ROLE OF COVARIANCE MATRIX IN SYMMETRIC MULTIQUBIT SYSTEMS

R. Prabhu[1,2], A. R. Usha Devi[2,3], M. S. Uma[2] and A. K. Rajagopal[3,4]

[1]Department of Physics, Kuvempu University, Shankaraghatta, Shimoga,
Karnataka, India – 577 451

[2]Department of Physics, Bangalore University, Bangalore,
Karnataka, India – 560 056

[3]Inspire Institute Inc. McLean, VA 22101, USA

[4]Department of Computer Science and Center for Quantum Studies,
George Mason University, VA 22030, USA

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"When two systems, of which we know the states by their respective representatives, enter into temporary **physical interaction** due to known forces between them, and when after a time of mutual influence the **systems separate again**, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would **not call that one** but rather the **characteristic trait of quantum mechanics**, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives have become **entangled**."

For the long period from 1935 to 1964, until Bell's work was published [J. S. Bell, Physics **1**, 195 (1964)] discussions about entanglement were purely meta-theoretical.

Quantum information theory has established entanglement as a physical resource and key ingredient for:

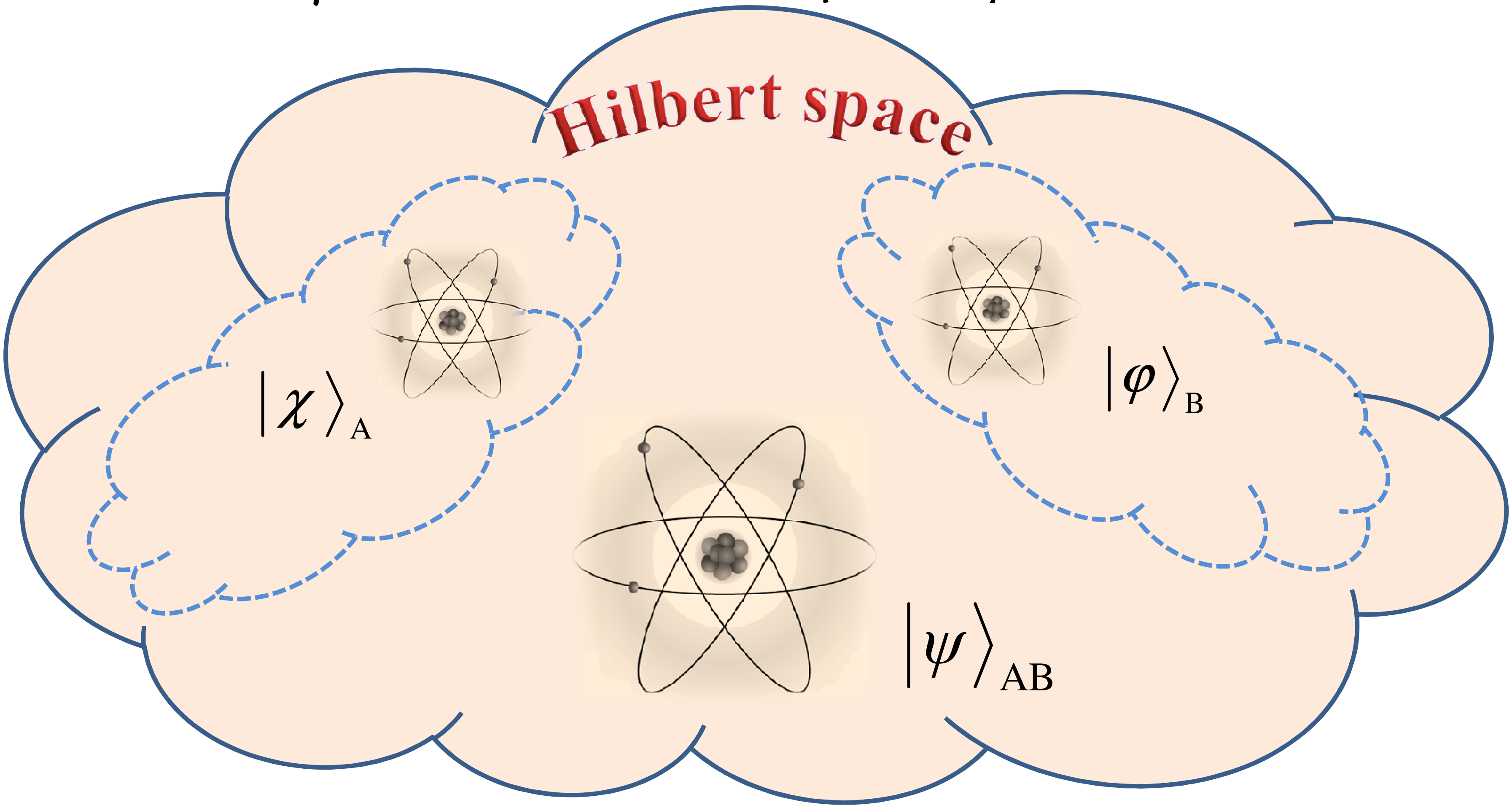
Quantum Computation

Quantum Teleportation

Quantum Cryptography

Quantum Communication …

Entanglement refers to the non-separability of quantum states of composite systems



$$|\psi\rangle_{AB} = |\chi\rangle_A \otimes |\varphi\rangle_B \Longrightarrow$$ **Separable**

$$|\psi\rangle_{AB} \neq |\chi\rangle_A \otimes |\varphi\rangle_B \Longrightarrow$$ **ENTANGLED**

Mathematical description of entanglement is necessary and involves its

CHARACTERIZATION

MANIPULATION

QUANTIFICATION

