Causality, Bell's inequality and quantum mechanics

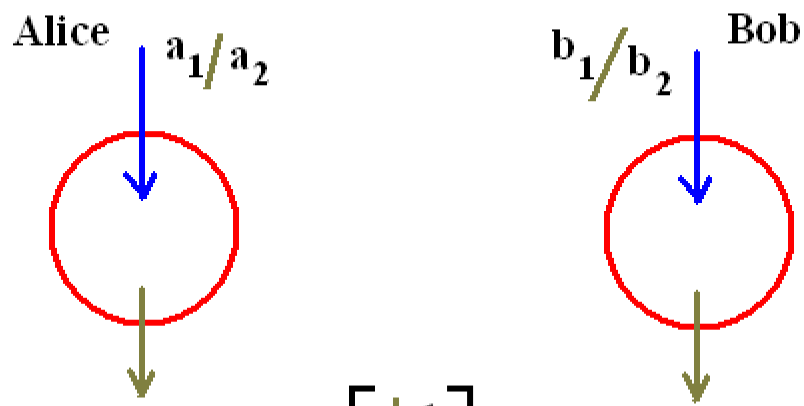
Guruprasad Kar

Physics & Applied Mathematics Unit

Indian Statistical Institute

Kolkata.

A Game



$$V(a_i) = [\pm 1] = V(b_j)$$

Winning condition :

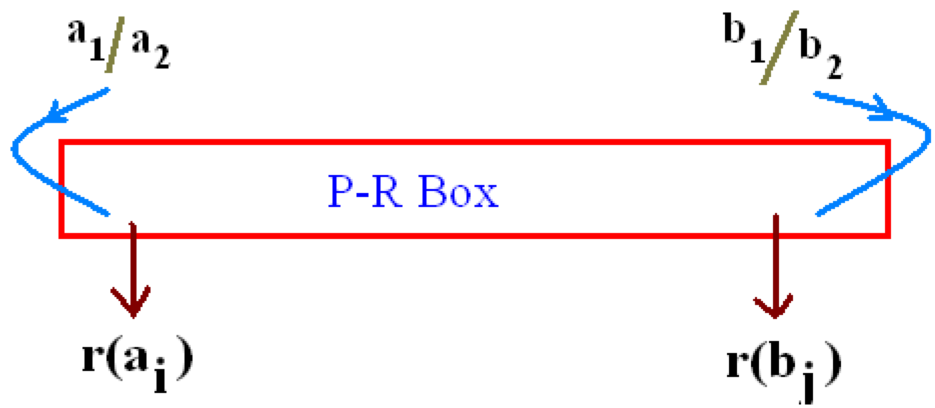
$$V(a_1)V(b_1) = +1$$
$$V(a_1)V(b_2) = +1$$
$$V(a_2)V(b_1) = +1$$
$$V(a_2)V(b_2) = -1$$

or

$$\langle V(a_1)V(b_1)\rangle + \langle V(a_1)V(b_2)\rangle + \langle V(a_2)V(b_1)\rangle - \langle V(a_2)V(b_2)\rangle = 4$$

To win this game in classical world, classical communication is necessary.

Does it imply that for winning the game classical communication is necessary?



$r(a_1)$	$r(b_1)$	Prob
+1	+1	$\frac{1}{2}$
−1	−1	$\frac{1}{2}$

$r(a_2)$	$r(b_1)$	Prob
+1	+1	$\frac{1}{2}$
−1	−1	$\frac{1}{2}$

$r(a_1)$	$r(b_2)$	Prob
+1	+1	$\frac{1}{2}$
−1	−1	$\frac{1}{2}$

$r(a_2)$	$r(b_2)$	Prob
+1	−1	$\frac{1}{2}$
−1	+1	$\frac{1}{2}$

1) Measurement values are ± 1

2) values are predetermined and revealed in measurement.

3) These values for a subsystem do not depend upon choice of measurement on other subsystem.

This assumptions leads to

The Bell-CHSH Expression :

$$|E(A_1,B_1)+ E(A_1,B_2)+ E(A_2,B_1)- E(A_2,B_2)|$$

where $E(A,B) =\langle AB \rangle$

The above expression is bounded by :

(a) 2 for local realistic theory as one can easily check that under this assumption:

$$A_1(B_1+B_2) + A_2(B_1 - B_2) = \pm 2$$

(b) $2\sqrt{2}$ for Quantum mechanical Correlations.

(c) No signalling condition does not put any restriction.

So a natural question arises:

(Q)**What makes Quantum mechanics to limit the CHSH expression by Tsirelson's bound ?**

OR

To which odd will it lead if one assumes that "Quantum correlations" violate the Bell-CHSH inequality by more than $2\sqrt{2}$.

This presentation (our recent work) is an attempt to answer this question.

