Generalized measurements to distinguish classical and quantum correlations

A. R. Usha Devi
Department of physics,
Bangalore University,
Bangalore-560 056, India

and

A. K. Rajagopal,
Department of Computer Science and
Center for Quantum Studies,
George Mason University, VA 22030, USA

Inspire Institute Inc. McLean, VA 22101, USA

International School and Conference on Quantum Information (ISCQI-08)
March 4 – 12, 2008, Bhubaneswar, Orissa, India

Outline

- Joint Probabilities, classical correlations and Shannon mutual information.
- Extension to bipartite quantum systems -

  density matrices of composite states and their marginals; von Neumann information entropy
- Measures of "quantumness of correlations": quantum discord (OZ), quantum deficit (RR) …
- Generalized measures to discern quantumness
- Summary

Two random variables are said to be correlated if their joint probability distributions cannot be expressed as a mere product of the marginal probabilities:

$$P(a,b) \neq P(a)P(b) \Longrightarrow \text{correlated}$$

Shannon Mutual information entropy:

$$H(A:B) = H(A) + H(B) - H(A,B)$$

$$= -\sum_a P(a)\log P(a) - \sum_b P(b)\log P(b) + \sum_{a,b} P(a,b)\log P(a,b)$$

$$\boxed{H(A:B) = 0 \quad \text{iff} \quad P(a,b) = P(a)P(b)}$$

Quantum description:

$$P(a,b) \rightarrow \rho_{AB}$$ Bipartite density matrix

$$P(a) \rightarrow \rho_A = Tr_B\, \rho_{AB}$$

$$P(b) \rightarrow \rho_B = Tr_A\, \rho_{AB}$$ Subsystem density matrices

Natural extension of the idea of correlation:

$$\rho_{AB} \neq \rho_A \otimes \rho_B \quad \Longrightarrow \quad \text{correlated}$$

von Neumann mutual information:

$$S(A:B) = S(A) + S(B) - S(A,B)$$

$$= S(\rho_{AB} \| \rho_A \otimes \rho_B) = -Tr\, \rho_A \log \rho_A - Tr\, \rho_B \log \rho_B$$

$$+ Tr\, \rho_{AB} \log \rho_{AB}$$

Relative entropy

$$\boxed{S(A:B) = 0 \quad \text{iff} \quad \rho_{AB} \neq \rho_A \otimes \rho_B}$$

Notion of correlation *per se* does not set a borderline between classical and quantum descriptions.

How do we distinguish between classical and quantum correlations in a bipartite quantum state?

