A study of the efficiency of the class of W-states as a quantum channel

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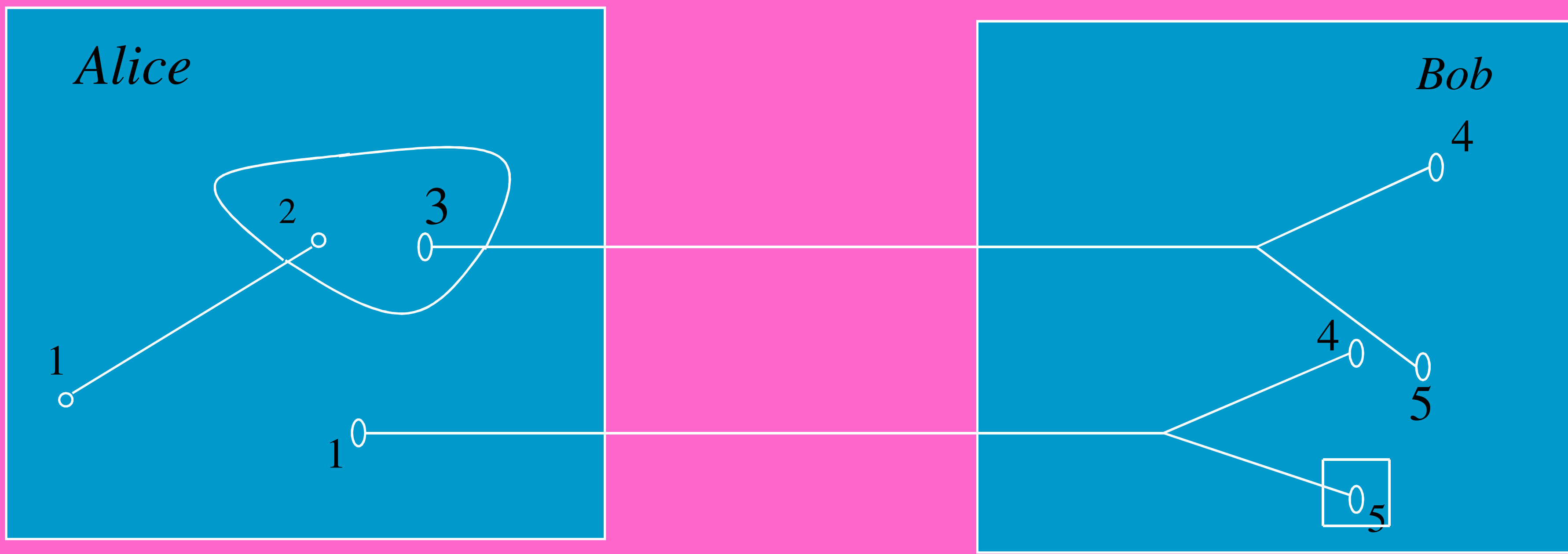
e-print quant-ph / 0803.0607