An inequality in general probabilistic theory under different assumptions:

Assumptions :

1) **Joint measurement is possible on system S_1**
2) **No-signalling condition holds**

The probability that Alice will obtain the result $A_J = A'_J$ can be written as

$$p(A_J = A'_J ;B) = p(A_J = A'_J = B)+p(A_J = A'_J = -B) \qquad (1)$$

As these probabilities are non-negative, hence:

$$p(A_J = A'_J = B) + p(A_J = A'_J = -B) \geq$$

$$|p(A_J = A'_J = B) - p(A_J = A'_J = -B| \qquad (2)$$

Now the term in the right hand side can be written as

$$p(A_J = A'_J =B) - p(A_J = A'_J = -B)| = ½|E(A_J, B)+E(A'_J, B)| \qquad (3)$$

where the correlation function E(A, B) is defined as :

$$E(A, B) = p(A = B) - p(A = -B) = AB$$

The above three equations finally give us

$$p(A_J = A'_J ;B) \geq 1/2|E(A_J, B)+E(A'_J , B)|$$

(4)

Similarly, if we assume that Bob measures for the observable B′ , we will obtain $p(A_J = -A'_J; B') \geq 1/2\ |E(A_J,B') - E(A'_J, B')|$ (5)

Adding eq. (4) and (5) We get :

$$p(A_J = A'_J; B) + p(A_J = -A'_J; B') \geq$$

$$1/2\ [\ |E(A_J,B) + E(A'_J,B)| + |E(A_J,B') - E(A'_J, B')|\ ] \quad (6)$$

The No-signalling constraint tells us that :

$$p(A_J = -A'_J;B) = p(A_J = -A'_J;B') \quad (7)$$

Otherwise by looking at the probabilities at her hand, Alice will know that what measurement Bob has performed on his subsystem in no time. Putting it in equation (6), we get:

$$p(A_J = A'J;B) + p(A_J = -A'_J;B) \geq$$

$$1/2\ [\ |E(A_J, B) + E(A'_J, B)| + |E(A_J, B') - E(A'_J, B')|\ ] \quad (8)$$

Now the LHS ,by the simple law of probability theory ,is equal to **1**. so the above inequality ultimately reduces to:

$$[\ |E(A_J, B)+E(A'_J, B)|+|E(A_J, B')-E(A'_J, B')|\ ] \le 2 \qquad (9)$$

If this inequality gets violated in a theory then we should conclude that :

(a) ***either there can be no joint measurement in that theory***

OR

(b) ***If there exists joint measurement in that theory then the theory permits signalling.***

Quantum measurements :

Theoretical Quantum measurements = Resolution of Identity Operator by means of non-negative self-Adjoint operators.

$\mathbf{I} = \sum \mathbf{E_i}$,where $0 \leq E_i \leq I$.

$\mathbf{E_i}$´s are called 'elements' of measurement.In a measurement, probability of clicking the *i*th result $= \mathbf{Tr}[\rho\, \mathbf{E_i}]$**,** $\boldsymbol{\rho}$ is the initial sate of the system.

Joint Measurement in QM

Consider two observable

$$F_1 + \overline{F_1} = I\,, \qquad F_2 + \overline{F_2} = I$$

Joint measurement exists iff

$$\{F_{12}\,, \quad F_{\bar{1}2}\,, \quad F_{1\bar{2}}\,, \quad F_{\overline{12}}\}(\geq 0)$$

$$such \quad that \qquad F_{12} + F_{\bar{1}2} = F_2$$

$$F_{12} + F_{1\bar{2}} = F_1 \qquad etc.$$

Unsharp spin observable :

$I = 1/2\,[I + \alpha.\sigma] + 1/2\,[I - \alpha.\sigma]$

The interpretation: If in a measurement for spin along direction **α**, **$1/2\,[I + \alpha.\sigma]$** clicks then , we will assign 1 as the value for spin and will conclude that spin of particle is sharply defined along the direction **α.** And If **$1/2\,[I - \alpha.\sigma]$** clicks then , we will assign -1 as the value for spin and will conclude that spin of particle is sharply defined along the direction **$-\alpha$**.

Another resolution(due to P.Busch) :

$I = E_{\lambda}(\alpha) + E_{\lambda}(-\alpha)$ where **$E_{\lambda}(\alpha) = 1/2\,[I + \lambda\,\alpha.\sigma]$** and ***$0 < \lambda \leq 1$***
The interpretation:The spectral decomposition of **$E_{\lambda}(\alpha)$** is given by

$$E_{\lambda}(\alpha) = \left(\frac{1+\lambda}{2}\right)\frac{1}{2}[I + \alpha.\sigma] + \left(\frac{1-\lambda}{2}\right)\frac{1}{2}[I - \alpha.\sigma] \qquad (10)$$

From this representation it is clear that the **POVM** **{$E_\lambda(\alpha)$, $E_\lambda(-\alpha)$ }** is a smeared version of the projective measurement

{1/2 [I + α.σ], 1/2 [I − α.σ]}.

