# Bounds on minimum-error discrimination between mixed quantum states


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# Remarks

In this talk, we would rather introduce the technical results, than expound the details for proving them.

# Contents

# Brief Review of Quantum State Discrimination (Roughly Speaking)

- Quantum state detection, namely, *Ambiguous Discrimination*, or called *Minimum-Error Discrimination*
- Unambiguous Discrimination
- Some other schemes combining the above two

# Quantum state detection—— *Ambiguous Discrimination*

What is the definition of Ambiguous Discrimination (Minimum-Error Discrimination)?

Given states $\rho_1, \rho_2, \ldots, \rho_m$

with respective probabilities

$$\eta_1, \eta_2, \ldots, \eta_m$$

# *How to define Ambiguous Discrimination?*

then for any POVM measurement, say $\Pi_i \quad i = 1,2,\ldots,m$

where $\Pi_i$ are positive semi-definite operators, and

$$\sum_{i=1}^{m} \Pi_i = I$$

# Continue

the average probability of correct discriminating these states is

$$P = \sum_{i=1}^{m} \eta_i Tr(\Pi_i \rho_i)$$

and the average probability of erroneous detection is then as

$$Q = 1 - P$$

# Continue

Notably, here, $Tr(\Pi_i \rho_j)$ may not be zero (that also is the reason called ambiguous discrimination), and thus error likely results unless

$$\rho_1, \rho_2, \ldots, \rho_m$$

are mutually orthogonal.

# Some existing results concerning ambiguous discrimination

- In 1970's, Helstrom, Holevo,Yuen etc began this study
- The first important result: *Helstrom limit*

$$Q_A = \frac{1}{2}(1 - Tr\,|\eta_2\rho_2 - \eta_1\rho_1|)$$

by Helstrom in 1976 for *ambiguously* discriminating **two mixed** states

$$\rho_1, \rho_2$$

- That is to say, the above bound on the minimum-error discrimination between TWO states can be precisely saturated

Reference:

C. W. Helstrom, **Quantum Detection and Estimation Theory** (Academic Press, New York, 1976).

# Continue

- However, for *ambiguously* discriminating more than two states, only some necessary and sufficient conditions have been derived for an optimum measurement maximizing the success probability of correct detection. For the details, see, e.g.,

**References**


- A.S. Holevo, J. Multivariate Anal. 3, 337(1973).
- H.P. Yuen, R.S. Kennedy, and M. Lax, IEEE Trans. Inform. Theory 21,125 (1975).
- Y.C. Eldar, A. Megretski, and G.C. Verghess, IEEE Trans. Inform. Theory 49,1007 (2003).

# Continue

Analytical solutions for an optimum measurement have been obtained only for some special cases (namely, the discriminated states

$$\rho_1, \rho_2, \ldots, \rho_m$$

satisfy certain conditions) . The details can be referred to:


- Y.C. Eldar and G.D. Forney, Jr., e-print arXiv: quant-ph/0211111.
- S.M. Barnett, Phys. Rev. A 64, 030303(R) (2001).
- E. Andersson, S.M. Barnett, C.R. Gilson, and K. Hunter, Phys. Rev. A 65, 052308 (2002).
- C.-L. Chou and L.Y. Hsu, Phys. Rev. A 68, 042305 (2003).
- U. Herzog and J.A. Bergou, Phys. Rev. A 65, 050305(R) (2002).

# Our lower bound on the minimum-error probability for discrimination

- D.W. Qiu, PRA 77, 012328 (2008): the minimum-error probability $P_A$ for discriminating $m$ states satisfies

$$P_A \geq \frac{1}{2}\left(1-\frac{1}{m-1}\sum_{1\leq i<j\leq m} Tr\left|\eta_i\rho_i-\eta_j\rho_j\right|\right)$$

- When $m=2$ it is the *Helstrom limit*

# Briefly introduce how to derive this lower bound

First, we have

$$(m-1)\sum_{i=1}^{m} Tr(\eta_i \rho_i \Pi_i) + \sum_{\substack{1\le i<j\le m \\ k\neq i,j}} [\eta_i Tr(\rho_i \Pi_k)] = \sum_{1\le i<j\le m} [\eta_i + Tr(\Lambda_{ij}\Pi_j)]$$

where

$$\Lambda_{ij} = \eta_j \rho_j - \eta_i \rho_i$$

# Continue

Indeed, we can prove the above equality as follows.