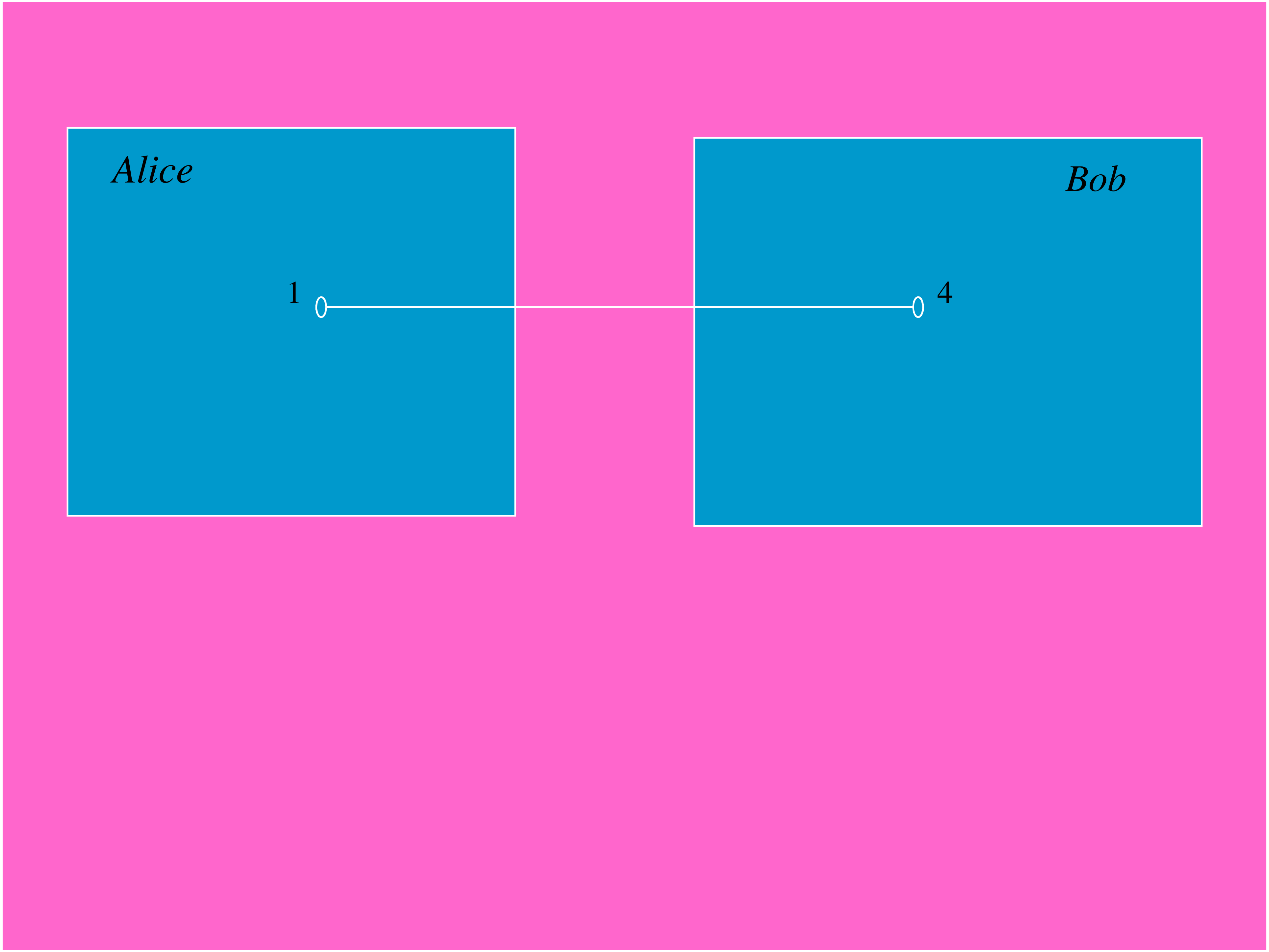
Motivation

It is known that GHZ-class and W-class are two important classes of three-qubit entangled systems.

GHZ state such as $|GHZ> = \frac{1}{\sqrt{2}}(|000> + |111>)$ is suitable for perfect teleportation and superdense coding but the prototype W state $|W> = \frac{1}{\sqrt{3}}(|100> + |010> + |001>)$ is not suitable for perfect teleportation and superdense coding.
Agarwal and Pati showed that there exist a class of states within the W states category which are suitable for the above mentioned purposes.
Our work identifies those three-qubit entangled states from the set of W states category which are most efficient for quantum entanglement teleportation.

Schematic diagram





Alice
Bob
1
4

Recently, the class of states $|W_n>$ was defined by

$$|W_n> = \frac{1}{\sqrt{(2+2n)}}(|100> + \sqrt{n}e^{i\gamma}|010> + \sqrt{n+1}e^{i\delta}|001>)$$

For simplicity we have ignored the phase factors and rewrite the above state as

$$|W_n>_{345} = 1\big/\sqrt{(2+2n)}\ (|100> + \sqrt{n}\ |010> + \sqrt{n+1}|001>)$$

where 'n' is a positive real number and γ and δ are phases.

Let us consider a two-qubit entangled state

$$|\psi>_{12} = \alpha|00> + \beta|11>$$

where $\alpha^2 + \beta^2 = 1$

Perfect teleportation and superdense coding with W states – P. Agarwal and A. K. Pati
Phys. Rev. A 74, 062320 (2006).

The combined system of five particles can be written as a tensor product of $|\psi>_{12}$ *and* $|W_n>_{345}$

$$
\begin{aligned}
|\chi>_{12345} &= |\psi>_{12} \otimes |W_n>_{345} \\
&= 1/2\sqrt{1+n}\,[|\Phi^+>_{23}(\sqrt{n}\,\alpha|010>_{145}+\sqrt{n+1}\,\alpha|001>_{145}+\beta|100>_{145}) \\
&\quad +|\Phi^->_{23}(\sqrt{n}\,\alpha|010>_{145}+\sqrt{n+1}\,\alpha|001>_{145}-\beta|100>_{145}) \\
&\quad +|\Psi^+>_{23}(\alpha|000>_{145}+\sqrt{n}\,\beta|110>_{145}+\sqrt{n+1}\,\beta|101>_{145}) \\
&\quad +|\Psi^->_{23}(\alpha|000>_{145}-\sqrt{n}\,\beta|110>_{145}-\sqrt{n+1}\,\beta|101>_{145})]
\end{aligned}
$$

