An Experimentally Accessible Geometric Measure for Entanglement in 3-qubit Pure states

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Right from its inception, quantum information fraternity is confronted with two basic questions :

(i) Given a multipartite quantum state (possibly mixed), how to find out whether it is entangled or separable?

(ii) Given an entangled state, how to decide how much entangled it is?

Answers to both these questions are known for bipartite pure states.

(i) If $\rho_A^2 = \rho_A$ then ρ is separable.

(ii) Typical measure is the entanglement entropy

$E(|\psi\rangle) = S(\rho_A) = -\sum_i \lambda_i ln \lambda_i$

Zero for separable states, lnN for maximally entangled states.

Multipartite states

General answers to both these questions are not knowns. Different types of entanglement. Many separability criteria are proposed Example: Generalizations of Peres-Horodecki criterion. The genuine entanglement of pure multipartite quantum state is established by checking whether it is entangled in all bipartite cuts, which can be tested using Peres-Horodecki criterion.

For mixed states this strategy does not work because there are mixed states which are separable in all bipartite cuts but are genuinely entangled [PRL 1999, **82**, 5385]. A direct and independent detection of genuine multipartite entanglement is lacking.

In this talk I present a new measure of entanglement for 3-qubit pure states.
I present all the results for N-qubit pure states except one which we could prove only for two and three qubit pure states.

Let ρ act on H; $dim(H) = d$. $\rho \in L(H)$; scalar product $(A, B) = Tr(A^\dagger B)$.
$dim(L(H)) = d^2$.
ρ can be expanded in any orthonormal basis of $L(H)$.
The basis comprising $d^2 - 1$ generators of $SU(d)$ is particularly useful:
$\{I_d,\ \lambda_i;\ i = 1, 2, \cdots, d^2 - 1\}$.

$\{\lambda_i\}$ are traceless Hermition operators satisfying

$Tr(\lambda_i \lambda_j) = 2\delta_{ij}$

and $\lambda_i \lambda_j = \frac{2}{d}\delta_{ij} I_d + i f_{ijk}\lambda_k + g_{ijk}\lambda_k$

f_{ijk}, g_{ijk} are completely antisymmetics (symm.) tensors.

$d = 2$:

$\lambda_i \leftrightarrow \sigma_i; f_{ijk} = \epsilon_{ijk}$ (Levi-civita) $\quad g_{ijk} = 0.$

ρ **expanded in this basis:**

$$\rho = \frac{1}{d}(I_d + \sum_i s_i \lambda_i) \qquad (A)$$

where $s_i = \langle \lambda_i \rangle = Tr(\rho \lambda_i)$ **is the average value of the** i**th generator** λ_i **in the state** ρ**.**

Bloch Vectors

The vector $\mathbf{s} = (s_1, s_2, \ldots, s_{d^2-1})$; $s_i = \langle \lambda_i \rangle$
is called the Bloch vector of state.
The correspondence $\mathbf{s} \leftrightarrow \rho$
via the expansion of ρ in (A) is one-to-one .
Thus we can use $\mathbf{s}$ to specify a quantum state.

Note that $\mathbf{s}$ is very easily accessible experimentally because all the averages can be directly computed using the outputs of measurements of $\{\lambda_i\}$; $i = 1, 2, \cdots, d^2 - 1$. In fact the Bloch vector $\mathbf{s}$ can be obtaind experimenttally even if the form of ρ is not known.

Bloch vector space

Bloch vectors for a given system live in $\mathbb{R}^{d^2-1}$.

If we put an arbitrary vector $\in \mathbb{R}^{d^2-1}$ in equation (A) we may not get a valid density operator.

A density operator has to satisfy

(i) $Tr\rho = 1$ (ii) $\rho = \rho^\dagger$

(iii) $x^\dagger \rho x \geq 0 \; \forall x \in \mathbb{C}$

So the problem is to find the set of Bloch vectors in $\mathbb{R}^{d^2-1}$, called Bloch vector space $B(\mathbb{R}^{d^2-1})$.
This problem is solved only for $d = 2$:
The Bloch vector space is a ball of unit radius in $\mathbb{R}^3$, known as the Bloch ball.

