Quantum Information Theory II

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Hilbert space formulation of Quantum Mechanics

- **(i) Every quantum mechanical system S is associated with a Hilbert space H_S.**
- **(ii) Every state of the system S is described by a density operator $\rho : H_S \to H_S$.**
- **The state ρ can be a part of a joint state $|\Psi\rangle$ of $H_S \otimes H_T$, i.e., $Tr_T(|\Psi\rangle\langle\Psi|) = \rho$. $|\Psi\rangle$ is a 'purification' of ρ. This purification is not a physical process!**

Hilbert space formulation of QM (continued)

- **(iii) As a generalization of projective measurement $\{P_i|P_iP_j = \delta_{ij}P_j; i, j = 1, 2, \ldots, d\}$, every measurement on S is associated to a POVM $\{E_i|i = 1, 2, \ldots, N\}$, where $E_i : H_S \to H_S$ is a positive operator and $\sum_{i=1}^N E_i = I$. The probability of 'clicking' E_i is $Tr(E_i\rho)$. The output state is $(E_i^{1/2}\rho E_i^{1/2})/(Tr(E_i\rho))$ in this case, while the average output state is $\sum_{i=1}^N E_i^{1/2}\rho E_i^{1/2}$ in a particular realization of the POVM.**

Hilbert space formulation of QM (continued)

- **Each POVM $\{E_i|i=1,2,\ldots,N\}$ on S can be realized by a projective measurement $\{P_i|i=1,2,\ldots,N\}$ on $S+S'$ where $Tr_S(E_i\rho)=Tr_{S+S'}(P_iU(\rho\otimes\sigma_0)U^\dagger)$, σ_0 being a fixed state of S' and U being a unitary evolution of $S+S'$ after which $\{P_i|i=1,2,\ldots,N\}$ is measured.**
- **(iv) As a generalization of the unitary Schrodinger dynamics, the dynamics of the is described by a completely positive (CP) map $T:\mathcal{D}(H_S)\to\mathcal{D}(H_{S'})$, where $\mathcal{D}(H_S)$ and $\mathcal{D}(H_{S'})$ are the convex sets of all density operators of S and S' respectively.**

Hilbert space formulation of QM (continued)

- **A linear, Hermitian, positive, trace-preserving map $T' : \mathcal{B}(H_S) \to \mathcal{B}(H_{S'})$ from the Hilbert space $\mathcal{B}(H_S)$ of bounded linear operators on H_S (with $(A, B) \equiv Tr(A^\dagger B)$) to $\mathcal{B}(H_{S'})$ is also completely positive if for every Hilbert space H_A, the linear, Hermitian, trace-preserving map $(T' \otimes I) : \mathcal{B}(H_S \otimes H_A) \to \mathcal{B}(H_{S'} \otimes H_A)$ is again positive. Restrict $\mathcal{B}(H_S)$ to $\mathcal{D}(H_S)$ for our purpose.**
- **Unitary Schrodinger dynamics, non-selective POVM as well as their copmositions are all CP maps.**

Kraus representation

- **Every CP map $T' : \mathcal{B}(H_S) \to \mathcal{B}(H_{S'})$ can be represented by $T(\rho) = \sum_{i=1}^{M} A_i \rho A_i^\dagger$ where $A_i : H_S \to H_{S'}$'s are linear maps and $\sum_{i=1}^{M} A_i^\dagger A_i = I_S$.**
- Any map $T : \mathcal{D}(H_S) \to \mathcal{D}(H_{S'})$, which has a Kraus form, is always a CP map.
- **Every quantum mechanical operation (e.g., non-selective measurement, unitary evolution, taking trace, taking partial trace over a sub-system of a composite system, etc.) is a CP map.**

Realization of CP maps

- **Every CP map $T : \mathcal{D}(H_S) \to \mathcal{D}(H_S)$ can be obtained by: $T(\rho) = Tr_A(U(\rho \otimes \sigma_0)U^\dagger)$ for every ρ in $\mathcal{D}(H_S)$, σ_0 is a fixed state of A and $U : H_S \otimes H_A \to H_S \otimes H_A$ is unitary, for some suitably chosen H_A.**
- **Gorini, Kossakowski, Sudarshan, Lindblad: Dynamics of any open quantum system in the Lindblad form:**

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \sum_j (2L_j \rho L_j^\dagger - \{L_j^\dagger L_j, \rho\})$$

corresponds to a CP map $\rho(0) \mapsto \rho(t) \equiv V(t)\rho(0)$.