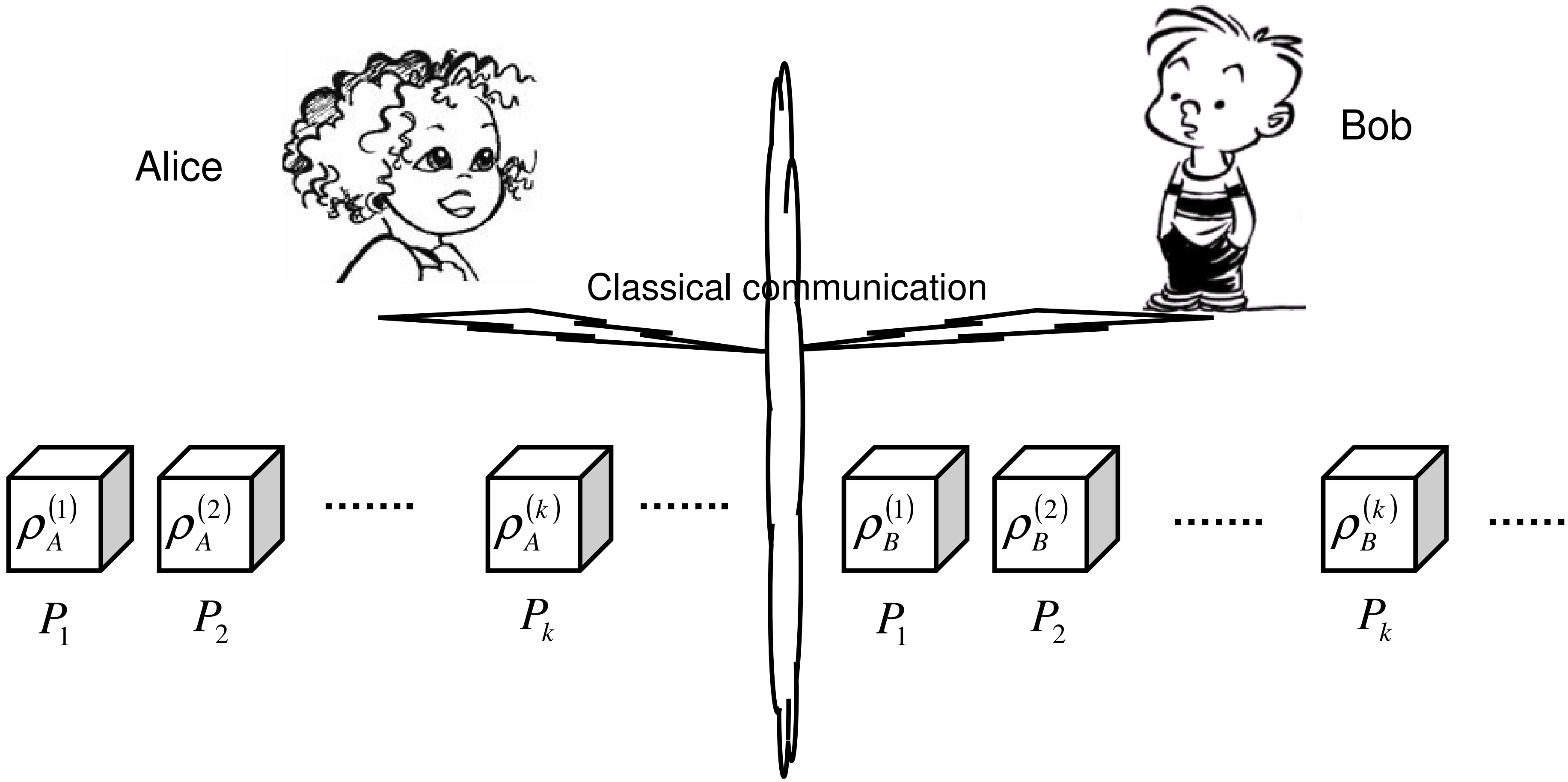
Can we express

$$S(A:B) = \begin{array}{c}\text{Contribution}\\ \text{from classical}\\ \text{correlations}\end{array} + \begin{array}{c}\text{Contribution}\\ \text{from quantum}\\ \text{correlations}\end{array} \; ?$$

R. F. Werner, Phys. Rev. A **40**, 4277 (1989)

A bipartite density operator ρ_{AB} is **classically** correlated (separable) if it admits a convex combination of product states:

$$\rho_{AB}^{(sep)} = \sum_i P_i \, \rho_A^{(i)} \otimes \rho_B^{(i)}; \quad 0 \le P_i \le 1, \quad \sum_i P_i = 1$$



Measurements on one part of the quantum system distinguishing classical and quantum correlation:

H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88,** 017901 (2001)

Measurements on one end disturbs the **quantum** correlated state in general:

$$\rho_{AB} \xrightarrow[\text{on A}]{\text{measurement}} \rho'_{AB} \neq \rho_{AB}$$

If an optimal measurement scheme (on one part) exists such that $\rho'_{AB} = \rho_{AB}$ the state is **classically correlated**

Are separable states classical?

OZ approach

Projective measurements on *A* $\{\Pi_\alpha^A \otimes I_B\}$

$$\sum_\alpha \Pi_\alpha^A = I_A \qquad \Pi_\alpha^A \Pi_{\alpha'}^A = \Pi_\alpha^A \delta_{\alpha\alpha'}$$