Study of **Entanglement Measures** serves as characterization and quantification of Entanglement

- Mathematical quantity that should capture the essential feature that we associate with entanglement
- Ideally should be related to some operational procedure

Pure states

For arbitrary pure bipartite state $\hat{\rho}_{AB}$ the entropy of entanglement $E(\hat{\rho}_{AB})$, namely von Neumann entropy of the reduced density matrix $\hat{\rho}_{A,B} = \mathrm{Tr}_{B,A}\hat{\rho}_{AB}$ is given by

$$\boxed{S = -\mathrm{Tr}\,\hat{\rho}_A \log \hat{\rho}_A = -\mathrm{Tr}\,\hat{\rho}_B \log \hat{\rho}_B}$$

serves as a good measure of entanglement

$S = 0 \longrightarrow$ SEPARABLE

$S \neq 0 \longrightarrow$ ENTANGLED

Mixed states

$$\hat{\rho} = \sum_i P_i \hat{\rho}^i_A \otimes \hat{\rho}^i_B \longrightarrow \text{SEPARABLE}$$

Classically correlated states

$$\hat{\rho} \neq \sum_i P_i \hat{\rho}^i_A \otimes \hat{\rho}^i_B \longrightarrow \text{ENTANGLED}$$

Can’t be represented as a arbitrary convex combination of direct product of single qubit states

Mixed states

Due to interaction of states with environment we always have mixed states in our labs

Greater difficulty !

Several entanglement measures have been proposed

Entanglement of formation $E_F(\hat{\rho}_{\mathrm{AB}})$

Entanglement cost $E_C(\hat{\rho}_{\mathrm{AB}})$

Quantify asymptotic pure-state required to create $\hat{\rho}_{\mathrm{AB}}$

Distillable entanglement $E_D(\hat{\rho}_{\mathrm{AB}})$

Quantify the states which can be extracted from $\hat{\rho}_{\mathrm{AB}}$

Relative entropy of Entanglement

Related measure that interpolates between $E_C(\hat{\rho}_{\mathrm{AB}})$ and $E_D(\hat{\rho}_{\mathrm{AB}})$

Peres' PPT criterion

"An elegant criterion to check whether a given state is entangled or not is given by Peres' Positivity under partial transpose (PPT) of $\hat{\rho}_{\mathrm{AB}}$ "