Quantum channels

- Quantum channel with input system S and output system $T \equiv$ CP map $T' : \mathcal{D}(H_S) \to \mathcal{D}(H_T)$.
- Bit-flip channel: $\rho \mapsto E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$, with

$$E_0 = \sqrt{p} I = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; E_1 = \sqrt{1-p}\sigma_x = \sqrt{1-p} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Quantum channels (continued)

- Phase-flip channel: $\rho \mapsto E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$, with

$$E_0 = \sqrt{p} I = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; E_1 = \sqrt{1-p}\sigma_z = \sqrt{1-p} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Quantum channels (continued)

- bit-phase flip channel: $\rho \mapsto E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$, with

$$E_0 = \sqrt{p} I = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; E_1 = \sqrt{1-p}\,\sigma_y = \sqrt{1-p} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Quantum channels (continued)

- **Depolarizing channel:** $\rho \mapsto p\frac{I}{2} + (1-p)\rho \equiv \sum_{i=0}^{3} E_i \rho E_i^\dagger$, with $E_0 = \sqrt{1-3p/4}I$, $E_1 = \sqrt{p}/2\sigma_x$, $E_2 = \sqrt{p}/2\sigma_y$, $E_3 = \sqrt{p}/2\sigma_z$.
- **Amplitude damping channel:** $\rho \mapsto E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$ with

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \cos\theta \end{pmatrix}, \; E_1 = \begin{pmatrix} 0 & \sin\theta \\ 0 & 0 \end{pmatrix} \quad (\theta \in [0, \pi/2]).$$

Why CP, why not just positive?

- Transposition map: Consider an orthonomal basis $\{|i\rangle : i = 1, 2, \ldots d = \text{ dim} H_S\}$ for H_S. For any $A \in \mathcal{B}(H_S)$, define its transposition $A^T : H_S \rightarrow H_S$ as $\langle i|A^T|j\rangle \equiv \langle j|A|i\rangle$. So the map $\mathcal{T} : A \mapsto A^T$ is a linear, Hermitian, trace-preserving, positive map. But the map $(\mathcal{T} \otimes I) : \mathcal{B}(H_S \otimes H_S) \rightarrow \mathcal{B}(H_S \otimes H_S)$, when acts on $|\Phi^+\rangle\langle\Phi^+|$, produces a <u>non-positive</u> operator, where $|\Phi^+\rangle = (1/\sqrt{d}) \sum_{i=1}^{d} |ii\rangle$.

Positive maps for testing entanglement

- Think of the map $(I \otimes \mathcal{T})$ as time-reversal (under Schrodinger evolution) of one subsystem of a composite system.
- Separable states: A state ρ of $H_S \otimes H_T$ is separable iff $\rho = \sum_{i=1}^{L} w_i \rho_i^{(S)} \otimes \sigma_i^{(T)}$, with $\rho_i^{(S)} \in \mathcal{D}(H_S)$, $\sigma_i^{(T)} \in \mathcal{D}(H_T)$, $0 \leq w_i \leq 1$, $\sum_{i=1}^{L} w_i = 1$. If ρ is not separable, then it is entangled.
- A state ρ of $S + T$ is separable iff for every positive map $\mathcal{A} : \mathcal{B}(H_S) \to \mathcal{B}(H_{S'})$, the operator $(\mathcal{A} \otimes I_T)(\rho) : H_S \otimes H_T \to H_{S'} \otimes H_T$ is positive.

Recalling Shannon (source coding)

- **For processing an arbitrarily large message of letters from the values of the random variable $X = \{x, p(x)\}$, its incompressible information content per letter is $H(X)$.**

Recalling Shannon (channel coding)