Completeness Orthogonality

The **conditional density operator** of subsystem *B* – when measurement $\Pi_\alpha^A \otimes I_B$ is known to have led to the value α - is given by,

$$\boxed{\rho_{AB}^\alpha = \frac{\Pi_\alpha^A \otimes I_B \;\; \rho_{AB} \;\; \Pi_\alpha^A \otimes I_B}{Tr\,[(\Pi_\alpha^A \otimes I_B\,)\,\rho_{AB}]}}$$

$$Tr\left[\left(\Pi_\alpha^A \otimes I_B\right)\rho_{AB}\right] = P_\alpha, \quad \sum_\alpha P_\alpha = 1$$

Given the results of the complete measurements $\{\Pi_\alpha^A \otimes I_B\}$ the **conditional information entropy** is given by,

$$S\left(B \mid A_{\{\Pi_\alpha^A\}}\right) = \sum_\alpha P_\alpha \, S\left(\rho_{AB}^{(\alpha)}\right)$$

positive

exhibits inherent dependence on measurements

A structural generalization of Shannon conditional entropy

$$H(B \mid A) = H(A,B) - H(A)$$

$$-\sum_{a,b} P(a,b) \log P(b \mid a)$$

This is a consequence of the **Bayes' rule**

$$\boxed{P(b \mid a) = \frac{P(a,b)}{P(a)}}$$

gives

$$\boxed{S(B \mid A) = S(A,B) - S(B)}$$

uncritical extension of Shannon form

Not necessarily a positive definite !!!!!

Quantum discord (OZ): optimal difference of two classically identical expressions for conditional entropies:

$$\delta(A,B) = \min_{\{\Pi_\alpha^A\}} S\left(B \mid A_{\{\Pi_\alpha^A\}}\right) - S(B \mid A)$$

Optimal measurement $\{\Pi_\alpha^A\}$ leaves the overall state with least disturbance and this is quantified by $\delta(A,B)$

Bipartite states, which are in conformity with Bayes' Rule have

$$\delta(A,B) = 0$$

OZ: $\delta(A,B) = 0 \quad \text{iff} \quad \rho'_{AB} = \sum_\alpha \left(\Pi_\alpha^A \otimes I_B \;\; \rho_{AB} \;\; \Pi_\alpha^A \otimes I_B\right) = \rho_{AB}$

i.e, only when the state is left undisturbed as a result of optimal projective measurement on one part of the system

Quantum Discord DOES NOT VANISH FOR ALL SEPARABLE STATES !!!!

Separability is not synonymous with classical correlations ?!

Quantum states with vanishing quantum discord:

$$\boxed{\rho_{AB}^{(classical)} = \sum_{\alpha} P_{\alpha} \;\; \Pi_{\alpha}^{A} \otimes \rho_{\alpha}^{B}}$$

A. K. Rajagopal and R. W. Rendell, Phys. Rev. A **66,** 022104 (2002)

Quantum Deficit: $\mathrm{D_{AB}}=S\left(\rho_{AB}\|\rho_{AB}^{d}\right)$

- a measure of quantumness of correlations

$\rho_{AB}^{(d)}$: classical decohered counterpart of ρ_{AB}

Subsystem eigen basis

$$\rho_{AB}=\sum_{\alpha,\beta}\rho_{\alpha'\beta';\alpha\beta}\,|\alpha'\rangle\langle\alpha|\otimes|\beta'\rangle\langle\beta|$$

$$\rho_{AB}^{(d)}=\sum_{\alpha,\beta}\rho_{\alpha\beta;\alpha\beta}\,\Pi_{\alpha}^{(A)}\otimes\Pi_{\beta}^{(B)}$$

$$=\sum_{\alpha}P_{\alpha}\Pi_{\alpha}^{(A)}\otimes\left[\frac{\sum_{\beta}\rho_{\alpha\beta;\alpha\beta}\Pi_{\beta}^{(B)}}{P_{\alpha}}\right];\qquad P_{\alpha}=\sum_{\beta}\rho_{\alpha\beta;\alpha\beta}$$

$$=\sum_{\alpha}P_{\alpha}\Pi_{\alpha}^{(A)}\otimes\rho_{\alpha}^{(B)}$$

$\Longrightarrow$ classical

$$\Pi_{\alpha}^{(A)}=|\alpha\rangle\langle\alpha|$$

$$\Pi_{\beta}^{(B)}=|\beta\rangle\langle\beta|$$

$$\Pi_{\alpha}^{(A)}\Pi_{\alpha'}^{(A)}=\Pi_{\alpha}^{(A)}\delta_{\alpha'\alpha}$$

$$\sum_{\alpha}\Pi_{\alpha}^{(A)}=I_{A}\quad \text{etc...}$$

L. Henderson and V. Vedral: J. Phys. A: Math. Gen. **34,** 6899 (2001)

Classical correlation: $$C_A(\rho_{AB}) = \max_{\{V_i^A\}} S(\rho_B) - \sum_i P_i S(\rho_B^i)$$

