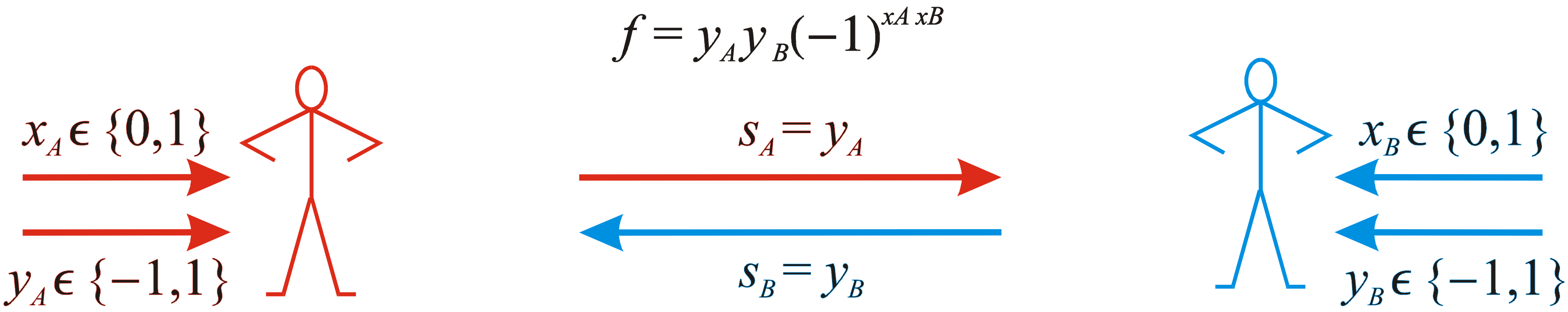
Entanglement-assisted reduction of communication

Rules



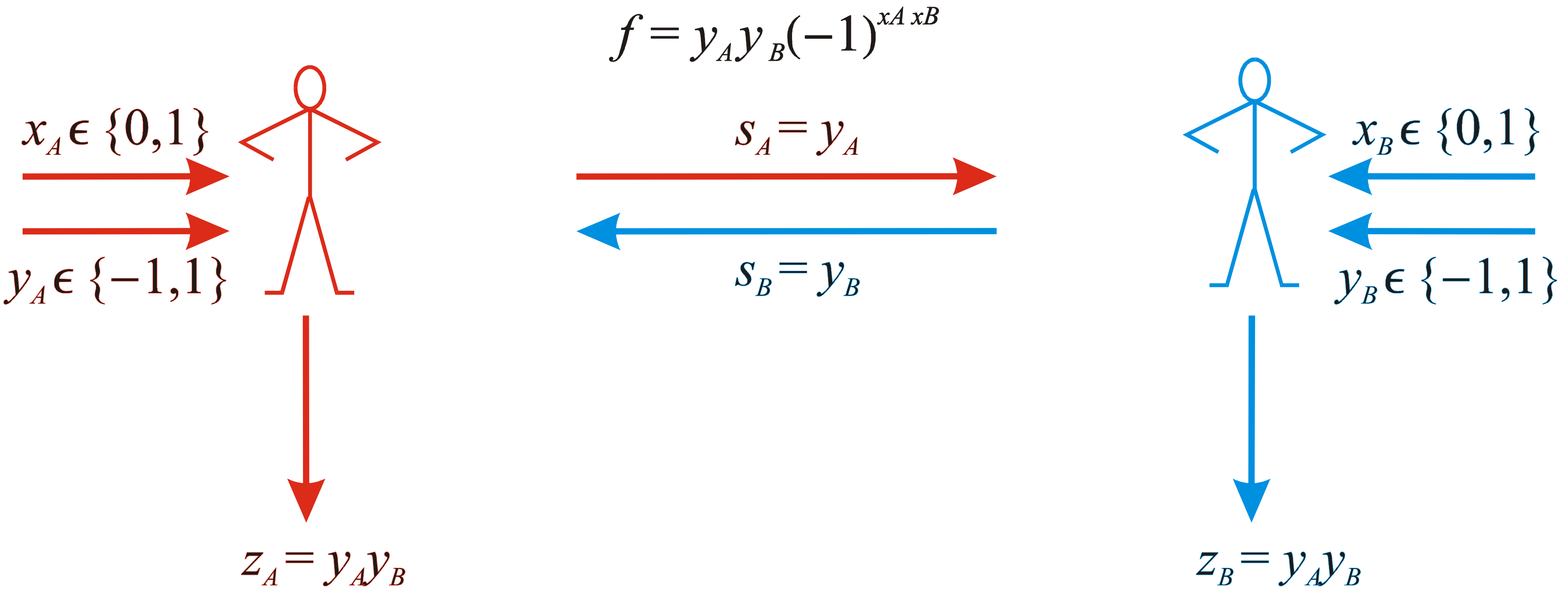
Entanglement-assisted reduction of communication