- **Given an output $Y = y$ of a noisy channel $\mathcal{N}$ for sending X, the incompressible information content for the probability distribution $\{\mathrm{Prob}(x|y) : x\}$ being $H(X|y)$, the average value of this information content is $H(X|Y) = \sum_y p(y) H(X|y)$. The information about X, gained by sending X through a channel $\mathcal{N}$ (characterized by the probabilities $\mathrm{Prob}(y|x)$, from which $\mathrm{Prob}(x|y)$'s can be obtained by using the Bayes' rule: $\mathrm{Prob}(x|y) = (\mathrm{Prob}(y|x) \times\ \mathrm{Prob}(x)) /\ \mathrm{Prob}(y)$ where $\mathrm{Prob}(y) \equiv \sum_x\ \mathrm{Prob}(y|x) \times\ \mathrm{Prob}(x)$) is $I(X;Y) = H(X) - H(X|Y) = H(X) + H(Y) - H(X,Y)$. The capacity of the channel $\mathcal{N}$ is $\max\{I(X;Y)|X\}$.**

von Neumann entropy

- **Now we encode the letter x by a density matrix ρ_x of S. So, the classical ensemble $X = \{x, p(x) : x\}$ is now replaced by the quantum ensemble $\rho = \{\rho_x, p(x) : x\}$, representing the density matrix $\rho = \sum_x p(x)\rho_x$. If we take the preparation $\{\rho_x, p(x) : x\}$ of ρ in its spectral decomposition (i.e., ρ_x's are pairwise orthogonal), then the information content about X, is $-\sum_x p(x) \log_2 p(x) = -Tr(\rho \log_2 \rho)$, as ρ_x's are distinguishable.**

von Neumann entropy (continued)

- **But what would be information content in ρ if its preparation is not known? von Neumann, considering phenomenological considerations of Thermodynamics, provided the formula for that information content. It is the von Neumann entropy $S(\rho) \equiv -Tr(\rho \log_2 \rho)$.**
- **Schumacher (1995) has shown that if we consider strings $\rho \otimes \rho \otimes \ldots \rho \equiv \rho^{\otimes n}$ of large length n, $S(\rho)$ gives the incompressible information content, in terms of qubits, of ρ.**

Shannon vs. von Neumann

- $S(\rho) \geq 0$, **and** $S(\rho) = 0$ **iff** ρ **is a pure state.**
- **For unitary** $U : H_S \to H_S$, $S(U\rho U^\dagger) = S(\rho)$.
- **For any** ρ, $S(\rho) \leq \log_2(\dim H_S)$. **(Recall that** $H(p_1, p_2, \ldots, p_n) \leq \log_2 n$.**)**
- $S(Tr_B(|\psi\rangle_{AB}\langle\psi|)) = S(Tr_A(|\psi\rangle_{AB}\langle\psi|))$.
- $S(\sum_{i=1}^N w_i\rho_i) \geq \sum_{i=1}^N w_i S(\rho_i)$, **where** w_i**'s are weights.**
- **If** $\{P_i | i = 1, 2, \ldots, d\}$ **is a projective measurement on** ρ, **then for the average output state** $\sum_{i=1}^d P_i\rho P_i \equiv \rho'$, $S(\rho') \geq S(\rho)$.

Shannon vs. von Neumann (continued)

- $H(X) \geq S(\rho = \sum_x p(x)|\psi_x\rangle\langle\psi_x|)$. **So the distingushability among the signals x by encoding them by non-orthogonal states $|\psi_x\rangle$. $S(\rho)$ is here a tight upper bound on the amount of classical information about the signals x that one can get by performing measurement on the state ρ (Holevo's bound).**
- **Subadditivity:** $S(\rho_{AB}) \leq S(Tr_B(\rho_{AB})) + S(Tr_A(\rho_{AB}))$**; equality iff** $\rho_{AB} = Tr_B(\rho_{AB}) \otimes Tr_A(\rho_{AB})$**. (Compare with the classical case:** $H(X,Y) \leq H(X) + H(Y)$**.)**

Shannon vs. von Neumann (continued)

- **Strong subadditivity:**

$S(\rho_{ABC})+S(Tr_{AC}(\rho_{ABC})) \leq S(Tr_C(\rho_{ABC}))+S(Tr_A(\rho_{ABC}))$. **(Compare with the classical case:**
$H(X,Y,Z)+H(Y) \leq H(X,Y)+H(Y,Z)$**.)**

- **Araki-Lieb inequality:**

$S(\rho_{AB}) \geq |S(Tr_B(\rho_{AB})) - S(Tr_A(\rho_{AB}))|$. **(Compare with the classical case:** $H(X,Y) \geq H(X), H(Y)$**.) But for a quantum state** ρ_{AB}**, it may happen that**
$S(\rho_{AB}) \leq S(Tr_B(\rho_{AB}))$.