Residual information entropy of B after carrying out a POVM measurement $\{V_i^A\}$ on the subsystem *A*

$$\rho_B^i = \frac{1}{P_i} Tr_A \left[V_i^A \otimes I_B \; \rho_{AB} \; V_i^{A^\dagger} \otimes I_B \right];$$

$$P_i = Tr_{AB} \left[V_i^A \otimes I_B \; \rho_{AB} \; V_i^{A^\dagger} \otimes I_B \right]$$

Classical and entangled correlations do not add up to give total correlations!

$$C_A(\rho_{AB}) + E_{RE}(\rho_{AB}) \leq S(A:B)$$

"Are different types of correlations not additive?"

Measurements play a crucial role in distinguishing and quantifying correlations as **classical** and **quantum**

Our approach:

(A. R. Usha Devi and A. K. Rajagopal, To appear in Phys. Rev. Lett.)

- Consider all tripartite extensions ρ_{CAB} of the state ρ_{AB} such that

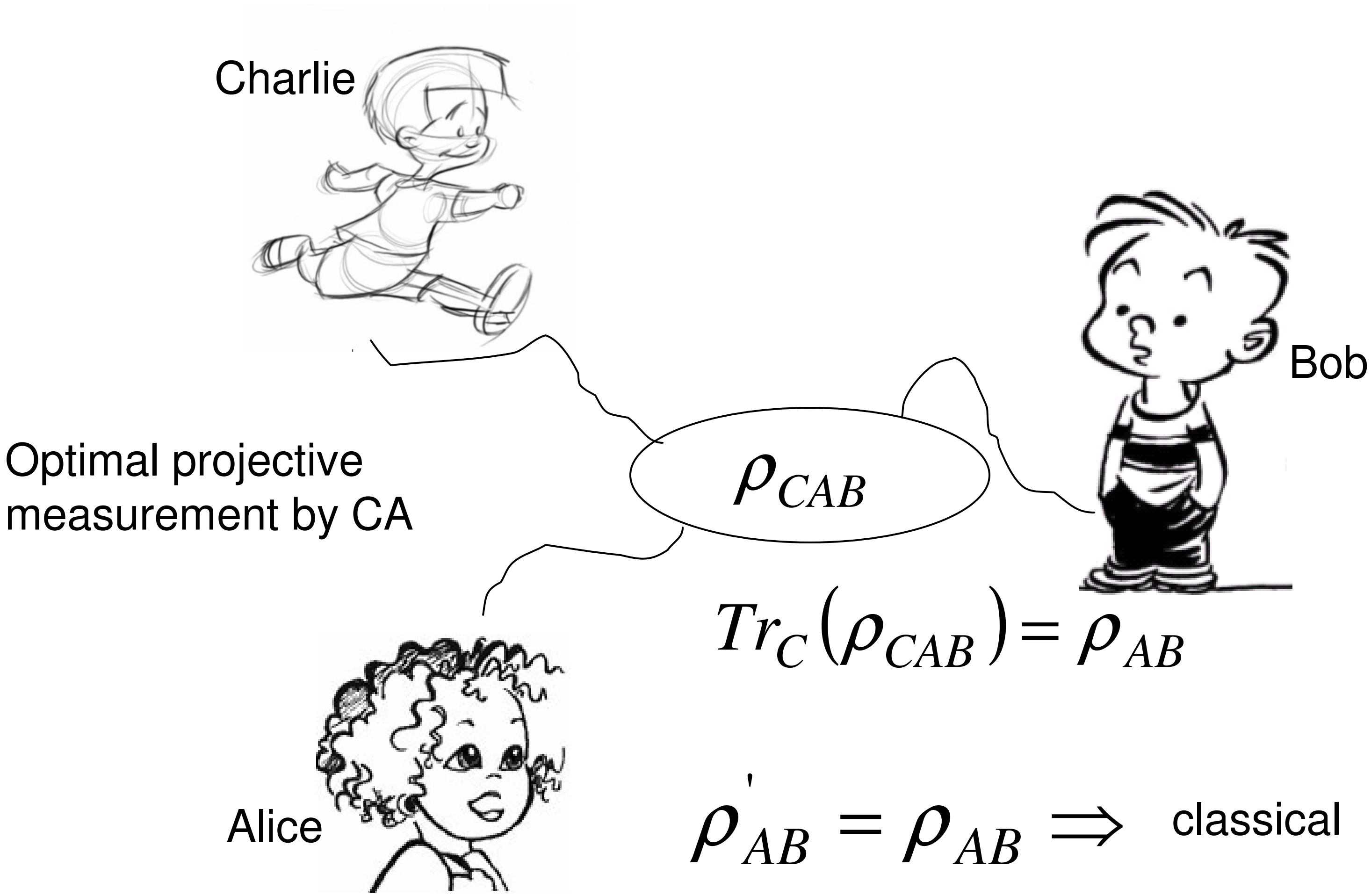
$$Tr_C\left[\rho_{CAB}\right] = \rho_{AB}$$

