A Study on Correlation measures of pure state entanglement (through incomparability)

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Bipartite system

- Pure states

$$\rho_{AB}{}^{2} = \rho_{AB} \quad \Rightarrow \quad \rho_{AB} = \left|\Psi\right\rangle_{AB}\left\langle\Psi\right|$$

- Every Pure state has unique Schmidt representation.

Schmidt form

Pure state of m $\times$ n system, can be expressed as,

$$\left|\Psi\right\rangle_{AB} = \sum_{i=1}^{d} \sqrt{\lambda_i}\left|i\right\rangle_A \left|i'\right\rangle_B \qquad \text{- - - - - (I)}$$

Where, $d \leq \min\{m,n\}$ Schmidt rank of $\left|\Psi\right\rangle_{AB}$

With Schmidt coefficients

$$\lambda_i \in (0,1) \quad \forall \quad i = 1,2,\cdots \mathrm{d}$$

are invariant under Local unitary operations

Entropy of Entanglement for Pure state

- For the pure bipartite state $\left|\Psi\right\rangle_{AB}$ with Schmidt form (I), the entropy of entanglement is given by the Von-Nuemann entropy of one of its reduced density matrices

$$E\left(\left|\Psi\right\rangle\right) = -\sum_{i=1}^{d} \lambda_i \log_2 \lambda_i \qquad \text{-----(II)}$$

Uniqueness

For pure states, Entanglement of Formation ($\mathbf{E_F}$)= E = Distillable Entanglement ($\mathbf{E_D}$) $\Rightarrow$ The process is reversible for all pure states under LOCC and the amount of Entanglement of a pure state is uniquely measured by the entropy function of its subsystems.

S. Popescu and D. Rohrlich, Phys. Rev. A **56**, 3319(R), 1997.

- Thermo-dynamical law of entanglement : Amount of Entanglement can’t be increased under LOCC

Preservation of Entanglement under deterministic LOCC

- Entropy of entanglement is invariant under local unitary operations.
- Does there any other LOCC which could, in deterministic sense, preserve entropy of entanglement?
- From (II), states having different Schmidt form (not always $U_1 \times U_2$ connected) may have same entropy of entanglement.

Comment: The operator form of Von-Neumann Entropy suggests that there are many states with different Schmidt vector, i.e., not locally unitarily Connected,but may have same amount of entanglement. Though the physical reasons behind this fact and interconnections between them are not very clear.

Transformation of pure bipartite states by deterministic LOCC

- A given pure bipartite entangled state $|\Psi\rangle_{AB}$ can be transformed to another state $|\Phi\rangle_{AB}$ by deterministic LOCC if and only if $\lambda_{|\Psi\rangle}$ majorizes $\lambda_{|\Phi\rangle}$ (denoted by $\lambda_{|\Psi\rangle} \prec \lambda_{|\Phi\rangle}$) where $\lambda_{|\Psi\rangle}$ is the Schmidt vector of $|\Psi\rangle_{AB}$ with decreasing order of Schmidt coefficients and so for the other.

M. A. Nielsen, Phys. Rev. Lett. **83**, 436, 1999.

Incomparability $\left(|\Psi\rangle_{AB} \not\leftrightarrow |\Phi\rangle_{AB}\right)$

The pair of states $\left(|\Psi\rangle_{AB}, |\Phi\rangle_{AB}\right)$, for which $|\Psi\rangle_{AB} \not\rightarrow |\Phi\rangle_{AB}$ and $|\Phi\rangle_{AB} \not\rightarrow |\Psi\rangle_{AB}$ (i.e., $\lambda_{|\Psi\rangle} \not\prec \lambda_{|\Phi\rangle}$ and $\lambda_{|\Psi\rangle} \not\succ \lambda_{|\Phi\rangle}$) both occur, are called incomparable to each other.

Incomparability as a detector

Like the thermo-dynamical law of entanglement, impossibility of converting one pair of incomparable states by deterministic LOCC, may act as a detector of many unphysical operations, such as,

Impossibility of cloning or deleting non-orthogonal states,

- Non-existence of anti-unitary operators acting perfectly on any three states not lying in one great circle.