A. Peres, Phys. Rev. Lett. **77**, 1473 (1996)

$$\boxed{\hat{\rho}_{\mathrm{AB}}^{\mathrm{T_A}}\ (\mathrm{or}\ \hat{\rho}_{\mathrm{AB}}^{\mathrm{T_B}})\ \geq 0}$$ ***SEPARABLE***

where $\hat{\rho}^{\mathrm{T_A}}_{i_{\mathrm{A}} j_{\mathrm{A}};i_{\mathrm{B}} j_{\mathrm{B}}} = \hat{\rho}_{i_{\mathrm{A}} j_{\mathrm{B}};i_{\mathrm{B}} j_{\mathrm{A}}}$

"PPT criterion is necessary and sufficient for separability in the 2×2 and 2×3 dimensional cases, but ceases to be sufficient condition in higher dimensions "

M. Horodecki, et.al, Phys. Lett. A **223**, 1 (1996)

Geometric interpretation of entanglement

for infinite dimension systems

"In continuous variable states (CV) the partial transpose operation acquires a beautiful geometric interpretation as mirror reflection in phase space".

R. Simon, Phys. Rev. Lett. **84**, 2726 (2000)

Using phase space variables and the Hermitian cannonical operators we have

$$\hat{\xi} = \begin{pmatrix} \hat{q}_1 \\ \hat{p}_1 \\ \hat{q}_2 \\ \hat{p}_2 \end{pmatrix}; \quad \hat{q}_i = \frac{a_i + a_i^\dagger}{2}, \quad \hat{p}_i = \frac{a_i - a_i^\dagger}{2i}; \quad i = 1,2$$

Commutation relation will be

$$[\hat{\xi}_\alpha, \hat{\xi}_\beta] = \mathrm{i}\Omega_{\alpha\beta}; \ \alpha, \beta = 1,2,3,4; \ \Omega = \begin{pmatrix} \mathrm{J} & 0 \\ 0 & \mathrm{J} \end{pmatrix}, \mathrm{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

4X4 real variance matrix

$$V_{\alpha\beta} = \frac{1}{2}\left\langle \left\{ \Delta\hat{\xi}_{\alpha}, \Delta\hat{\xi}_{\beta} \right\} \right\rangle, \quad V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$$

where $\Delta\hat{\xi}_{\alpha} = \hat{\xi}_{\alpha} - \left\langle \hat{\xi}_{\alpha} \right\rangle$

$$\left\{ \Delta\hat{\xi}_{\alpha}, \Delta\hat{\xi}_{\beta} \right\} = \Delta\hat{\xi}_{\alpha}\Delta\hat{\xi}_{\beta} + \Delta\hat{\xi}_{\beta}\Delta\hat{\xi}_{\alpha}$$

Under canonical transformation

preserves the commutation relation

$$\hat{\xi} \rightarrow \hat{\xi}' = S\hat{\xi}; \quad S \in SP(4R)$$

$$V \rightarrow V' = SVS^T$$

Entanglement properties are unaltered

Elements of variance matrix under canonical transformation

$$A \to A' = SAS^T$$
$$B \to B' = SBS^T$$
$$C \to C' = SCS^T$$

$$I_1 = \det A$$
$$I_2 = \det B$$
$$I_3 = \det C$$
$$I_4 = \mathrm{Tr}\,[AJCJBJC^T J]$$

Peres-Horodecki PPT criterion:

$$I_1 I_2 + \left(\tfrac{1}{4} - |I_3|\right)^2 - I_4 \geq \tfrac{1}{4}\left(I_1 + I_2\right)$$

Gaussian states with $\det C \geq 0$ are necessarily separable, where as those with $\det C < 0$ and violating above inequality are entangled

OUTLINE OF OUR WORK

- Identification of structural parallelism between CV states and two-qubit states.

- Necessary and sufficient inseparability conditions imposed on variance matrix of symmetric two qubits.