State under investigation

- Perform generalized projective measurements $\left\{\Pi_i^{(CA)} \otimes I_B\right\}$ on **one part** (CA) of the system.

$$\rho_{AB}^{'} = Tr_C\left[\sum_i \Pi_i^{(CA)} \otimes I_B \rho_{CAB} \Pi_i^{(CA)} \otimes I_B\right] = Tr_C(\rho_{CAB}^{'})$$

state left after generalized measurement



$$Tr_C(\rho_{CAB}) = \rho_{AB}$$

$$\rho'_{AB} = \rho_{AB} \Rightarrow \text{ classical}$$

$$\rho'_{AB} \neq \rho_{AB} \Rightarrow \text{ quantum}$$

Quantumness:

$$Q_{\mathrm{AB}} = \min_{\left\{\Pi_i^{(CA)} \otimes I_B\ ,\ \rho_{CAB}\right\}} S\left(\rho_{AB} \,||\, \rho'_{AB}\right)$$

Minimization is over the set of all tripartite extensions and the set of all projective measurements at the CA end

$$Q_{AB} = 0 \quad \text{iff} \quad \rho_{\mathrm{AB}} = \rho'_{AB}$$

Separability and Quantumness

Quantumness vanishes when

$$\rho_{CAB} = \sum_i P_i \, \Pi_i^{(CA)} \otimes \rho_i^{(B)}$$

$$i.e., \ \rho_{AB} = Tr_C\,[\,\rho_{CAB}] = \sum_i P_i \, \rho_i^{(A)} \otimes \rho_i^{(B)}$$

$$\text{where} \quad Tr_C\,[\,\Pi_i^{(CA)}\,] = \rho_i^{(A)}$$

Separable states are insensitive to generalized measurement (optimal) $\Longrightarrow$ **quantumness vanishes**

Generalized measurements are NOT necessarily POVMs

An example:

$$\rho_{AB} = P|0_A,0_B\rangle\langle 0_A,0_B| + (1-P)|+_A,+_B\rangle\langle +_A,+_B|$$

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right); \quad 0 \le \mathrm{P} \le 1$$

Three qubit extended state

$$\rho_{CAB} = P|1_C,0_A,0_B\rangle\langle 1_C,0_A,0_B| + (1-P)|0_C,+_A,+_B\rangle\langle 0_C,+_A,+_B|$$

$$Tr_C[\rho_{CAB}] = \rho_{AB}$$

An optimal measurement on *CA:*

$$\begin{cases} \Pi_1^{(CA)} = |1_C,0_A\rangle\langle 1_C,0_A|, \ \Pi_2^{(CA)} = |1_C,1_A\rangle\langle 1_C,1_A|, \\ \Pi_3^{(CA)} = |0_C,+_A\rangle\langle 0_C,+_A|, \ \Pi_4^{(A'A)} = |0_C,-_A\rangle\langle 0_C,-_A| \end{cases}$$