Incomparability in 3×3 is strong

- Any incomparable pair of 3×3 system of states is strongly incomparable, i.e., not convertible by increasing no. of copies finitely and not also catalyzable(catalytic transformation is transformation by presence of extra amount of recoverable entanglement) .
- Only by using non-recoverable amount of entanglement such a pair can be deterministically transformed by LOCC

States with same entanglement

In Schmidt rank 2, $E\left(|\Psi\rangle_{AB}\right) = E\left(|\Phi\rangle_{AB}\right)$ imply

$\lambda_{|\Psi\rangle} \equiv \lambda_{|\Phi\rangle}$ i.e., $|\Psi\rangle \xrightarrow{U_1 \otimes U_2} |\Phi\rangle$

Theorem: In Schmidt rank 3, $|\Psi\rangle_{AB} \rightarrow |\Phi\rangle_{AB}$ with $\lambda_{|\Psi\rangle} \not\equiv \lambda_{|\Phi\rangle}$ imply $E\left(|\Psi\rangle_{AB}\right) > E\left(|\Phi\rangle_{AB}\right)$

Proof: $|\Psi\rangle_{AB} \rightarrow |\Phi\rangle_{AB} \Rightarrow \lambda_{|\Psi\rangle} \prec \lambda_{|\Psi\rangle}$ Let Schmidt vectors $\lambda_{|\Psi\rangle} = \left(\mu_1, \mu_2, \mu_3\right), \lambda_{|\Phi\rangle} = \left(\eta_1, \eta_2, \eta_3\right)$

- Then, $$\mu_1 < \eta_1$$ $$\mu_1 + \mu_2 < \eta_1 + \eta_2$$

- $\Rightarrow$ $$\mu_1 < \eta_1$$ $$\mu_3 > \eta_3$$

- $\Rightarrow$ $$\eta_1 = \mu_1 + \varepsilon,\ \eta_3 = \mu_3 - \xi \ ; \ \ \varepsilon, \xi > 0$$

- So,

$$E\left(|\Psi\rangle_{AB}\right)-E\left(|\Phi\rangle_{AB}\right)=\left(-\sum_{i=1}^{3}\mu_i\log_2\mu_i\right)-\left(-\sum_{i=1}^{3}\eta_i\log_2\eta_i\right)$$

$$=(\mu_1+\varepsilon)\log_2(\mu_1+\varepsilon)+(\mu_2-\varepsilon+\xi)\log_2(\mu_2-\varepsilon+\xi)$$

$$+(\mu_3-\xi)\log_2(\mu_3-\xi)-\mu_1\log_2\mu_1-\mu_2\log_2\mu_2-\mu_3\log_2\mu_3$$

$$=\mu_1\log_2\left(1+\frac{\varepsilon}{\mu_1}\right)+\mu_2\log_2\left(1-\frac{\varepsilon-\xi}{\mu_2}\right)+\mu_3\log_2\left(1-\frac{\xi}{\mu_3}\right)$$

$$+\varepsilon\log_2\left(\frac{\mu_1+\varepsilon}{\mu_2-\varepsilon+\xi}\right)+\xi\log_2\left(\frac{\mu_2-\varepsilon+\xi}{\mu_3-\xi}\right)>0$$