$$
\begin{aligned}
& \sum_{1\le i<j\le m} [\eta_i + \mathrm{Tr}(\Lambda_{ij}\Pi_j)] && (1)\\
= & \sum_{1\le i<j\le m} \eta_i + \mathrm{Tr}[(\eta_j\rho_j - \eta_i\rho_i)\Pi_j] && (2)\\
= & \sum_{1\le i<j\le m} \eta_j \mathrm{Tr}(\rho_j\Pi_j) + \sum_{1\le i<j\le m} \eta_i\,(1-\mathrm{Tr}(\rho_i\Pi_j)) && (3)\\
= & \sum_{1\le i<j\le m} \eta_j \mathrm{Tr}(\rho_j\Pi_j) + \sum_{1\le i<j\le m} \eta_i \mathrm{Tr}[\rho_i(I-\Pi_j)] && (4)\\
= & \sum_{1\le i<j\le m} \eta_j \mathrm{Tr}(\rho_j\Pi_j) + \sum_{1\le i<j\le m} \eta_i \mathrm{Tr}[\rho_i(\Pi_i + \sum_{k\ne i,j}\Pi_k)] && (5)\\
= & \sum_{1\le i<j\le m} (\eta_i\mathrm{Tr}(\rho_i\Pi_i) + \eta_j\mathrm{Tr}(\rho_j\Pi_j)) + \sum_{\substack{1\le i<j\le m\\ k\ne i,j}} (\eta_i\mathrm{Tr}(\rho_i\Pi_k)) && (6)\\
= & \ (m-1)\sum_{i=1}^{m} \mathrm{Tr}(\eta_i\rho_i\Pi_i) + \sum_{\substack{1\le i<j\le m\\ k\ne i,j}} (\eta_i\mathrm{Tr}(\rho_i\Pi_k))\,. && (7)
\end{aligned}
$$

# Continue

Suppose the decomposition of positive semi-definite operators:

$$\Lambda_{ij} = A_{ij} - B_{ij}$$

where the spectral decompositions:

$$A_{ij} = \sum_k a_k^{(ij)} \left| \phi_k^{(ij)} \right\rangle \left\langle \phi_k^{(ij)} \right|$$

$$B_{ij} = \sum_l b_l^{(ij)} \left| \varphi_l^{(ij)} \right\rangle \left\langle \varphi_l^{(ij)} \right|$$

# Continue

Then we have

$$\sum_{i=1}^{m} Tr(\eta_i \rho_i \Pi_i) = \frac{1}{m-1} \sum_{1 \le i < j \le m} [\eta_i + Tr(A_{ij}\Pi_j) - Tr(B_{ij}\Pi_j)] - \frac{1}{m-1} \sum_{\substack{1 \le i < j \le m \\ k \ne i, j}} [\eta_i Tr(\rho_i \Pi_k)]$$

$$\le \frac{1}{m-1} \sum_{1 \le i < j \le m} \left( \eta_i + \sum_k a_k^{(ij)} \left\langle \phi_k^{(ij)} \middle| \Pi_j \middle| \phi_k^{(ij)} \right\rangle - \sum_l b_l^{(ij)} \left\langle \varphi_l^{(ij)} \middle| \Pi_j \middle| \varphi_l^{(ij)} \right\rangle \right)$$

$$\le \frac{1}{m-1} \sum_{1 \le i < j \le m} \left( \eta_i + \sum_k a_k^{(ij)} \right)$$

The last term is the upper bound on the success probability for discrimination. Therefore, 1 minus it is the lower bound on the minimum-error probability we stated before.