Entropy vs. thermodynamics

- $S(\rho'_A)+S(\rho'_B)) \geq S(\rho'_{AB}) = S(U(\rho_A \otimes \rho_B)U^\dagger) = S(\rho_A \otimes \rho_B)$. **So the total entropy of the system $A+B$ increases under the interaction of A and B (Second law?).**

Schumacher compression

- **Consider the density matrix ρ given by the ensemble $\{p_x, |\phi_x\rangle\langle\phi_x|\}$, where $|\phi_x\rangle$'s are not necessarily orthogonal to each other. Consider now a string $\rho^{\otimes n} = \rho \otimes \rho \otimes \ldots n \text{ times}$ of length n, for large n.)**

Schumacher compression (continued)

- **Consider now the spectral decomposition of ρ: $\rho = \sum_{i=1}^{d} w_i |\psi_i\rangle\langle\psi_i|$. We have basically now a classical probability distribution: $X = \{i, w_i : i = 1, 2, \ldots, d\}$. Consider now the typical sequences $i_1 i_2 \ldots i_n$ will have probability $w_{i_1} w_{i_2} \ldots w_{i_n}$, which satisfies (for given $\delta, \epsilon > 0$) $2^{-n(H(X)-\delta)} \geq w_{i_1} w_{i_2} \ldots w_{i_n} \geq 2^{-n(H(X)+\delta)}$ and the sum of these probabilities exceeds $1 - \epsilon$.**

Schumacher compression (continued)

- **Thus we see that the subspace $S_{typical} \equiv \Lambda$ of $H_S^{\otimes}$, spanned by the pairwise orthogonal states $|\psi_{i_1}\rangle \otimes |\psi_{i_2}\rangle \otimes \ldots |\psi_{i_n}\rangle$, corresponding to the typical sequence $i_1 i_2 \ldots i_n$, has dimension equal to the total number $N(\epsilon, \delta; n)$ of such typical sequences, and for the projector P on this subspace Λ, $Tr(\rho^{\otimes n} P) =$ the total prob. of typical sequences $> 1 - \epsilon$.**
- **So we have: $2^{n(H(X)+\delta)} \geq N(\epsilon, \delta; n) \geq (1-\epsilon)2^{n(H(X)-\delta)}$, where $H(X) = S(\rho)$.**

Schumacher compression (continued)

- We now use some encoding unitary operation: $U|\Psi_{typical}\rangle = |\Psi_{comp}\rangle \otimes |0\rangle_{useless}$, where $|\Psi_{typical}\rangle$ is any state the typical subspace, $|\Psi_{comp}\rangle$ is a $n(S(\rho)+\delta)$-qubit state and $|0\rangle$ is a fixed state of $\Lambda^{\perp}$. Once we have $\Psi_{comp}\rangle$, we can now get back $|\Psi_{typical}\rangle$ by applying U^{-1} by appending $|0\rangle_{useless}$.

Schumacher compression (continued)

- **Consider now the states** $|\Phi_{i_1 i_2 \ldots i_n}\rangle = |\phi_{i_1}\rangle \otimes |\phi_{i_2}\rangle \otimes \ldots |\phi_{i_n}\rangle$ **from the ensemble** $\rho^{\otimes n} = \{w_{i_1} w_{i_2} \ldots w_{i_n}; |\Phi_{i_1 i_2 \ldots i_n}\rangle\}$**. We now perform measurement of** $\{P, I - P\}$**. Under this measurement,** $|\Phi_{i_1 i_2 \ldots i_n}\rangle\langle\Phi_{i_1 i_2 \ldots i_n}| \mapsto P|\Phi_{i_1 i_2 \ldots i_n}\rangle\langle\Phi_{i_1 i_2 \ldots i_n}|P + \rho^{junk}_{i_1 i_2 \ldots i_n}\langle\Phi_{i_1 i_2 \ldots i_n}|(I - P)|\Phi_{i_1 i_2 \ldots i_n}\rangle \equiv \rho'_{i_1 i_2 \ldots i_n}$ **(after applying above coding-decoding scheme).**

Schumacher compression (continued)