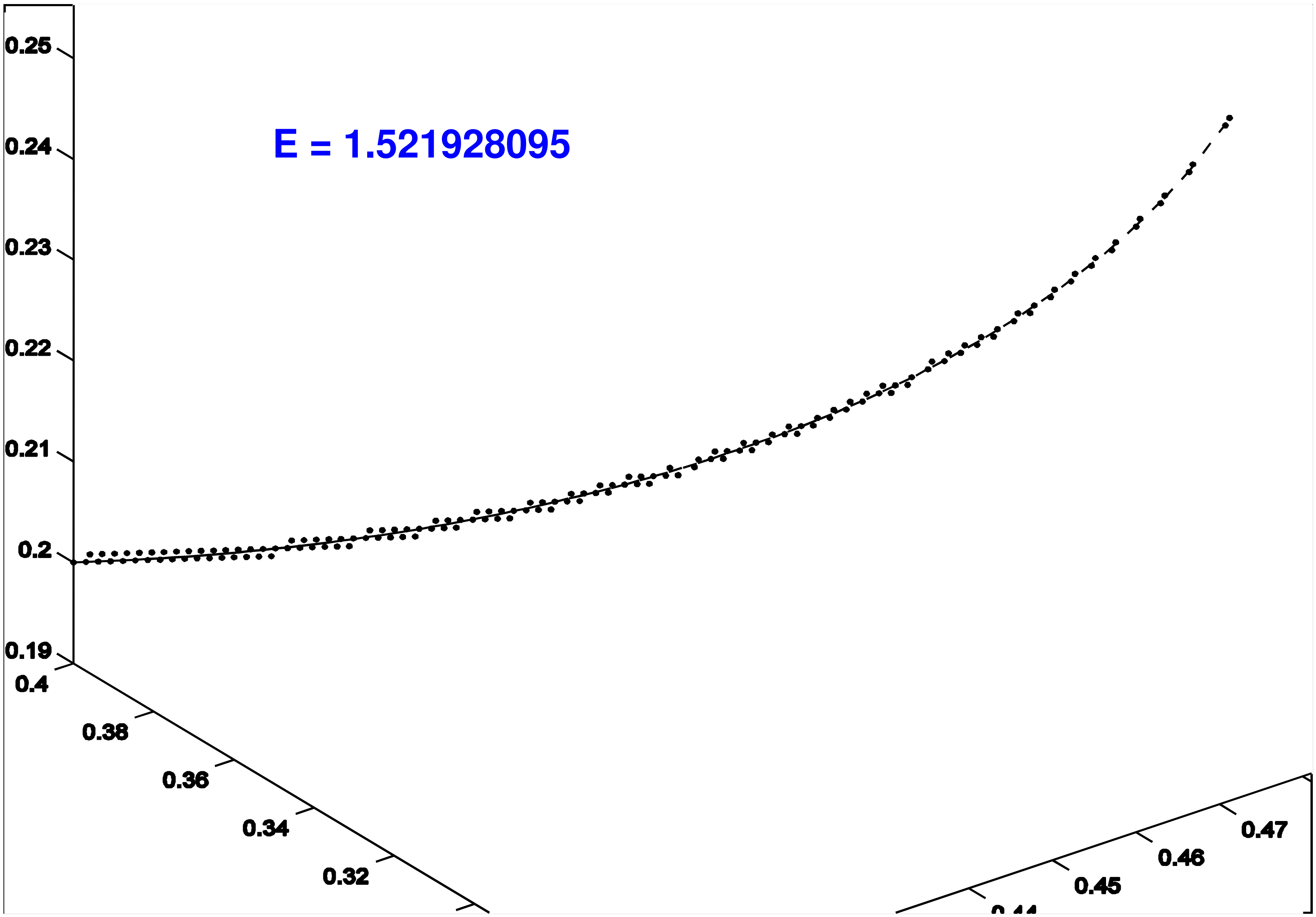
- For both cases $\varepsilon<\xi$ and $\varepsilon>\xi$

Theorem: In Schmidt rank d>3, $|\Psi\rangle_{AB} \to |\Phi\rangle_{AB}$ with $\lambda_{|\Psi\rangle} \not\equiv \lambda_{|\Phi\rangle}$ imply $E(|\Psi\rangle_{AB}) > E(|\Phi\rangle_{AB})$

Corr.- In Schmidt rank d≥3, $E(|\Psi\rangle_{AB}) = E(|\Phi\rangle_{AB})$ with $\lambda_{|\Psi\rangle} \not\equiv \lambda_{|\Phi\rangle}$ imply $|\Psi\rangle_{AB} \not\leftrightarrow |\Phi\rangle_{AB}$