Optimal classical protocol



Entanglement-assisted reduction of communication

Optimal classical protocol



Entanglement-assisted reduction of communication

Optimal classical protocol

$$P_{\mathrm{win}} = \frac{1}{4}\left[1+1+1+0\right] = \frac{3}{4} = 0.75$$

Entanglement-assisted reduction of communication

Quantum protocol

If Alice and Bob share pairs in the state

$$\left|\psi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle - \left|10\right\rangle\right)$$

Entanglement-assisted reduction of communication

Quantum protocol

Entanglement-assisted reduction of communication

Quantum protocol

Entanglement-assisted reduction of communication

Quantum protocol

Entanglement-assisted reduction of communication

Quantum protocol

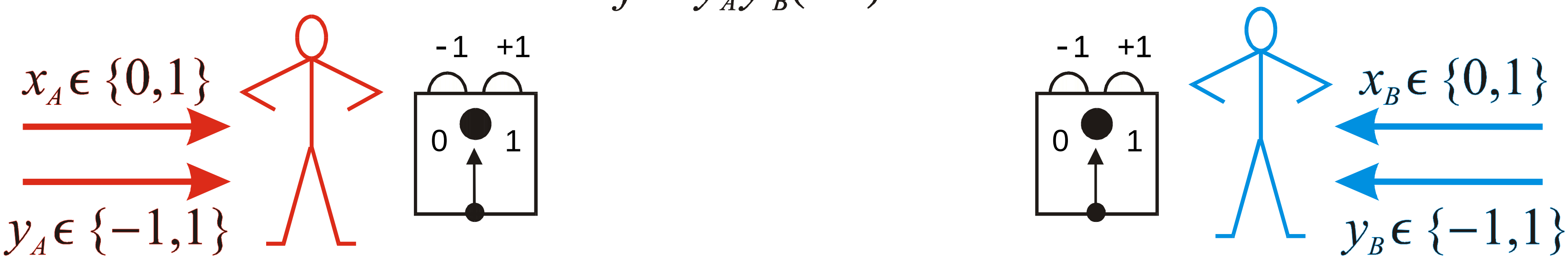
Entanglement-assisted reduction of communication