For $d > 2$, the problem is still open.
However for pure states ($\rho^2 = \rho$) the following relations hold,

$$||\mathbf{s}||_2 = \sqrt{\frac{d(d-1)}{2}}; \quad s_i s_j g_{ijk} = (d-2)s_k \qquad (A')$$

It is known that

$$D_r(\mathbb{R}^{d^2-1}) \subseteq B(\mathbb{R}^{d^2-1}) \subseteq D_R(\mathbb{R}^{d^2-1})$$

$$r = \sqrt{\frac{d}{2(d-1)}} \qquad R = \sqrt{\frac{d(d-1)}{2}}$$

Bloch Representation of a Multipartite state

We construct the basis of $L(H)$ which is the product of individual bases comprising generators of $SU(d_k)$; $k = 1, 2, \cdots, N$.
k,k_i : a subsystem chosen from N subsys.
$\{I_{d_{k_i}}, \lambda_{\alpha_{k_i}}\}$; $\alpha_{k_i} = 1, 2, \cdots, d_{k_i}^2 - 1$ is the basis of $\mathbb{C}^{d_{k-i}^2}$, comprising the generetors of $SU(d_{d_{k_i}})$.

Define, for subsystems k_1 and k_2

$$\lambda^{(k_1)}_{\alpha_{k_1}} = (I_{d_1} \otimes I_{d_2} \otimes \cdots \otimes \lambda_{\alpha_{k_1}} \otimes I_{d_{k_1+1}} \otimes \cdots \otimes I_{d_N})$$

$$\lambda^{(k_2)}_{\alpha_{k_2}} = (I_{d_1} \otimes I_{d_2} \otimes \cdots \otimes \lambda_{\alpha_{k_2}} \otimes I_{d_{k_2+1}} \otimes \cdots \otimes I_{d_N})$$

$$\lambda^{(k_1)}_{\alpha_{k_1}} \lambda^{(k_2)}_{\alpha_{k_2}} = (I_{d_1} \otimes I_{d_2} \otimes \cdots \otimes \lambda_{\alpha_{k_1}} \otimes I_{d_{k_1+1}} \otimes \cdots \otimes \lambda_{\alpha_{k_2}} \otimes I_{d_{k_2+1}} \otimes I_{d_N}$$

$\lambda_{\alpha_{k_1}}$ and $\lambda_{\alpha_{k_2}}$ occur at the k_1th and k_2th places and are the $\lambda_{\alpha_{k_1}}$th and $\lambda_{\alpha_{k_2}}$th generators of $SU(d_{k_1})$, $SU(d_{k_2})$ respectively.

In this basis we can expand ρ as

$\rho = \frac{1}{\Pi_k^N d_k}\{\bigotimes_k^N I_{d_k} + \sum_{k\in\mathcal{N}}\sum_{\alpha_k} s_{\alpha_k}\lambda_{\alpha_k}^{(k)} + \sum_{\{k_1,k_2\}}\sum_{\alpha_{k_1}\alpha_{k_2}} t_{\alpha_{k_1}\alpha_{k_2}}\lambda_{\alpha_{k_1}}^{(k_1)}\lambda_{\alpha_{k_2}}^{(k_2)} + \cdots + \sum_{\{k_1,k_2,\cdots,k_M\}}\sum_{\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_M}} t_{\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_M}}\lambda_{\alpha_{k_1}}^{(k_1)}\lambda_{\alpha_{k_2}}^{(k_2)}\cdots \lambda_{\alpha_{k_M}}^{(k_M)} + \cdots + \sum_{\alpha_1\alpha_2\cdots\alpha_N} t_{\alpha_1\alpha_2\cdots\alpha_N}\lambda_{\alpha_1}^{(1)}\lambda_{\alpha_2}^{(2)}\cdots\lambda_{\alpha_N}^{(N)}\}. \qquad (B)$

(B) is called the Bloch representation of ρ.

$\mathbf{s}^{(k)} = [s_{\alpha_k}]_{\alpha_k=1}^{d_k^2-1}$: Bloch vector for kth subsystem.