Numerical example of Existence

- Let $\lambda_{|\Psi\rangle}=\{.4,.4,.2\} \Rightarrow E(|\Psi\rangle)=1.521928.$
- Consider the pair of bipartite states $(|\Phi\rangle, |\Omega\rangle)$, with $\lambda_{|\Phi\rangle}= \{.45, .3377591, .2122409\}$ and $\lambda_{|\Omega\rangle}= \{.45, .3377592, .2122408\}$.
- Then $E(|\boldsymbol{\Psi}\rangle) = E(|\boldsymbol{\Phi}\rangle) = E(|\boldsymbol{\Omega}\rangle)$ (approx. to 6 decimal figure). From continuity of Von Neumann entropy, there exists a pure bipartite state between $(|\boldsymbol{\Phi}\rangle, |\boldsymbol{\Omega}\rangle)$ in the sense of Schmidt vectors, with exactly same amount of entanglement contained in $|\Psi\rangle$.


E = 1.521928095
0.25
0.24
0.23
0.22
0.21
0.2
0.19
0.4
0.38
0.36
0.34
0.32
0.47
0.46
0.45

Entanglement of Formation

is not in general a Monotonic function of Concurrence for pure bipartite system

E_F as function of concurrence

- In bipartite system, $C(\rho_{AB}) = \sqrt{2\left(1-\rho_A{}^2\right)}$
- For pure state $|\Psi\rangle$ with Schmidt vector

$\{\lambda_i \in (0,1);\; i=1,2,\cdots d\}$

Concurrence is $$C(\rho_{AB}) = \sqrt{2\left(1-\sum_i \lambda_i{}^2\right)}$$

- For 2×2 states $E_F = \Im(C)$, where $\Im$ is Monotonic.

E_F is monotonic function of Concurrence for comparable states

- Consider an arbitrary pair of pure bipartite system. If $|\Psi\rangle_{AB} \to |\Phi\rangle_{AB}$ by deterministic LOCC, then $\lambda_{|\Psi\rangle} \prec \lambda_{|\Phi\rangle}$ i.e.,

$$\sum_{i=1}^{k} \lambda_{\mathrm{i}} \leq \sum_{\mathrm{i}=1}^{\mathrm{k}} \mu_{\mathrm{i}} \quad \Rightarrow \quad \sum_{i=1}^{k} \lambda_{\mathrm{i}} = \sum_{\mathrm{i}=1}^{\mathrm{k}} \mu_{\mathrm{i}} - \varepsilon_{\mathrm{k}}$$

$$\forall \quad \mathrm{k} = 1, 2, \cdots d - 1$$

$$\Rightarrow \quad C(|\Psi\rangle) - C(|\Phi\rangle) = \left\{ \frac{1}{2}\left(1 - \sum_{i=1}^{d} \mu_i^{\,2}\right) - \frac{1}{2}\left(1 - \sum_{i=1}^{d} \eta_i^{\,2}\right) \right\}$$

$$= \frac{1}{2} \sum_{i=1}^{d} \left(\eta_i^{\,2} - \mu_i^{\,2}\right)$$

$$= \frac{1}{2} \sum_{i=1}^{d} \{(\eta_i - \eta_{i+1}) + (\mu_i - \mu_{i+1})\} \varepsilon_i > 0$$

Again $|\Psi\rangle_{AB} \rightarrow |\Phi\rangle_{AB}$ by deterministic LOCC, imply $E(|\Psi\rangle_{AB}) > E(|\Phi\rangle_{AB})$, i.e., $E_F(|\Psi\rangle_{AB}) > E_F(|\Phi\rangle_{AB})$ so E_F is monotonic function of concurrence, considering the comparable pure states only.

Incomparability causes Non-Monotonicity of $\Im$

- In pure bipartite system, **for comparable states Monotonicity of $\Im$ is assured,** so non-monotonicity can only occurs for some set of incomparable states.
- For every value of non-maximal entanglement, there exists infinite number of states with different Schmidt vector (thus having different values of ***Concurrence***), all are mutually incomparable.

From this idea we may explicitly construct many examples supporting non-monotonic nature of $\Im$.

It shows, even in pure bipartite system,

`E_F is not a monotonic function of concurrence' .