- **So the fidelity of this scheme:**

$F = \sum_{i_1 i_2 \ldots i_n} p_{x_{i_1}} p_{x_{i_2}} \ldots p_{x_{i_n}} \langle \Phi_{i_1 i_2 \ldots i_n} | \rho'_{i_1 i_2 \ldots i_n} | \Phi_{i_1 i_2 \ldots i_n} \rangle =$
$\sum_{i_1 i_2 \ldots i_n} p_{x_{i_1}} p_{x_{i_2}} \ldots p_{x_{i_n}} || P | \Phi_{i_1 i_2 \ldots i_n} \rangle ||^4 +$
$\sum_{i_1 i_2 \ldots i_n} p_{x_{i_1}} p_{x_{i_2}} \ldots p_{x_{i_n}} \langle \Phi_{i_1 i_2 \ldots i_n} | \rho^{junk}_{i_1 i_2 \ldots i_n} | \Phi_{i_1 i_2 \ldots i_n} \rangle \times$
$\langle \Phi_{i_1 i_2 \ldots i_n} | (I - P) | \Phi_{i_1 i_2 \ldots i_n} \rangle \geq$
$\sum_{i_1 i_2 \ldots i_n} p_{x_{i_1}} p_{x_{i_2}} \ldots p_{x_{i_n}} || P | \Phi_{i_1 i_2 \ldots i_n} \rangle ||^4 \geq$
$\sum_{i_1 i_2 \ldots i_n} p_{x_{i_1}} p_{x_{i_2}} \ldots p_{x_{i_n}} (2 || P | \Phi_{i_1 i_2 \ldots i_n} \rangle ||^2 - 1) =$
$2Tr(\rho^{\otimes n} P) - 1 > 1 - 2\epsilon$.

- **It can be shown to be optimal!**

Appendix: Completely positive maps

- Any dynamical operation $\mathcal{N}$ on states of a quantum system S must have the following properties:

(a) $\mathcal{N}$ must be linear; in other words, it should respect superposition principle.

(b) It should be Hermiticity preserving; in other words, observable should be transformed into a bonafide observable (think of the Heisenberg picture).

(c) It should be positivity as well as trace-preserving; in other words, each density matrix should be transformed into a bonafide density matrix.

- Sudarshan, Mathews and Rau [*Phys. Rev.* (1961)] has taken the above-mentioned three conditions (a), (b) and (c) as the <u>defining</u> conditions for the most general quantum dynamical operation.

Appendix: Completely positive maps (continued)

- **In that direction, let us consider a linear map $\mathcal{N} : \mathcal{B}(H_S) \to \mathcal{B}(H_{S'})$ from the Hilbert space $\mathcal{B}(H_S)$ of bounded linear operators on H_S to a Hilbert space $H_{S'}$ of bounded linear operators on $H_{S'}$. Let dim $H_S = d$ and dim $H_{S'} = d'$. Let us fix an orthonormal basis (ONB) $\{|e_i\rangle : i = 1, 2, \ldots, d\}$ for H_S and an ONB $\{|f_k\rangle : k = 1, 2, \ldots, d'\}$ for $H_{S'}$.**
- **For any density matrix ρ of S, let us write $\rho_{ij} = \langle e_i|\rho|e_j\rangle$. Also, with respect to the ONB $\{|f_k f_l\rangle\langle e_i e_j| : i, j = 1, 2, \ldots, d; k, l = 1, 2, \ldots, d'\}$, let us write $\langle e_i e_j|\mathcal{N}|f_k f_l\rangle = \mathcal{N}_{kl,ij}$.**
- **So, for the mapping $\rho \mapsto \mathcal{N}(\rho) \equiv \rho'$, we have the following matrix equations: $\rho'_{kl} = \sum_{i,j=1}^{d} \mathcal{N}_{kl,ij}\rho_{ij}$.**

Appendix: Completely positive maps (continued)

- **Condition (a) is automatically satisfied here via the above-mentioned matrix equations.**
- **Condition (b) implies that** $(\rho'_{lk})^* = \rho'_{kl}$ **if** $(\rho_{ji})^* = \rho_{ij}$**. So we have** $\sum_{i,j=1}^{d} (\mathcal{N}_{lk,ij})^* \rho_{ji} = \sum_{i,j=1}^{d} \mathcal{N}_{kl,ji} \rho_{ji}$**. Thus:**

$$(1)\quad \mathcal{N}_{kl,ij} = (\mathcal{N}_{lk,ji})^*, \quad i,j = 1,2,\ldots,d;\ k,l = 1,2,\ldots,d'.$$