$\binom{N}{M}$ terms in the sum $\sum_{\{k_1,k_2,\cdots,k_M\}}$
Each contains a tensor (M-way array) of order M

$t_{\alpha_{k_1}\alpha_{k_2}\ldots\alpha_{k_M}} = \frac{d_{k_1}d_{k_2}\ldots d_{k_M}}{2^M} Tr[\rho\lambda^{(k_1)}_{\alpha_{k_1}}\lambda^{(k_2)}_{\alpha_{k_2}}\cdots\lambda^{(k_M)}_{\alpha_{k_M}}]$
$= \frac{d_{k_1}d_{k_2}\ldots d_{k_M}}{2^M} Tr[\rho_{k_1k_2\ldots k_M}(\lambda_{\alpha_{k_1}}\otimes\lambda_{\alpha_{k_2}}\otimes\cdots\otimes\lambda_{\alpha_{k_M}})]$
$\mathcal{T}^{\{k_1,k_2,\cdots,k_M\}} = [t_{\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_M}}]$

The tensor in the last term is $\mathcal{T}^{(N)}$.

Outer product of vectors

Let $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(M)}$ be vectors in $\mathbb{R}^{d_1^2-1}, \mathbb{R}^{d_2^2-1}, \cdots, \mathbb{R}^{d_M^2-1}$.
The outer product $\mathbf{u}^{(1)} \circ \mathbf{u}^{(2)} \circ \cdots \circ \mathbf{u}^{(M)}$ is a tensor of order M, (M-way array), defined by $t_{i_1 i_2 \cdots i_M} = \mathbf{u}^{(1)}_{i_1} \mathbf{u}^{(2)}_{i_2} \ldots \mathbf{u}^{(M)}_{i_M};\ 1 \leq i_k \leq d_k^2 - 1,\ k = 1, 2, \cdots, M.$

We need the following result

A pure N-partite state with Bloch representation (B) is fully separable (product state) If and only if $\mathcal{T}^{(N)} = \mathbf{s}^{(1)} \circ \mathbf{s}^{(2)} \circ \cdots \circ \mathbf{s}^{(N)}$ where $\mathbf{s}^{(k)}$ is the Bloch vector of kth subsystem reduced density matrix.

We propose the following measure for N-qubit pure state entanglement.

$E(\rho) = \frac{(||\mathcal{T}^{(N)}||-1)}{R}$

where the normalization constant R is given by

$R = (1 + \frac{1}{4}(1+(-1)^N)^2 + \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \binom{N}{2k})^{1/2} - 1$

where $R = ||\mathcal{T}^{(N)}|| - 1$ calculated for N-qubit GHZ state as shown below.

The general *GHZ* state is
$|\psi\rangle = \sqrt{p}|0\cdots 0\rangle + \sqrt{1-p}|1\cdots 1\rangle$
For this state the elements of $\mathcal{T}^{(N)}$ are given by $t_{i_1 i_2 \cdots i_N} = \langle\psi|\sigma_{i_1} \otimes \cdots \otimes \sigma_{i_N}|\psi\rangle$
Using this, the norm of $\mathcal{T}^{(N)}$ for the state $|\psi\rangle\langle\psi|$ is given by
$||\mathcal{T}^{(N)}||^2 = 4p(1-p) + (p + (-1)^N(1-p))^2 + 4p(1-p)\sum_{k=1}^{\lfloor\frac{N}{2}\rfloor}\binom{N}{2k}$

For maximally entangled state $p = \frac{1}{2}$
$R = ||\mathcal{T}^{(N)}|| - 1$
$= (1 + \frac{1}{4}(1 + (-1)^N)^2 + \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \binom{N}{2k}))^{1/2} - 1$
$E(\rho_{GHZ}) = \frac{1}{R}[(4p(1-p) + (p + (-1)^N(1 - p))^2 + 4p(1-p) \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \binom{N}{2k}))^{1/2} - 1]$
E as a function of p is plotted in the next slide. Note that $E(\rho) \geq 0$ for general N-qubit GHZ state.