A Study on Measures of correlations for Pure bipartite State

- Consider different measures of pure state entanglement, that does not coincide with Entropy of Entanglement.
- Such measures directly depends on Schmidt coefficients of the state, thus expected to differentiate between the class of incomparable states with same entanglement.
- We have taken 3 different values of entanglement.
- Plot the graph of measures with the largest Schmidt coefficient of the corresponding state.

Measures of Pure state Entanglement

- Negativity: $$N(\rho)=\frac{1}{2}\left(\left(\sqrt{\lambda_1}+\sqrt{\lambda_2}+\sqrt{\lambda_3}\right)^2-1\right)$$

  or
- Logarithmic Negativity: $$LN(\rho)=2\log_2\left(\sqrt{\lambda_1}+\sqrt{\lambda_2}+\sqrt{\lambda_3}\right)$$
- Linear Entropy : $$S_2=-\log_2\left(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2\right)$$
- Renyi Entropy or α-Entropy: $$S_3=-\log_2\left(\lambda_1{}^3+\lambda_2{}^3+\lambda_3{}^3\right)$$

(There are many α-Entropies(H_α). Each can differentiate between the non-localities of states with same entanglement, here we only consider the simplest one.)

- Concurrence : $C\left(\left|\Psi\right\rangle_{AB}\right) = \sqrt{2\left(1 - \sum_i \lambda_i^{\ 2}\right)}$

- Concurrence Hierarchy: $C_2\left(\left|\Psi\right\rangle\right) = \lambda_1\lambda_2\lambda_3$

- Maximum Fidelity:

$$F\left(\left|\Psi\right\rangle\right) = \frac{1}{3}\left(\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3}\right)^2$$

- Robustness: $R\left(\left|\Psi\right\rangle\right) = \left(\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3}\right)^2 - 1$

Distance Measures

- Trace distance: $D_{tr}\left(\left|\Psi\right\rangle\right) = 2\sqrt{1-\lambda_1}$

- Hilbert-Schmidt distance:

$$D_{HS}\left(\left|\Psi\right\rangle\right) = \sqrt{2\left(1-\lambda_1\right)}$$

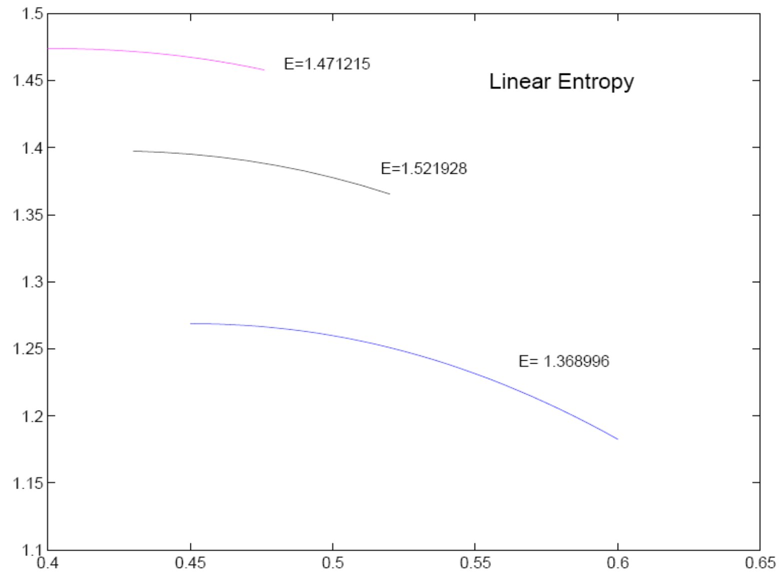
- Fubini-Study distance:

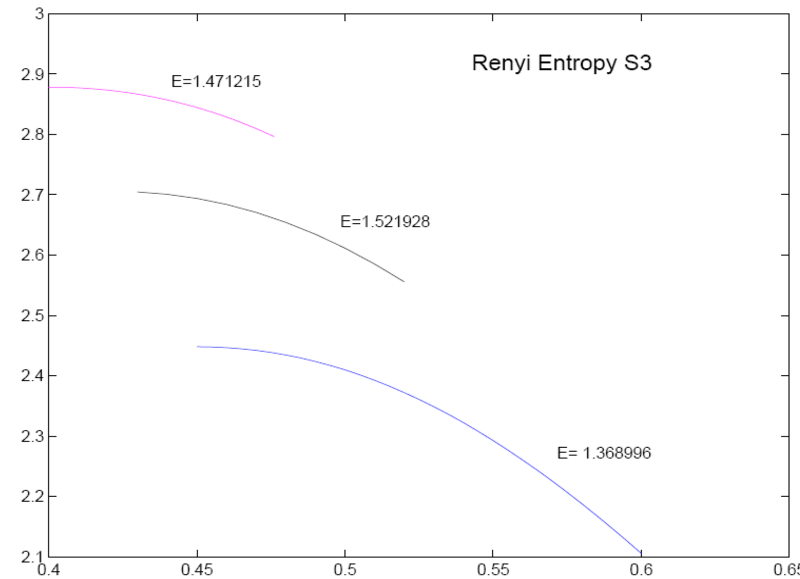
$$D_{FS}\left(\left|\Psi\right\rangle\right) = \arccos\left(\sqrt{\lambda_1}\right)$$

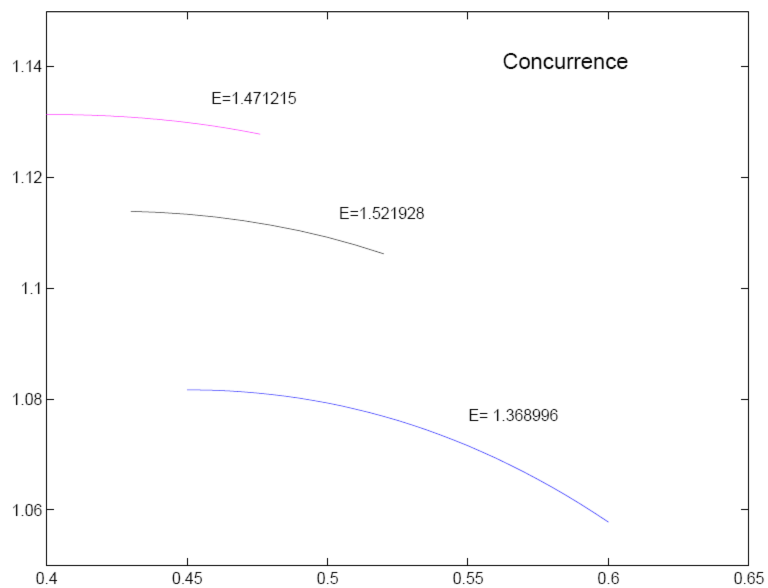
- We consider 3 states of Schmidt rank 3 with Schmidt vectors $\lambda_{|\Psi\rangle}=(.4, .4, .2)$, $\lambda_{|\Phi\rangle}=(.45, .39, .16)$, $\lambda_{|\Omega\rangle}= (.45, .45, .1)$.

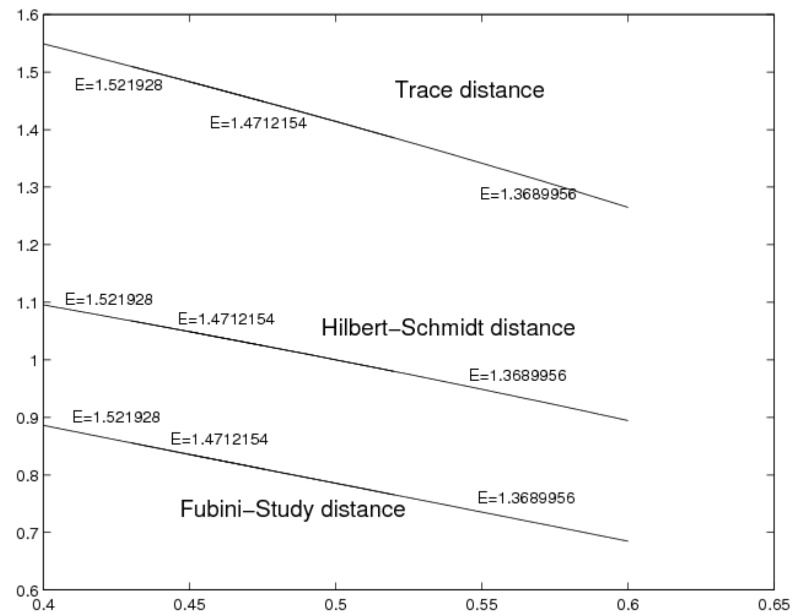
  Then, $E(|\Psi\rangle) = 1.471215$, $E(|\Phi\rangle) = 1.521928$ $E(|\Omega\rangle) = 1.368996$