- Extension of covariance matrix method to symmetric *N*-qubit state – when individual qubits are accessible

- Collective multipole like signatures of entanglement is symmetric *N*-qubit systems (involves bulk observables, generalizes the concept of spin squeezing)

Arbitrary two qubit density operator in the Hilbert space $\mathcal{H} = C^2 \otimes C^2$

$$\rho = \tfrac{1}{4}\left[I \otimes I + \sum_{i=x,y,z} \left(\sigma_{1i} s_{1i} + \sigma_{2i} s_{2i} \right) + \sum_{i,j=x,y,z} \sigma_{1i} \sigma_{2i} t_{ij} \right]$$

where

$$\sigma_{1i} = \sigma_i \otimes I \qquad s_{\alpha i} = \mathrm{Tr}\left(\rho \sigma_{1i} \right)$$

$$\sigma_{2i} = I \otimes \sigma_i \qquad t_{ij} = \mathrm{Tr}\left(\rho \sigma_{1i} \sigma_{2j} \right)$$

15 state parameters
$s_{1i}(3)$, $s_{2j}(3)$ and $t_{ij}(9)$

Symmetric two qubit density operator in the Hilbert space $\mathcal{H}_s = \mathrm{Sym}\left(C^2 \otimes C^2 \right)$

$$s_{1i} = s_{2i} = s_i \qquad t_{ij} = t_{ji} \qquad \mathrm{Tr}(T) = 1$$

8 state parameters

$s_{1i} = s_{2i} = s_i$ (3)

$t_{ij} = t_{ji}$ (5)

$\mathrm{Tr}(T) = 1$

Basic variables of two qubit systems are

$$\hat{\zeta} = \begin{pmatrix} \sigma_{1i} \\ \sigma_{2j} \end{pmatrix}$$

The covariance matrix is given by

$$\boxed{V_{\alpha\beta} = \tfrac{1}{2}\left\langle \{\Delta\hat{\zeta}_{\alpha i}, \Delta\hat{\zeta}_{\beta\,\mathrm{j}}\}\right\rangle; \;\alpha,\beta = 1,2 \quad i,j = x, y, z}$$

In 3×3 block form

$$V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$$

where
$$A_{ij} = \tfrac{1}{2}\left[\left\langle \sigma_{1i}, \sigma_{1j} \right\rangle - \left\langle \sigma_{1i} \right\rangle \left\langle \sigma_{1j} \right\rangle\right] = \delta_{ij} - s_{1i}s_{1j} = I - s_1 s_1^T$$
$$B_{ij} = \tfrac{1}{2}\left[\left\langle \sigma_{2i}, \sigma_{2j} \right\rangle - \left\langle \sigma_{2i} \right\rangle \left\langle \sigma_{2j} \right\rangle\right] = \delta_{ij} - s_{2i}s_{2j} = I - s_2 s_2^T$$
$$C_{ij} = \tfrac{1}{2}\left[\left\langle \sigma_{1i}, \sigma_{2j} \right\rangle - \left\langle \sigma_{1i} \right\rangle \left\langle \sigma_{2j} \right\rangle\right] = t_{ij} - s_{1i}s_{2j}$$

In symmetric states each block assumes the form

$$\boxed{\begin{aligned} A = B &= I - ss^T \\ C &= T - ss^T \end{aligned}}$$

A. R. Usha Devi, M. S. Uma, R. Prabhu and A. K. Rajagopal, Phys. Lett. A **364,** 203 (2007)

For separable symmetric state $C = T - ss^T \geq 0$

A two qubit separable symmetric state is given by

$$\varrho = \sum_w P_w \rho_w \otimes \rho_w; \quad \sum_w P_w = 1, \quad 0 \leq P_w \leq 1$$

where $\rho_w = \frac{1}{2}\left(1 + \sum_{i=x,y,z} \sigma_i s_{iw}\right)$ single qubit density matrix

The state variables are given by

$$s_i = \langle \sigma_{\alpha i} \rangle = Tr(\varrho \sigma_{\alpha i}) = \sum_w p_w s_{iw}$$

$$t_{ij} = \langle \sigma_{1i} \sigma_{2i} \rangle = Tr(\varrho \sigma_{1i} \sigma_{2i}) = \sum_w p_w s_{iw} s_{jw}$$