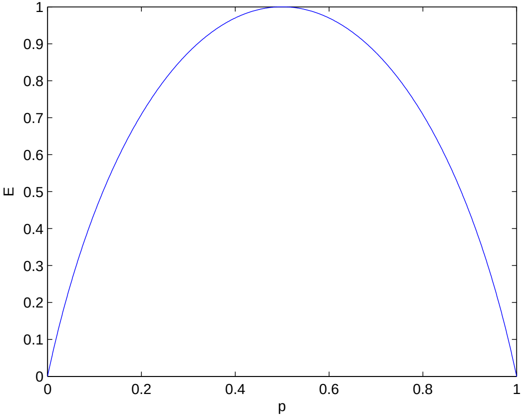

figure 1

$|W\rangle$ state

$|W\rangle = \frac{1}{\sqrt{N}} \sum_j |0 \cdots 01_j 0 \cdots 0\rangle$
$|\widetilde{W}\rangle = \frac{1}{\sqrt{N}} \sum_j |1 \cdots 10_j 1 \cdots 1\rangle$
where jth summand has a single 1 for $|W\rangle$ and sigle 0 for $|\widetilde{W}\rangle$ at the jth bit.

For both the states we get

$$||\mathcal{T}^{(N)}||^2 = 1 + 4\frac{N-1}{N}$$

so that,

$$E(|W\rangle) = E(|\widetilde{W}\rangle) = \frac{1}{R}\left(\sqrt{1 + 4\frac{N-1}{N}} - 1\right).$$

$E(|W\rangle) = E(|\widetilde{W}\rangle)$ is to be expected as these are LU equivalent.

Figure 2 shows the variation of E with weight s in the state $|\psi_s\rangle = \sqrt{s}|W\rangle + \sqrt{1-s}\, e^{i\phi}|\widetilde{W}\rangle$
Note that the entanglement is independent of ϕ.

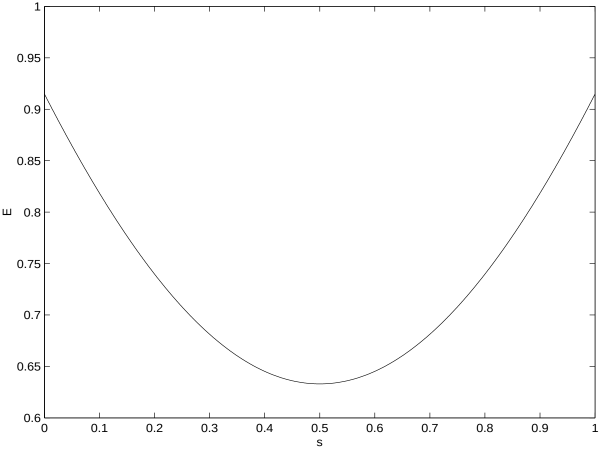

figure 2

Figure 3 shows the variation of E with weight s in the state

$$|\chi_s\rangle = \sqrt{s}|GHZ\rangle + \sqrt{1-s}\, e^{i\phi}|W\rangle$$

Note agian that the entanglement is independent of ϕ.

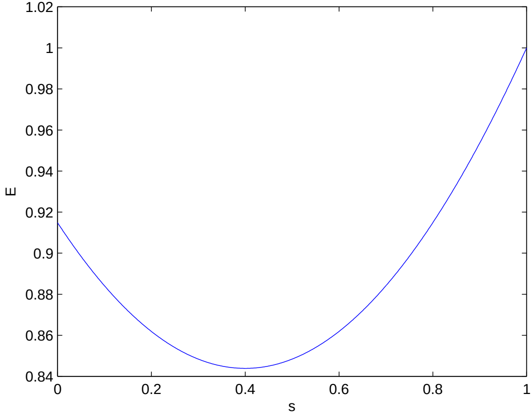

figure 3

An entanglement measure must have the following basic properties
(a) (i) $E(\rho)) \geq 0$ (ii) $E(\rho) = 0$ if and only if ρ is separable
(b) Monotonicity under probabilistic $LOCC$.
(c) Convexity,
$E(p\rho + (1-p)\sigma) \leq pE(\rho) + (1-p)E(\sigma)$ with $p \in [0,1]$.
We prove these properties for our measure one by one.

Proposition 1 : Let ρ be a N-qubit pure state with Bloch representation (B). Then, $||\mathcal{T}^{(N)}|| = 1$ if and only if ρ is a product state.