This leaves the overall state unperturbed:

$$\sum_{i=1}^{4} \Pi_i^{(CA)} \otimes I_B \ \rho_{CAB} \ \Pi_i^{(CA)} \otimes I_B = \rho_{CAB}$$

and $$\rho'_{AB} = \rho_{AB} \Longrightarrow \mathrm{Q}_{AB} = 0$$

Operational aspects of quantumness

$$\rho'_{CAB} = \sum_i \Pi_i^{(CA)} \otimes I_B \;\rho_{CAB}\; \Pi_i^{(CA)} \otimes I_B$$

$$= \sum_{i,b',b} \langle i\, b' | \rho_{CAB} | i\, b \rangle \; |i\rangle\langle i| \otimes |b'\rangle\langle b|$$

$$= \sum_i P_i \;\Pi_i^{(CA)} \otimes \sum_{b',b} \langle i\, b' | \rho_{CAB} | i\, b \rangle |b'\rangle\langle b|$$

$$= \sum_i P_i \;\Pi_i^{(CA)} \otimes \rho_i^{(B)}$$

$$\Pi_i^{(CA)} = |i\rangle\langle i|; \qquad \rho_i^{(B)} = \sum_{b',b} \frac{\langle i\, b' | \rho_{CAB} | i\, b \rangle}{P_i} \;|b'\rangle\langle b|$$

$$P_i = Tr_{CAB}\left[\Pi_i^{(CA)} \otimes I_B \;\rho_{CAB}\; \Pi_i^{(CA)} \otimes I_B\right] = \sum_b \langle i\, b | \rho_{CAB} | i\, b \rangle$$

$$\boxed{Tr_C \;\rho'_{CAB} = \sum_i P_i \;\rho_i^{(A)} \otimes \rho_i^{(B)} \qquad \text{is a separable state}}$$

with same marginal: $Tr_{CA}\,[\rho'_{CAB}] = \sum_i P_i \;\rho_i^{(B)} = \sum_{i,b',b} \langle i\, b' | \rho_{CAB} | i\, b \rangle \;|b'\rangle\langle b| = \rho_B$

$$\boxed{\begin{aligned} \mathrm{Q}_{\mathrm{AB}} &= \min_{\{\Pi_i^{CA} \otimes I_B,\, \rho_{CAB}\}} S\left(\rho_{AB} \,\|\, \rho'_{AB}\right) \\ &= \min_{\{\rho_{AB}^{(sep)}\}} S\left(\rho_{AB} \,\|\, \rho_{AB}^{(sep)}\right) \end{aligned}}$$

Minimum entropic *distance* between ρ_{AB} and the closest separable state $\rho_{AB}^{(sep)}$ which shares the same marginal ρ_B

* $\mathrm{Q}_{\mathrm{AB}} = 0$ iff ρ_{AB} is separable

* $\mathrm{Q}_{\mathrm{AB}} \neq 0$ for all entangled states

Classical correlations:

$$C_A(\rho_{AB}) = S(\rho_{AB} \| \rho_A \otimes \rho_B) - \min_{\{\rho_{AB}^{(sep)}\}} S(\rho_{AB} \| \rho_{AB}^{(sep)}) \geq 0$$

so that total correlations (mutual information) is equal to a sum of classical correlations $C_A(\rho_{AB})$ and quantumness Q_{AB}

Summary

- Importance of generalized measurements in discerning quantumness of correlations.
- A physical approach to this fundamental problem, based on the basic concept of a quantum measurement and the corresponding information content
- Entangled states get projected to their closest separable states (with same marginal for one of the subsystems) by an optimal generalized projective measurement on one part
- Our new measure **Quantumness** is the minimum entropic distance of the bipartite state with its closest separable state; it serves as an upper bound of relative entropy of entanglement
- Flawless merger of quantumness of correlations with quantum entanglement itself – based on a measurement based approach.