# Continue

By the following equation we complete the proof, but we leave out the proof of the equation (see PRA 77, 2008,issue 1)

$$\frac{1}{2}\left(1+\frac{1}{m-1}\sum_{1\leq i<j\leq m} Tr\left|\Lambda_{ij}\right|\right)=\frac{1}{m-1}\sum_{1\leq i<j\leq m}\left(\eta_i+\sum_k a_k^{(ij)}\right)$$

# The reachability for this lower bound

If the bound can be attained, there are two equations to be satisfied. We here say an equation, that is,

$$\frac{1}{m-1}\sum_{1\le i<j\le m}\left[\eta_i + Tr(\Lambda_{ij}\Pi_j)\right] = \frac{1}{2}\left(1-\frac{1}{m-1}\sum_{1\le i<j\le m} Tr\left|\Lambda_{ij}\right|\right)$$

In [D.W. Qiu, PRA 77, 012328 (2008)] We have given a sufficient and necessary condition for holding this equation but we do not explain further the reachability here.

# An upper bound on the minimum-error probability for discrimination

- Under certain conditions we also have

$$P_A \leq$$

$$\frac{1}{2}\left(1-\frac{1}{m-1}\sum_{1\leq i<j\leq m} Tr\left|\eta_i\rho_i-\eta_j\rho_j\right|\right)$$

$$+\frac{1}{2(m-1)}\sum_{2\leq i<j\leq m}\left(\eta_i+\eta_1-Tr\left|\eta_1\rho_1-\eta_i\rho_i\right|\right)$$

# What are the conditions for deriving the above upper bound?

We would like to point out that the lower bound has been derived without using any premise condition, but the upper bound is based on certain conditions [D.W. Qiu, PRA 77,012328 (2008)]. We here omit the details.

# Comparisons with some recent results

Recently, in [A. Montanaro, arXiv:0711.2012], another lower bound on $P_A$ has been derived

$$P_A \geq \sum_{1 \leq i < j \leq m} \eta_i \eta_j F(\rho_1, \rho_2)^2$$

where

$$F(\rho_1, \rho_2) = Tr(\sqrt{\rho_2}\rho_1\sqrt{\rho_2})^{\frac{1}{2}}$$

# Continue

We have verified that when

$$\eta_1 = \eta_2 = \cdots = \eta_m = \frac{1}{m}$$

$$\frac{1}{2}\left(1-\frac{1}{m-1}\sum_{1\le i<j\le m} Tr\left|\eta_i\rho_i - \eta_j\rho_j\right|\right)$$

$$\geq \sum_{1\le i<j\le m} \eta_i\eta_j F(\rho_1,\rho_2)^2$$

and only for mutually orthogonal states the above inequality is equivalent. So, in a way, our bound is still better.

# Continue

- In [H. Barnum and E. Knill, Reversing quantum dynamics with near-optimal quantum and classical fidelity, J. Math. Phys., 43 (5): 2097-2106, 2002],
  an upper bound on $P_A$ has been derived

$$P_A \leq \sum_{1 \leq i < j \leq m} \sqrt{\eta_i \eta_j} F(\rho_1, \rho_2)$$

# Continue

We have shown that when

$$\eta_1 = \eta_2 = \cdots = \eta_m = \frac{1}{m}$$

$$\frac{1}{2}\left(1 - \frac{1}{m-1}\sum_{1\le i<j\le m} Tr\left|\eta_i\rho_i - \eta_j\rho_j\right|\right) + \frac{1}{2(m-1)}\sum_{2\le i<j\le m}\left(\eta_i + \eta_1 - Tr\left|\eta_1\rho_1 - \eta_i\rho_i\right|\right)$$

$$\le \sum_{1\le i<j\le m}\sqrt{\eta_i\eta_j}\,F(\rho_1,\rho_2)$$

and the equivalence holds only if they are mutually orthogonal. So, to a certain extent, our upper bound is also better.

# Comparison with unambiguous discrimination

- First it was considered by I. D. Ivanovic, D. Dieks, A. Peres in 1980's for unambiguously discriminating pure states

References


- I. D. Ivanovic, Phys. Lett. A123, 257 (1987).
- D. Dieks, Phys. Lett. A126, 303 (1988).
- A. Peres, Phys. Lett. A128, 19 (1988).