By the result we have quoted

$\mathcal{T}^{(N)} = \mathbf{s}^{(1)} \circ \mathbf{s}^{(2)} \circ \cdots \circ \mathbf{s}^{(N)}$

Taking norm on both sides

$||\mathcal{T}^{(N)}||^2 = \langle \mathcal{T}^{(N)}, \mathcal{T}^{(N)} \rangle = \Pi_i \langle \mathbf{s_i}, \mathbf{s_i} \rangle = \Pi_i ||\mathbf{s_i}||^2 = 1$

Immediatly it follows that N-qubit pure state ρ has $E(\rho) = 0$ if and only if ρ is a product state.

Proposition 2 : For two and three qubit states $||\mathcal{T}^{(N)}|| \geq 1$
We prove this by directly computing $||\mathcal{T}^{(N)}||$ for the general two and three qubit states.

Consider, the general two qubit state

$|\psi\rangle = a_1|00\rangle + a_2|01\rangle + a_3|01\rangle + a_4|11\rangle,$

$\sum_i |a_i|^2 = 1.$

$$||\mathcal{T}^{(2)}||^2 = 1 + 8(a_2 a_3 - a_1 a_4)^2 \geq 1$$

Consider, the general Schmidt form of three qubit state

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\phi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \ \lambda_i \geq 0, \ \sum_i |\lambda_i|^2 = 1.$$

By direct calculation of $||\mathcal{T}^{(3)}||$ we get

$$||\mathcal{T}^{(3)}||^2 \geq 1 + 12\lambda_0^2\lambda_4^2 + 8\lambda_0^2\lambda_2^2 + 8\lambda_0^2\lambda_3^2 + 8(\lambda_0^2\lambda_3 - \lambda_1\lambda_4)^2 \geq 1$$

For any two and three qubit pure states ρ
$E(\rho) \geq 0$.

We conjecture that $||\mathcal{T}^{(N)}|| \geq 1$ for any N-qubit pure state.

Proposition 3 : Let U_i be a local unitray operator acting on the Hilbert space of ith subsystem.

If $\rho' = (\bigotimes_{i=1}^{N} U_i)\rho(\bigotimes_{i=1}^{N} U_i^{\dagger})$

then $||\mathcal{T}'^{(N)}|| = ||\mathcal{T}^{(N)}||$.

Proposition 4 : $E(\rho)$ is $LOCC$ invariant.

This follows from proposition 3 and the result due to Bennett et al. that N-partite pure state is $LOCC$ invariant if and only if it is LU invariant [PRA 2000,**63** 012307].

Convexity

$$E(p|\psi\rangle\langle\psi| + (1-p)|\phi\rangle\langle\phi|)$$

$$= \frac{1}{R}(||p\mathcal{T}^{(N)}_{|\psi\rangle} + (1-p)\mathcal{T}^{(N)}_{|\phi\rangle}|| - 1)$$

$$\leq \frac{1}{R}(p||\mathcal{T}^{(N)}_{|\psi\rangle}|| + (1-p)||\mathcal{T}^{(N)}_{|\phi\rangle}|| - 1)$$

$$= pE(|\psi\rangle) + (1-p)E(|\phi\rangle)$$

Continuity

$$||(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|)|| \to 0$$
$$\Rightarrow \Big|E(|\psi\rangle) - E(|\phi\rangle)\Big| \to 0$$

$$||(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|)|| \to 0$$
$$\Rightarrow ||\mathcal{T}^{(N)}_{|\psi\rangle} - \mathcal{T}^{(N)}_{|\phi\rangle}|| \to 0$$

But $||\mathcal{T}^{(N)}_{|\psi\rangle} - \mathcal{T}^{(N)}_{|\phi\rangle}|| \geq \Big| ||\mathcal{T}^{(N)}_{|\psi\rangle}|| - ||\mathcal{T}^{(N)}_{|\phi\rangle}|| \Big|$

Therefore $||\mathcal{T}^{(N)}_{|\psi\rangle} - \mathcal{T}^{(N)}_{|\phi\rangle}|| \to 0$

$$\Rightarrow ||\mathcal{T}^{(N)}_{|\psi\rangle}|| - ||\mathcal{T}^{(N)}_{|\phi\rangle}|| \to 0$$

$$\Rightarrow \Big| E(|\psi\rangle) - E(|\phi\rangle) \Big| \to 0$$