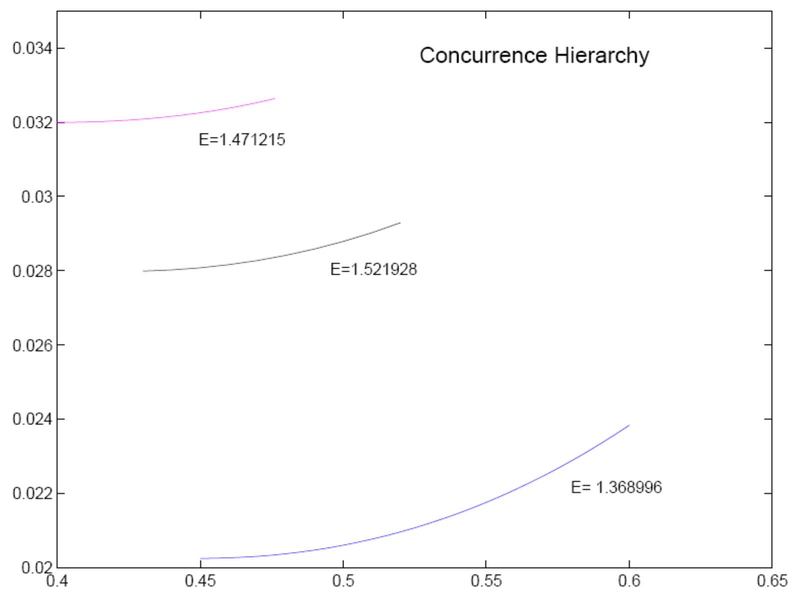
  We search the three sets of states with these three values of entanglement and try to discriminate the states of any one the set, through the above measures.

