On Local Distinguishability and Indistinguishability of Orthogonal Composite States

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Basic Tasks

- To distinguish a class of states in a composite system by LOCC.
- The states given are orthogonal to each other.
- Therefore, all are distinguishable globally.
- The number of states given is finite.

Our considerations

- Pure bipartite States.
- Single copy case and with certainty.
- Mainly on generalized Bell states.

Short Review

- The subject starts with one of the most striking invention by Bennett group **[Quantum non-locality without entanglement, PRA, 59(1999) 1070-1091]**
- A complete orthogonal basis of 3x3 system, $|0\otimes(0\pm1)\rangle$, $|2\otimes(1\pm2)\rangle$, $|(1\pm2)\otimes0\rangle$, $|(0\pm1)\otimes2\rangle$, $|1\otimes1\rangle$ are not locally distinguishable with certainty in the single copy case.

Short review (continued)

- Another most striking example of an orthogonal product basis is the discovery of Unextendible product basis by Bennett et. al.[PRL, 82(1999),5385], $|0\otimes(0-1)\rangle$,$|2\otimes(1-2)\rangle$, $|(1-2)\otimes 0\rangle$, $|(0-1)\otimes 2\rangle$, $|(0+1+2)\otimes(0+1+2)\rangle$ in 3x3.This set is not exactly locally distinguishable. Later many classes of UPB have been formed, and all have the same property regarding distinguishability.

Short review (continued)

- Quite contrary to the above cases, it is found by Hardy, Walgate,…**[PRL, 85 (2000), 4972]**, that any two orthogonal pure states of bipartite or multipartite systems(whether entangled or not) are locally distinguishable with certainty. They have shown that any two orthogonal pure bipartite states can be expressed as;
- $|\psi\rangle = |1 \otimes a\rangle + |2 \otimes b\rangle + |3 \otimes c\rangle$……………..
- $|\phi\rangle = |1 \otimes a'\rangle + |2 \otimes b'\rangle + |3 \otimes c'\rangle$…………….. where $|1\rangle, |2\rangle, |3\rangle$,……are orthogonal to each other for the first system and $|a\rangle$, $|a'\rangle$, $|b\rangle$, $|b'\rangle$, $|c\rangle$, $|c'\rangle$ are pair wise orthogonal.

Short review (continued)

- It is then found that in 2x2, four Bell states, $\sqrt{½}|00+11\rangle$, $\sqrt{½}|00-11\rangle$,
- $\sqrt{½}|01+10\rangle$, $\sqrt{½}|01-10\rangle$ are not locally indistinguishable with certainty in the single copy case **[PRL, 87 (2001),277902].**
- It is also proved by using bound on distillable entanglement that even any three of them are locally indistinguishable. Later the result extended for non-maximally entangled states.

Short review (continued)

- The problem of local distinguishability or indistinguishability in 2x2 system is completely solved by Hardy, Walgate, **[PRL, 89 (2002),147901],**
- i.e., which three orthogonal pure states would be locally distinguishable or which four states.

Short review (continued)

- Later, Horodecki et.al. **[PRL,90(2003),047902]** gave a condition for a complete orthogonal basis in bipartite system to be locally distinguishable or not.
- If at least one of the vectors is entangled then the basis is not locally distinguishable and the basis is probabilistically distinguishable iff all are product.
- Thereafter lots of work done in this field.

Some recent interesting works:

Watrous **[PRL, 95(2005), 080505]** constructed a class of bipartite subspace having no bases distinguishable by LOCC.

- S. Bandyopadhyay and J. Walgate **[quant-ph/0612013]** showed that any three linearly independent pure quantum states are locally unambiguously distinguishable.
- Duan et al **[PRL, 98(2007),230502]** provided a lower bound on the number of locally unambiguously distinguishable members in an arbitrary basis.
- Walgate and Scott **[quant-ph/0709.4238]** showed a condition for unambiguously locally distinguishable random quantum pure states.

Local discrimination of generalized Bell States

- Consider the full set of orthogonal maximally entangled states in dxd:
- $|\psi_{nm}\rangle = \sqrt{1/d}\sum_{j=0}^{d-1} e^{\frac{2\pi ijn}{d}} |j\rangle \otimes |j+m \bmod d\rangle$, where n,m= 0,1,…,d-1.
- All are connected with $|\psi_{00}\rangle$ locally unitarily by $U_{nm} \otimes I$, where,
- $U_{nm} = \sqrt{1/d}\sum_{j=0}^{d-1} e^{\frac{2\pi ijn}{d}} |j\rangle\langle j+m \bmod d|$ are trace orthogonal.

Generalized Bell State discrimination (continued)

- The question is:
- What are the maximum no. of states that are locally distinguishable with certainty in single copy case?
- By using bound on distillable entanglement it is shown by Ghosh et al **[PRA, 70(2004), 022304]** that no $d+1$ states taken from d^2 states are locally distinguishable.