Quantum protocol

Entanglement-assisted reduction of communication

Quantum protocol

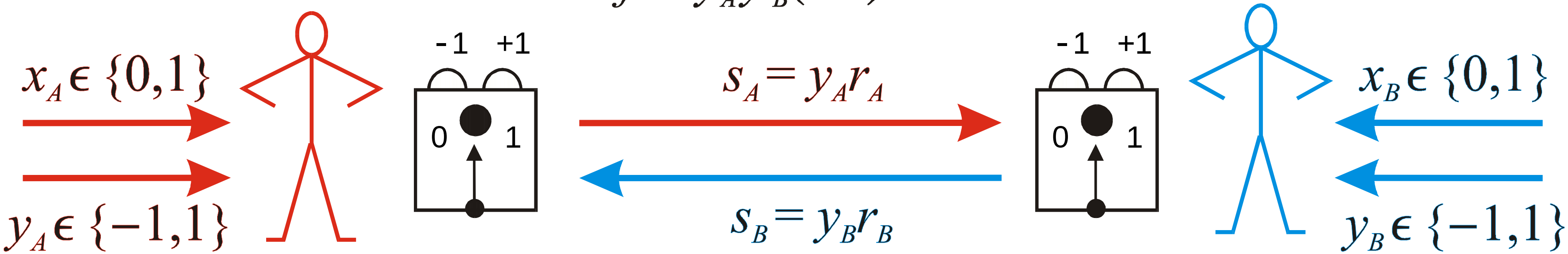
$$f = y_A y_B (-1)^{xA\, xB}$$



Entanglement-assisted reduction of communication

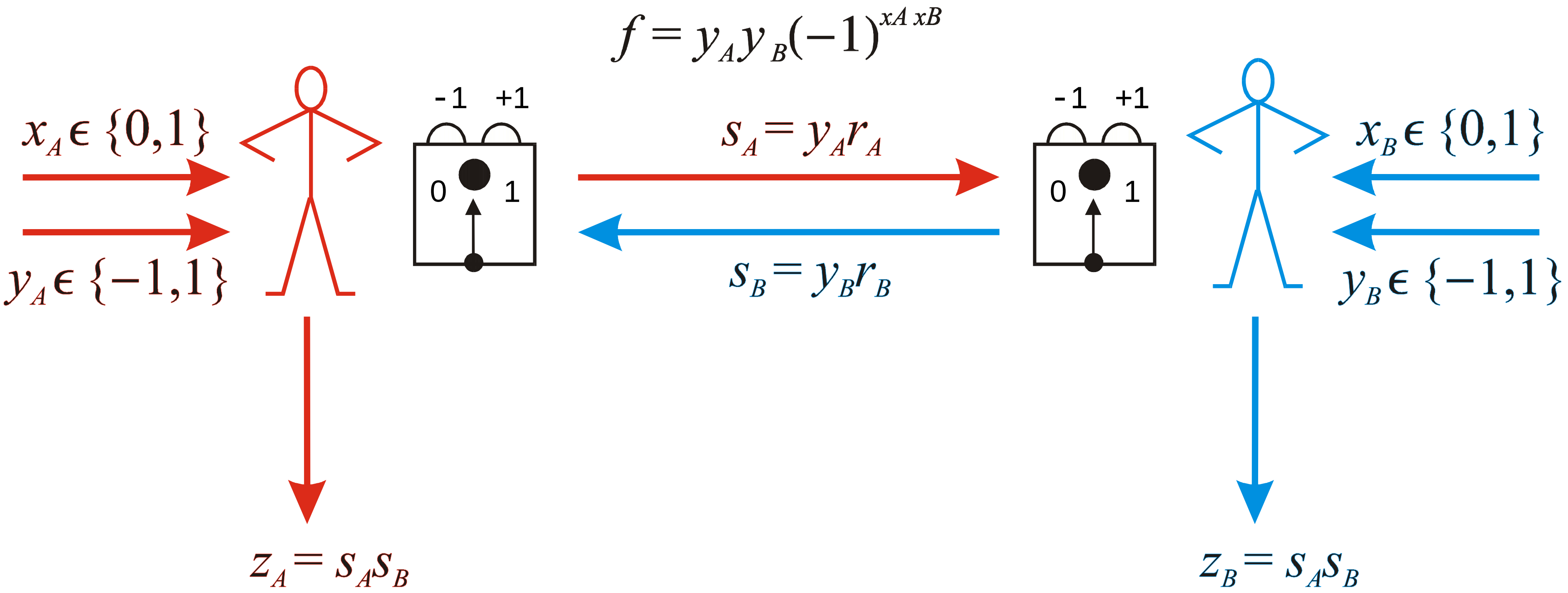
Quantum protocol

$$f = y_A y_B (-1)^{x_A x_B}$$



Entanglement-assisted reduction of communication

Quantum protocol



Entanglement-assisted reduction of communication

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

$$A_0 = \sigma_x \qquad B_0 = -\left(\sigma_x + \sigma_y\right)/\sqrt{2}$$

$$A_1 = \sigma_y \qquad B_1 = \left(\sigma_y - \sigma_x\right)/\sqrt{2}$$

$$P_{\text{win}} = \frac{1}{4}\left[P\left(A_0B_0 = 1\right) + P\left(A_0B_1 = 1\right) + P\left(A_1B_0 = 1\right) + P\left(A_1B_1 = -1\right)\right]$$

$$= \frac{1}{2} + \frac{1}{8}\left(\langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) \approx 0.85$$

In the next lecture

- GHZ: Beating EPR using their own weapons (perfect correlations)
- Mermin inequality: Violation that grows exponentially
- Bipartite AVN
- Bipartite AVN with only single-qubit measurements