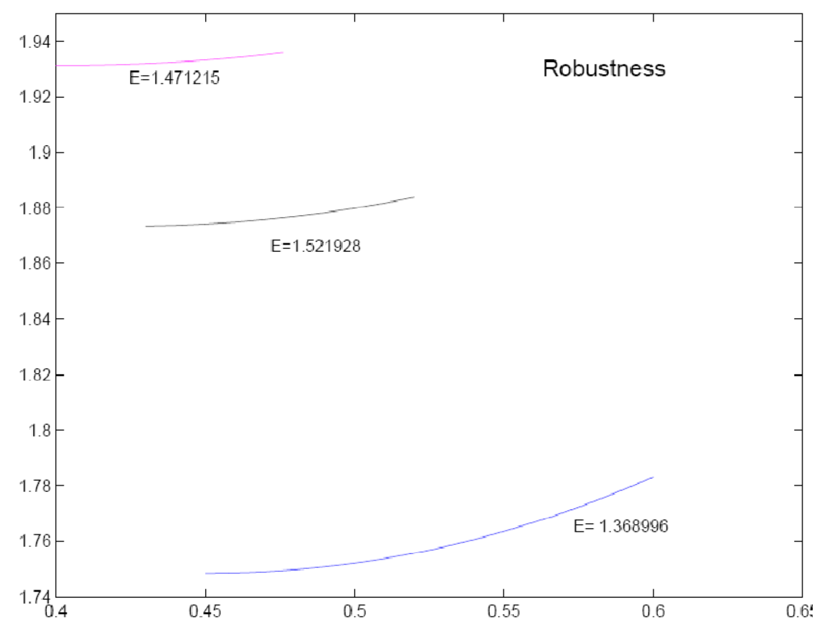

Linear Entropy
E=1.471215
E=1.521928
E= 1.368996

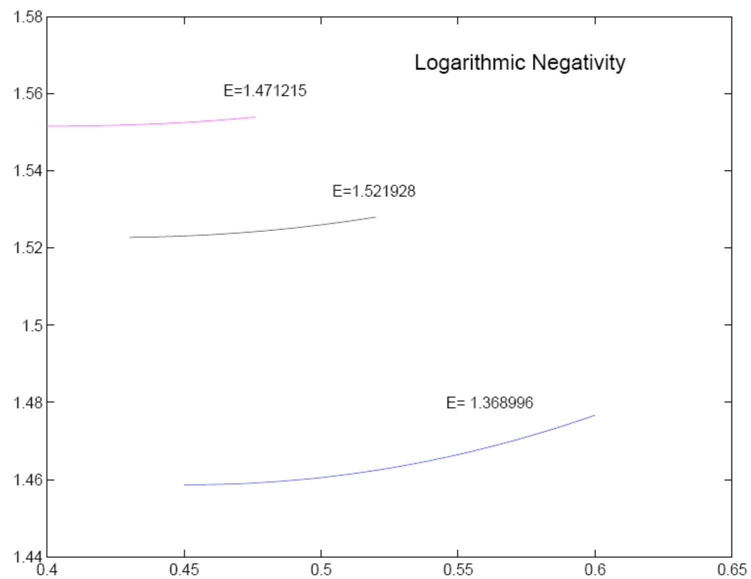

Renyi Entropy S3
E=1.471215
E=1.521928
E= 1.368996

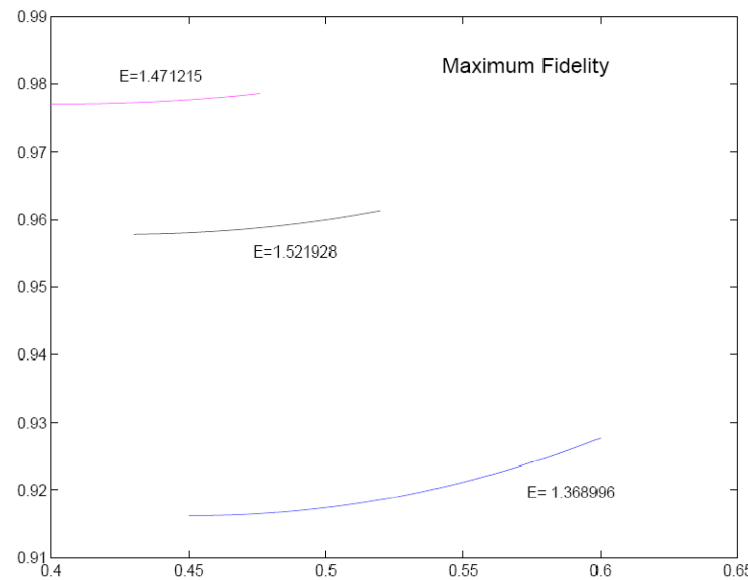

Concurrence
E=1.471215
E=1.521928
E= 1.368996


Trace distance
E=1.521928
E=1.4712154
E=1.3689956
Hilbert−Schmidt distance
E=1.521928
E=1.4712154
E=1.3689956
Fubini−Study distance
E=1.521928
E=1.4712154
E=1.3689956


Concurrence Hierarchy
E=1.471215
E=1.521928
E= 1.368996


Robustness
E=1.471215
E=1.521928
E= 1.368996


Logarithmic Negativity
E=1.471215
E=1.521928
E= 1.368996


Maximum Fidelity
E=1.471215
E=1.521928
E= 1.368996

Observations :

- Measures that are function of H_α for same value of α will show similar pattern such as Negativity, Robustness, Maximal Fidelity while the curvatures are different for different measures
- Distance Measures depending only on largest Schmidt coefficient shows only phase shifted parallel straight lines.

Thanks to all

for your kind attention.