# Continue

- Then it was dealt with by Chefles and others for pure states, for the details, see [A. Chefles, Contemp. Phys. 41, 401 (2000).]
- In 2003, T. Rudolph, et al. dealt with unambiguous discrimination for mixed states.
- Since then, many authors have dealt with this issue, see, e.g.,

[J.A. Bergou, U. Herzog, and M. Hillery,Quantum State Estimation, Lecture Notes in Physics Vol. 649 (Springer, Berlin, 2004), p. 417]

# What is the definition of *unambiguous discrimination*?

Given states $\rho_1, \rho_2, \ldots, \rho_m$

with respective probabilities

$$\eta_1, \eta_2, \ldots, \eta_m$$

then for any POVM measurement, say $\Pi_i$ $i = 0,1,2,\ldots,m$

where $\Pi_i$ are positive semi-definite operators, and

$$\sum_{i=0}^{m} \Pi_i = I$$

# Continue

and satisfies

$$Tr(\Pi_i \rho_j) = 0$$

for $i \neq j$, and $i, j > 0$ .

Note that this condition is ***not*** required in *ambiguous discrimination*.

# Continue

Similarly, the average probability of correct discriminating these states is

$$P = \sum_{i=1}^{m} \eta_i Tr(\Pi_i \rho_i)$$

and the average probability of erroneous detection is then as

$$Q = 1 - P$$

# The relation of *unambiguous and ambiguous* discrimination?

Given states $\rho_1, \rho_2, \ldots, \rho_m$

with respective probabilities

$$\eta_1, \eta_2, \ldots, \eta_m$$

Let $P_U$ denote the failure probability for unambiguous discrimination

Let $P_A$ denote the minimum-error probability for ambiguous discrimination

# Continue

Let $P_U$ denote the failure probability for unambiguous discrimination

Let $P_A$ denote the minimum-error probabilities for ambiguous discrimination. Then:

(1) For discriminating two states, we always have

$$P_U \geq 2P_A$$

(2) For discriminating more than two states, under certain conditions, we also have

$$P_U \geq 2P_A$$

# Problem I

What are the sufficient and necessary conditions for the following inequality for the case of more than two states? since it was proved under certain conditions.

$$P_U \geq 2P_A$$

# Problem II

How about the relation between our bounds and the existing ones for the general case? We have only dealt with them for the case of equality probability.

# Thank You!

$$
\begin{aligned}
& \sum_{1\le i<j\le m} [\eta_i + \mathrm{Tr}(\Lambda_{ij}\Pi_j)] && (1)\\
= & \sum_{1\le i<j\le m} \eta_i + \mathrm{Tr}[(\eta_j\rho_j - \eta_i\rho_i)\Pi_j] && (2)\\
= & \sum_{1\le i<j\le m} \eta_j \mathrm{Tr}(\rho_j\Pi_j) + \sum_{1\le i<j\le m} \eta_i \left(1 - \mathrm{Tr}(\rho_i\Pi_j)\right) && (3)\\
= & \sum_{1\le i<j\le m} \eta_j \mathrm{Tr}(\rho_j\Pi_j) + \sum_{1\le i<j\le m} \eta_i \mathrm{Tr}[\rho_i(I - \Pi_j)] && (4)\\
= & \sum_{1\le i<j\le m} \eta_j \mathrm{Tr}(\rho_j\Pi_j) + \sum_{1\le i<j\le m} \eta_i \mathrm{Tr}[\rho_i(\Pi_i + \sum_{k\ne i,j}\Pi_k)] && (5)\\
= & \sum_{1\le i<j\le m} (\eta_i \mathrm{Tr}(\rho_i\Pi_i) + \eta_j \mathrm{Tr}(\rho_j\Pi_j)) + \sum_{\substack{1\le i<j\le m\\ k\ne i,j}} (\eta_i \mathrm{Tr}(\rho_i\Pi_k)) && (6)\\
= & (m-1)\sum_{i=1}^{m} \mathrm{Tr}(\eta_i\rho_i\Pi_i) + \sum_{\substack{1\le i<j\le m\\ k\ne i,j}} (\eta_i \mathrm{Tr}(\rho_i\Pi_k))\,. && (7)
\end{aligned}
$$