Alice then performs measurement on the qubits 2 and 3 in the Bell-basis $\{|\Phi^+>, |\Phi^->, |\Psi^+>, |\Psi^->\}$

where

$$
|\Phi^{\pm}> = \frac{1}{\sqrt{2}}(|00> \pm |11>)
$$

$$
|\Psi^{\pm}> = \frac{1}{\sqrt{2}}(|01> \pm |10>)
$$

After Bell state measurement the outcomes are

$$|\chi> \xrightarrow{|\Phi^{+}>} \sqrt{n}\,\alpha|010> + \sqrt{n+1}\,\alpha|001> + \beta|100>$$

$$|\chi> \xrightarrow{|\Phi^{-}>} \sqrt{n}\,\alpha|010> + \sqrt{n+1}\,\alpha|001> - \beta|100>$$

$$|\chi> \xrightarrow{|\psi^{-}>} \alpha|000> - \sqrt{n}\,\beta|110> - \sqrt{n+1}\,\beta|101>$$

$$|\chi> \xrightarrow{|\psi^{+}>} \alpha|000> + \sqrt{n}\,\beta|110> + \sqrt{n+1}\,\beta|101>$$

To give the information about the measurement result, Alice send 2-bits of classical information to her distant partner Bob.

Let us suppose that if the measurement result on particles 2 and 3 is $|\Phi^{\pm}>$ then

$$|\chi> \xrightarrow{|\Phi^{\pm}>} \sqrt{n}\,\alpha\,|010> + \sqrt{n+1}\,\alpha\,|001> \pm \beta\,|100>$$

Thereafter Bob performs von-Neumann measurement on particle 5.

(i) If the measurement result is $|0>$ then the final two-qubit state will be

$$\sqrt{n}\,\alpha|01> \pm \beta|10>$$

(ii) If the measurement result is $|1>$ then the reduced two-qubit state will become separable state.

To measure the efficiency of the quantum channel, we have to measure the concurrence of the final two-qubit state.

Therefore, the final concurrence and the initial concurrence are related by

$$C^{final} = C^{initial} \left[\frac{\sqrt{n}}{(n-1)\alpha^2 + 1}\right]$$

where

$$C^{initial} = 2\alpha\sqrt{1-\alpha^2}$$

Observations:

1. For $n=1$, the initial and final concurrences are equal and hence $|W_1>$ serve as a better state independent quantum channel for entanglement teleportation.

2. For $n \neq 1$, the initial and final concurrences are equal but in this case the quantum channels are state dependent.

The channel parameter and the input state parameter are related by $\alpha^2 = \frac{\sqrt{n}-1}{n-1}$

$$C^{final} = C^{initial}\left[\frac{\sqrt{n}}{(n-1)\alpha^2+1}\right]$$

3. There also exist other three-qubit quantum channels $|W_n>$ that can be used for entanglement teleportation but they cannot retain the initial entanglement in the final two-qubit state i.e. there exists positive real values of ‘n’ for which $C^{final} < C^{initial}$.

(i) For $0 < n < 1$, the range for α^2 is given by

$$0 < \alpha^2 < \frac{\sqrt{n}-1}{n-1}$$

(ii) For $n > 1$, the range for α^2 is given by

$$\frac{\sqrt{n}-1}{n-1} < \alpha^2 < 1$$

When the Bell-state measurement result is $|\Psi^{\pm}>$ followed by Von-Neumann measurement result $|0>$, the concurrence of the final state is given by

$$C^{final} = C^{initial} \, [\frac{\sqrt{n}}{(n-1)\beta^2+1}]$$

Note: Whatever be Bell-state measurement result, the final concurrence (i.e. the entanglement of the final state) vanishes if the von-Neumann measurement result is $|1>$.

Next we investigate the performance of $|W_n>$ as a quantum channel when considering the teleportation of mixed state entanglement.

We start with the family of Werner state which is of the form

$$\rho_{12} = p|\Phi^{+}><\Phi^{+}| + \frac{(1-p)}{4} I$$

$$= \begin{pmatrix} (1+p)/4 & 0 & 0 & p/2 \\ 0 & (1-p)/4 & 0 & 0 \\ 0 & 0 & (1-p)/4 & 0 \\ p/2 & 0 & 0 & (1+p)/4 \end{pmatrix}$$

where $0 \le p \le 1$.