Generalized Bell State discrimination (continued)

- Fan [PRL, 92(2004), 177905] showed that if d is prime then any k states are locally distinguishable if $k(k-1)/2 \leq d$. Clearly, any three from the set of d^2 ($d>2$) states are locally distinguishable. Later, Nathanson [JMP, 46(2005), 062103] generalised this result to any set of three maximally orthogonal states $C^3 \otimes C^3$. Using the existence of mutually unbiased basis he also found the condition for a set of k maximally entangled states. Result of Fan is reproduced and also established the result for d+1 states.

Generalized Bell State discrimination (continued)

- All the above results do not provide us the condition that exactly which d or less than d number of states from d^2 generalized Bell states are locally distinguishable.
- To proceed further we now go back to the protocol given by Ghosh et al.

Local discrimination by teleportation protocol

- The protocol runs as follows:
- Take an arbitrary qudit $|\phi\rangle$. Use any state from the whole set of generalized Bell states. Teleport the above qudit using the maximally entangled state as Channel with standard Bennett protocol.
- Since the protocol is fixed, therefore the qudit obtained after teleportation are different.
- Now take any d or less than d no. of generalized Bell states.

Local discrimination by teleportation protocol(continued)

- If all the states found after teleportation are orthogonal for at least one qudit $|\phi\rangle$, then the states are surely locally distinguishable. So it is a sufficient condition for local distinguishability.
- It is interesting to note that the condition for existence of such $|\phi\rangle$ for d or less than d states is equivalent to the finding of solution for discriminating corresponding unitary operators by a qudit.

Local discrimination by teleportation protocol(continued)

- Two observations:
- If two copies of each generalized Bell states are supplied then by teleportation protocol it is shown that the full set of d^2 states are locally distinguishable.
- If a set of d or less than d states are locally distinguishable, then considering the teleported states as basis, we can rearrange the states in such a manner so that they could be distinguishable by 1-way LOCC.

Examples where teleportation protocol fails

- Consider the following 4 states in 6x6:
- $|\psi_{00}\rangle = 1/\sqrt{6}[|0\rangle\otimes|0\rangle+|1\rangle\otimes|1\rangle+\ldots+|5\rangle\otimes|5\rangle]$
- $|\psi_{10}\rangle = 1/\sqrt{6}[|0\rangle\otimes|0\rangle+\omega|1\rangle\otimes|1\rangle+\ldots+ \omega^5|5\rangle\otimes|5\rangle]$
- $|\psi_{30}\rangle = 1/\sqrt{6}[|0\rangle\otimes|0\rangle+\omega^3|1\rangle\otimes|1\rangle+\ldots+ \omega^3|5\rangle\otimes|5\rangle]$
- $|\psi_{03}\rangle=1/\sqrt{6}[|0\rangle\otimes|3\rangle+|1\rangle\otimes|4\rangle+|2\rangle\otimes|5\rangle+|3\rangle\otimes|0\rangle+$
- $|4\rangle\otimes|1\rangle+|5\rangle\otimes|2\rangle]$, where ω=sixth root of unity.
- Teleportation protocol fails for this set of states. Also, in 4x4, 5x5, there are examples of 4 states where the protocol fails.

Towards the solution

- Theorem. If a set of orthogonal bipartite states are locally distinguishable with certainty, then there is a set of orthogonal product vectors by which any state can written as sum of the product vectors.
- (This is obtained from a result of Chen & Li **[PRA, 68(2003), 062107]**).
- Note that this criterion does not help us to infer for UPB or such class of states.

(continued)

- Take any d states from d^2 generalized Bell states.
- Suppose the given set is locally distinguishable.
- Use the above theorem to rearrange the states into orthogonal product vectors.
- Since each state is a maximally entangled states, therefore, new representations are also in Schmidt form.

(continued)

- Now, the states are locally unitarily connected in one side with $|\psi_{00}\rangle$.
- Take any one of the d states as first one with the rearranged form. All other are locally unitarily connected with it.
- By dimensional analysis and using linear independency, we can further rearrange the other d-1 states by the orthogonal vectors which are connected with the product vectors of the first state by the corresponding unitary operations.

(continued)

- The above rearrangement makes the states in the form by which we could distinguish them by 1-way LOCC.
- So, every set of d no. of states from the generalized Bell basis, if they are LOCC distinguishable,then also distinguishable with1-way LOCC.
- But, for the above class of states discrimination by teleportation protocol implies and implied by 1-way LOCC.

Ultimate result

- Thus any set of d no. of states from the basis of generalized Bell states are locally distinguishable if the corresponding unitary operations are distinguishable by a $|\phi\rangle$.
- And conversely, for a set of d no. of states if such a solution does not exist, then they are locally indistiguishable.

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