Evaluating the quadratic form $n^T\left(T - ss^T\right)n$

$$n^T\left(T - ss^T\right)n = \sum_{i,j}\left(t_{ij} - s_i s_j\right)n_i n_j$$

$$= \sum_{i,j}\left[\sum_w p_w s_{iw} s_{jw} - \sum_w p_w s_{iw} \sum_{w'} p_{w'} s_{iw'}\right] n_i n_j$$

$$= \sum_w p_w (\vec{s}\cdot\hat{n})^2 - \left(\sum_w p_w (\vec{s}\cdot\hat{n})\right)^2$$

which is of the form $\left\langle A^2 \right\rangle - \left\langle A \right\rangle^2$

$$\Longrightarrow \quad n^T\left(T - ss^T\right)n \geq 0$$

"off diagonal block C of the covariance matrix is necessarily positive semi-definite for separable symmetric states"

$C < 0 \Longrightarrow$ **Entangled**

The necessary condition for the inseparability of an arbitrary symmetric two qubit mixed state is given by $C < 0$

$$\rho = \tfrac{1}{4}\left[I \otimes I + \sum_{i=x,y,z} \left(\sigma_{1i} s_{1i} + \sigma_{2i} s_{2i} \right) + \sum_{i,j=x,y,z} \sigma_{1i} \sigma_{2i} t_{ij} \right]$$

Unitary operation involving Clebsch Gordan coefficients

$$\rho = \begin{pmatrix} \rho_s & 0 \\ 0 & 1 \end{pmatrix}$$

Following PT operation by a local rotation about the spin operators of the second qubit completely reverse their signs:

$$\boxed{\sigma_{2y} \rightarrow -\sigma_{2y}}$$

$$\rho^{PT} = \tfrac{1}{4}\left[I \otimes I + \sum_{i=x,y,z} \left(\sigma_{1i} s_{1i} - \sigma_{2i} s_{2i} \right) - \sum_{i,j=x,y,z} \sigma_{1i} \sigma_{2i} t_{ij} \right]$$

Unitary operation involving Clebsch Gordan coefficients

$$\rho_s^{T_2} = \tfrac{1}{2} \begin{pmatrix} T & s \\ s^T & 1 \end{pmatrix}$$

with the 3×3 real symmetric correlation matrix T, and 3×1 column s

Applying a congruence operation on partially transposed density matrix we get

$$L\rho_s^{T_2}L^{\dagger} = \tfrac{1}{2}\begin{pmatrix} T - ss^T & 0 \\ 0 & 1 \end{pmatrix}$$

where congruence operator is $L = \begin{pmatrix} I & -s \\ 0 & 1 \end{pmatrix}$

$$\rho_s^{T_2} < 0 \quad \Leftrightarrow \quad C = T - ss^T < 0$$

"Necessary condition for the inseparability of an arbitrary symmetric two qubit mixed state"

Realizing inseparability criterion $C < 0$ in symmetric N-qubit systems

Collective observables of a N-qubit system are given by

$$\vec{J} = \sum_{\alpha=1}^{N} \tfrac{1}{2}\vec{\sigma}_{\alpha}$$

Collective correlation matrix involving first and second moments of $\vec{J}$ may be defined as

$$\boxed{V_{ij}^{(N)} = \tfrac{1}{2}\left\langle J_i J_j + J_j J_i \right\rangle - \left\langle J_i \right\rangle \left\langle J_j \right\rangle; \quad i, j = x, y, z}$$

Collective observables when expressed in terms of the constituent qubit variables as

$$\tfrac{1}{2}\left\langle J_i J_j + J_j J_i \right\rangle = \tfrac{1}{4}\sum_{\alpha,\beta=1}^{N}\left\langle \sigma_{\alpha i}\sigma_{\beta j} \right\rangle = \tfrac{N}{4}\left[\delta_{ij} + (N-1)t_{ij}\right]$$

$$\left\langle J_i \right\rangle = \tfrac{1}{2}\sum_{\alpha=1}^{N}\left\langle \sigma_{\alpha i} \right\rangle = S_i = \tfrac{N}{2} s_i$$

Correlation matrix $V^{(N)}$ assumes the form

$$V^{(N)} = \tfrac{N}{4}\left(I - ss^T + (N-1)\left(T - ss^T\right)\right)$$

OR

$$\boxed{V^{(N)} + \tfrac{1}{N} SS^T = \tfrac{N}{4}\left(I + (N-1)C\right)}$$

using $\langle J_i \rangle = \tfrac{N}{2} s_i = S_i$

C is positive semi-definite for all separable symmetric two-qubit states $\Longrightarrow$ $V^{(N)} + \tfrac{1}{N} SS^T < \tfrac{N}{4} I$