This is the formal sense in which the former represents unsharp spin measurement in the direction **α** . Noteworthy here is that for **λ = 1**, it represents the usual sharp (projective) spin measurement along **α**. The eigen values **r** and **u** of **$E_\lambda(\alpha)$** where;

$$r = 1/2(1+\lambda) > 1/2$$

and

$$u = 1/2(1 - \lambda) < 1/2$$

are interpretated respectively as reality degree and the degree of unsharpness of the spin property along **α**.

Joint measurement of spin :

Projective measurements are too restrictive. In the framework of projective measurements, there are observables which cannot be measured jointly. This distinguishing feature of quantum mechanics is popularly known as Complementarity. Examples of complementary observables are position and momentum observables, spin bservables in different directions etc. But in the more general framework, it

has been shown that certain complementary observables (in standard measurement) can be measured jointly if they are represented by a particular form of POVM (having an interpretation in terms of unsharpness) instead of being represented by projection operators.

Joint measurement of spin observables in different directions has been extensively studied by *P. Busch* . He, by exploiting the necessary and sufficient condition for co-existence of two effects as given by Kraus , showed that a pair of unsharp spin properties $\mathbf{E}_{\lambda 1}(\alpha_1)$ and $\mathbf{E}_{\lambda 2}(\alpha_2)$ are co-existent (i.e. can be jointly measured) if and only if :

$$|(\lambda_1 \ \alpha_1 + \lambda_2 \ \alpha_2)| + |(\lambda_1 \ \alpha_1 - \lambda_2 \ \alpha_2)| \le 2 \qquad (11)$$

For $\lambda_1 = \lambda_2 = \lambda$ i.e for equal unsharpness for both the spin properties, the condition reduces to :

$$\lambda[\,|\alpha_1 + \alpha_2| + |\alpha_1 - \alpha_2|\,] \le 2 \qquad (12)$$

$$\lambda[\ |\alpha_1 + \alpha_2| + |\alpha_1 - \alpha_2|\ ] \leq 2$$

(12)

The term in brackets has maximum value $2\surd 2$. Hence the coexistence condition is satisfied for all pairs of directions α_1 and α_2 if and only if $\lambda \leq 1\ \surd 2$.
i.e. The overall structure of Quantum mechanics is such that it permits joint-measurement of spin along two different directions upto a degree of unsharpness $\lambda = 1\ \surd 2$

Violation of TSirelson's bound implies signalling in QM.

Now we consider a situation where the system consists of two, two level quantum systems in a state ρ (say). Out of the two observers Alice and Bob, Alice; on her subsystem, measures for the unsharp spin observables A_U or A'_U (*whose joint measurement is possible in quantum mechanics*)

Where : $\mathbf{A_U = 1/2\,[\,I + \lambda a.\sigma\,]}$

and

$\mathbf{A'_U = ½\,[\,I + \lambda a'.\sigma\,]}$.

We will denote the sharp counterparts of these observables by **A** and **A′** respectively.

Bob on his subsystem measures either

$\mathbf{B = 1/2\,[\,I + b.\sigma\,]}$

or

$\mathbf{B' = 1/2\,[I + b'.\sigma]}$

For these observables ***inequality*** **(9)** reduces to :

$$|E(A_U,B)+E(A'_U,B)|+|E(A_U,B')-E(A'_U,B')| \leq 2 \quad (13)$$

where $E(A_U, B)$ stands for $Tr(\rho A_U B)$;

$E(A'_U, B)$ for $Tr(\rho A'_U B)$ and so on.

$$A_U = (+1)\frac{1}{2}[I+\lambda\alpha.\sigma] \quad + \quad (-1)\frac{1}{2}[I-\lambda\alpha.\sigma] = \lambda(\alpha.\sigma)$$

$E(A_U, B) = Tr(\rho A_U B) = \lambda Tr(\rho AB) = \lambda E(A, B)$,
Similarly $E(A'_U, B) = \lambda E(A',B)$ and so on. It is noteworthy here that **E(A, B), E(A′, B)** ***etc***. denote the usual quantum-mechanical expectations.

With the help of above analysis equation **(13)** can be rewritten as

$$\lambda[\ |\ E(A, B)+E(A', B)\ |\ +\ |\ E(A, B')-E(A', B')\ |\ ] \leq 2 \quad (14)$$

As we have seen in the previous discussion that value of **λ** can go maximum up to **1/ √2** in order to make joint measurement of spin along any two different directions possible within quantum mechanics. Hence, for no violation of the 'no signalling condition' the term in the parentheses of equation **(14)** should be either less than or equal to **2√2**; *i.e* there will be no superluminal signalling in quantum mechanics as long as :

$$[\ |\ E(A,B) + E(A',B)\ |\ +\ |\ E(A,B')-E(A',B')\ |\ ] \leq 2\sqrt{2} \quad (15)$$

$$\Rightarrow |\ E(A,B) + E(A',B) + E(A,B') - E(A',B')\ | \leq 2\sqrt{2}$$

i.e as long as quantum correlations satisfy Tsirelson's bound.

THANK YOU