If the Bell-state measurement outcome is $|\Phi^{\pm}>$ and the von Neumann measurement result is $|0>$ then the final two-qubit state will be

$$\rho_{14}=\begin{pmatrix} (1-p)/8 & 0 & 0 & 0 \\ 0 & n(1+p)/8 & \pm\sqrt{n}p/4 & 0 \\ 0 & \pm\sqrt{n}p/4 & n(1+p)/8 & 0 \\ 0 & 0 & 0 & n(1-p)/8 \end{pmatrix}$$

The concurrence of the state described by the density matrix ρ_{14} is given by

$$\begin{aligned} C(\rho_{14}) &= C(\rho_{12})\ [8\sqrt{n}/(n+1)^2] \quad , \ if \ \ p>1/3 \\ &= 0 \quad , \ if \ \ p\leq 1/3 \end{aligned}$$

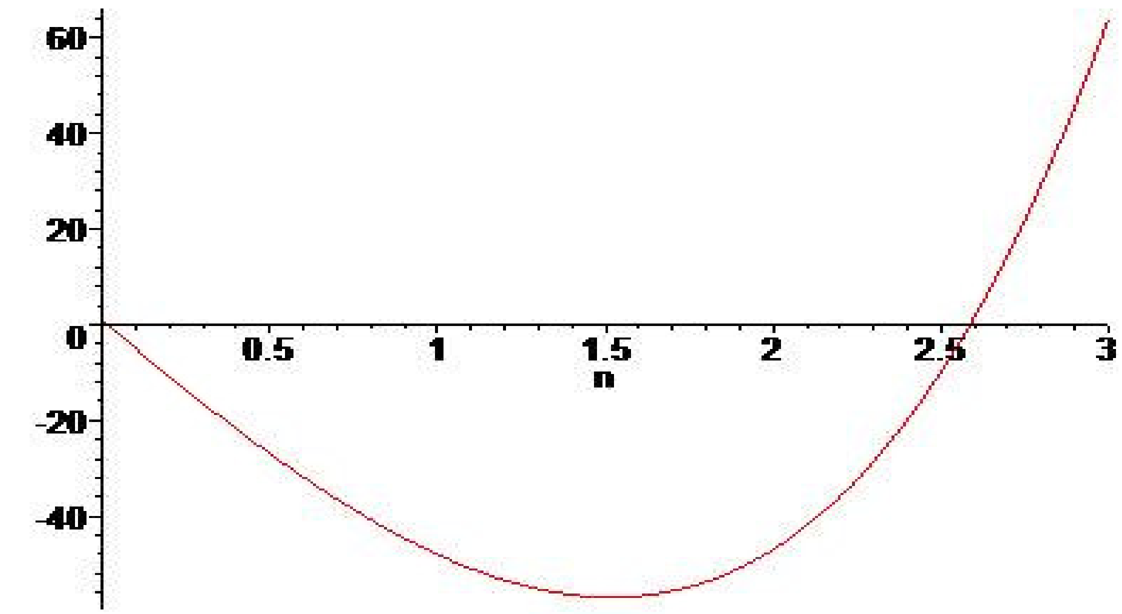
Where

$$C(\rho_{12}) = \frac{3p-1}{2}$$

Since entanglement cannot increase by LOCC so

$$C(\rho_{14}) \le C(\rho_{12})$$

$$\Rightarrow \quad n^4 + 4n^3 + 6n^2 - 60n + 1 \ge 0$$



If the Bell-state measurement outcome is $|\Psi^{\pm}>$ and the von-Neumann measurement result is $|0>$, then the two-qubit state will read

$$\rho'_{14} = \begin{pmatrix} (1+p)/8 & 0 & 0 & \pm\sqrt{n}\,p/4 \\ 0 & n(1-p)/8 & 0 & 0 \\ 0 & 0 & (1-p)/8 & 0 \\ \pm\sqrt{n}\,p/4 & 0 & 0 & n(1+p)/8 \end{pmatrix}$$

In this case, the concurrence is exactly equal to the concurrence obtained previously and hence the behaviour of the quantum channel is same as in the previous case.

Note: For any Bell-state measurement, if the von-Neumann measurement result is $|1>$ then the concurrence of the resultant state vanishes and hence $|W_n>$ state fails to act as a quantum channel.

To summarize, we have presented a protocol that measures the efficiency of the class of W-state. We observe that there exists two types of $|W_n>$ state which can serve as a quantum channel for entanglement swapping.
They can be classified as: (i) State independent channel and (ii) State dependent channel.
$|W_1>$ is regarded as a state independent channel because with some non-zero probability it helps in swapping the initial entanglement to the final two-qubit state for any arbitrary entangled two-qubit input states. On the other hand , there exist quantum channels which retains the initial entanglement in the final two-qubit state only for few entangled two-qubit state. Hence we call these type of channels as state dependent channel. We finally extend our analysis to the mixed state also.

Thank you