Under identical local unitary transformations $U \otimes U \otimes .... \otimes U$

$$V^{(N)'} = O V^{(N)} O^T \quad \text{and} \quad S' = OS$$

The symmetric N-qubit system is pairwise entangled iff the least eigenvalue of the real symmetric matrix

$$V^{(N)} + \tfrac{1}{N} SS^T > \tfrac{N}{4}$$

Characterizing multiparticle entanglement in symmetric N-qubit states

An arbitrary N-qubit system density matrix:

$$\rho = \frac{1}{2^N} \sum T_{\alpha_1 \alpha_2 \ldots \alpha_N} (\sigma_{1\alpha_1} \sigma_{2\alpha_2} \ldots \sigma_{N\alpha_N})$$

where,

$$\sigma_{\mu\alpha} = (I \otimes I \otimes \ldots \otimes \sigma_\alpha \otimes I \otimes \ldots)$$

α appearing at μ^{th} position

$$T_{\alpha_1 \alpha_2 \ldots \alpha_N} = \mathrm{Tr}[\rho(\sigma_{1\alpha_1} \sigma_{2\alpha_2} \ldots \sigma_{N\alpha_N})]$$

$$= \left\langle \sigma_{1\alpha_1} \sigma_{2\alpha_2} \ldots \sigma_{N\alpha_N} \right\rangle$$

Total number of parameters of the density matrix

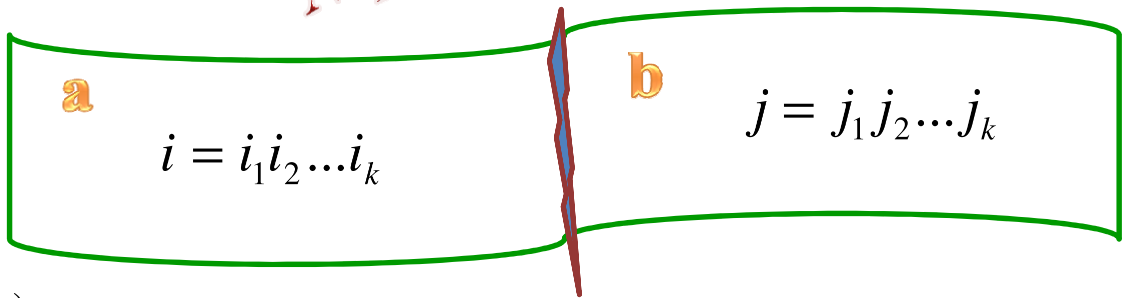
$$2^{2N} - 1$$

permutation symmetry

$$(N+1)^2 - 1$$

A. R. Usha Devi, R. Prabhu and A. K. Rajagopal,
Phys. Rev. Lett. **98,** 060501 (2007)

N qubit system



Moments $T_{ij}^{(2k)}$ of even order *2k* may be arranged as

$$T_{ij}^{(2k)} = T_{i_1 i_2 \ldots i_k ; j_1 j_2 \ldots j_k}^{(2k)} \quad \& \quad T_i^{(k)} = T_{i_1 i_2 \ldots i_k}^{(k)}$$

k qubit operator associated with each groups:

$$\boxed{\begin{aligned} \hat{A}_i^{(k)} &= \sigma_{a_1 i_1} \sigma_{a_2 i_2} \ldots \sigma_{a_k i_k} \\ \hat{B}_i^{(k)} &= \sigma_{b_1 j_1} \sigma_{b_2 j_2} \ldots \sigma_{b_k j_k} \end{aligned}} \Rightarrow \hat{\xi}^{(k)} = \begin{pmatrix} \hat{A}^{(k)} \\ \hat{B}^{(k)} \end{pmatrix}$$

2kth order variance matrix is given by

$$\boxed{V^{(2k)} = \tfrac{1}{2}[\Delta\hat{\xi}^{(k)} \Delta\hat{\xi}^{(k)\dagger} + H.C]}$$

Matrix form:

$$V^{(2k)} = \begin{pmatrix} \hat{A}^{(k)} & \hat{C}^{(k)} \\ \hat{C}^{(k)T} & \hat{B}^{(k)} \end{pmatrix}$$

Off diagonal block

$$\hat{C}^{(k)}_{ij} = \left\langle \hat{A}^{(k)}_i \hat{B}^{(k)}_j \right\rangle - \left\langle \hat{A}^{(k)}_i \right\rangle \left\langle \hat{B}^{(k)}_j \right\rangle$$
$$= T^{(2k)}_{ij} - T^{(k)}_i T^{(k)}_j$$

corresponds to $2k^{\text{th}}$ order covariance among the inter group of multiqubits

For every separable symmetric multiqubit state of various order $\hat{C}^{(2k)}$, $2k \leq N$ are necessarily positive semidefinite

To establish $\hat{C}^{(2k)} < 0$ Sylvester criterion may be used

Principle minor of Hermitian matrix is negative → Matrix is said to be positive definite

Therefore a series of sufficient conditions for entanglement of *2k* qubits could be extracted from negativity of principle minors of $\hat{C}^{(2k)}$

Separable conditions involving correlation observables making our criterion as experimentally amenable

Test for inseparability conditions $\hat{C}^{(2k)} < 0$

N qubit GHZ state

$$|GHZ\rangle_N = \tfrac{1}{\sqrt{2}}\left(|00\ldots0\rangle + |11\ldots1\rangle\right)$$

$C^{(N)}$ has one negative eigenvalue:

$$\lambda^{(-)} = -2^{\left[\left(N/2\right)-1\right]}$$

The diagonal element of $C^{(N)}$ with index $i = \{xxx\ldots xy\}$ will be negative

$$C_{ii}^{(N)} = T_{ii}^{N} - \left(T_i^{N/2}\right)^2$$

$$= \begin{cases} -1, & \text{if } \mathrm{N/2} = \text{even integer} \\ -2, & \text{if } \mathrm{N/2} = \text{odd integer} \end{cases}$$

N qubit W state

$$|W\rangle_N = \frac{1}{\sqrt{N}}\left(|10\ldots0\rangle + |010\ldots0\rangle + \ldots\right)$$

$C^{(2k)}$ has one negative eigenvalue:

$$\lambda^{(-)} = -\frac{2k(k-1)}{N^2}$$

2k-qubit entanglement for all values of *k*

We have observed that the diagonal element of $C_{ii}^{(2k)}$ with index $i = \{zzz\ldots z\}$ will be negative

$$C_{ii}^{(N)} = T_{ii}^{(2k)} = -1$$

N qubit mixed state

$$\rho^{(N)}_{Noisy} = \frac{1-x}{N+1} P_N + x|\psi\rangle\langle\psi|; 0 \le x \le 1$$

$$P_N = \sum_{M=-N/2}^{N/2} \left|\tfrac{N}{2} M\right\rangle\left\langle\tfrac{N}{2} M\right|$$

Projection operator

GHZ noisy state:

$N = 2$	$0.25 < x \le 1$
$N = 4$	$0.0625 < x \le 1$
$N = 6$	$0.014 < x \le 1$

W noisy state:

$N = 2$	$0.25 < x \le 1$
$N = 4$	$0.089 < x \le 1$
$N = 6$	$0.042 < x \le 1$

For large *N*, both will remain entangled for all values of x

A more general trend is found by examining the lowest order principal minor

GHZ noisy state:

$\frac{1}{N^2} < x \leq 1$

$T_{ii}^{N} = \frac{1-x}{N^2-1} - x$

$i = \{xxx \ldots xy\}$

Inseparability range

Negative diagonal element

Index

W noisy state:

$\frac{1}{N+2} < x \leq 1$

$T_{ii}^{N} = \frac{1-x}{N+1} - x$

$i = \{zzz \ldots z\}$

CONCLUSION

"Covariance matrix techniques are extremely useful in characterizing entanglement in symmetric multiqubit states"

Thank you