- **Trace-preservation in condition (c) implies that** $\sum_{k=1}^{d'} \sum_{i,j=1}^{d} \mathcal{N}_{kk,ij} \rho_{ij} = \sum_{i,j=1}^{d} \rho_{ij} \delta_{ij}$**. Thus:**

$$(2)\qquad \sum_{k=1}^{d'} \mathcal{N}_{kk,ij} = \delta_{ij}, \quad i,j = 1,2,\ldots,d.$$

Appendix: Completely positive maps (continued)

- **The positivity demand in (c) implies that for all $(y_1, y_2, \ldots, y_{d'}) \in \mathbb{C}^{d'}$ and for all $(x_1, x_2, \ldots, x_d) \in \mathbb{C}^d$: $\sum_{k,l=1}^{d'} \sum_{i,j=1}^{d} y_k^* \mathcal{N}_{kl,ij} \rho_{ij} y_l \geq 0$ whenever $\sum_{i,j=1}^{d} x_i^* \rho_{ij} x_j$.**
- **At this point, Sudarshan et al. considered a $(d'd) \times (d'd)$ matrix $\mathcal{T} \equiv (\mathcal{T}_{ki,lj})$, defined as: $\mathcal{T}_{ki,lj} \equiv \mathcal{N}_{kl,ij}$.**
- **Then, in terms of the linear operator $\mathcal{T}$, equation (1) says that $\mathcal{T}$ is Hermitian:**

$$\mathcal{T}_{ki,lj} = (\mathcal{T}_{lj,ki})^*. \tag{3}$$

Appendix: Completely positive maps (continued)

- **Equation (2) becomes:**

$$\sum_{k=1}^{d'} \mathcal{T}_{ki,kj} \equiv (Tr_{S'}(\mathcal{T}))_{ij} = \delta_{ij}, \quad i,j = 1,2,\ldots,d. \tag{4}$$

- **Now by equation (3), we have the spectral decomposition of $\mathcal{T}$:**

$$\mathcal{T} = \sum_{k=1}^{d'} \sum_{i=1}^{d} \lambda_{ki} |\Psi_{ki}\rangle\langle\Psi_{ki}|, \tag{5}$$

Appendix: Completely positive maps (continued)

- **where** $\lambda_{ki} \in I\!R$, $|\Psi_{ki}\rangle \equiv \sum_{l=1}^{d'} \sum_{j=1}^{d} a_{lj}^{(ki)} |e_j f_l\rangle$**, and** $\sum_{l=1}^{d'} \sum_{j=1}^{d} (a_{lj}^{(ki)})^* a_{lj}^{(k'i')} = \delta_{kk'}\delta_{ii'}$**.**
- **Thus we see that:** $\mathcal{N}_{kl,ij} = \mathcal{T}_{ki,lj} = \langle e_i f_k | \mathcal{T} | e_j f_l \rangle = \sum_{k'=1}^{d'} \sum_{i'=1}^{d} \lambda_{k'i'} a_{ki}^{(k'i')} (a_{lj}^{(k'i')})^*$**.**
- **We then have** $\sum_{k,l=1}^{d'} \sum_{i,j=1}^{d} y_k^* \mathcal{N}_{kl,ij} \rho_{ij} y_l = \sum_{k,l=1}^{d'} \sum_{i,j=1}^{d} \sum_{k'=1}^{d'} \sum_{i'=1}^{d} \lambda_{k'i'} y_k^* a_{ki}^{(k'i')} \rho_{ij} y_l (a_{lj}^{(k'i')})^* = \sum_{k'=1}^{d'} \sum_{i'=1}^{d} \lambda_{k'i'} \{ \sum_{i,j=1}^{d} (\sum_{k=1}^{d'} a_{ki}^{(k'i')} y_k^*) \rho_{ij} (\sum_{l=1}^{d'} a_{lj}^{(k'i')} y_l^*)^* \} \equiv \sum_{k'=1}^{d'} \sum_{i'=1}^{d} \lambda_{k'i'} \{ \sum_{i,j=1}^{d} (x_i^{(k'i')})^* \rho_{ij} (x_j^{(k'i')}) \}$**, where** $x_i^{(k'i')} = (\sum_{k=1}^{d'} a_{ki}^{(k'i')} y_k^*)^*$ **for** $i = 1, 2, \ldots, d$**.**
- **As** ρ **is a state, therefore,** $\sum_{k,l=1}^{d'} \sum_{i,j=1}^{d} y_k^* \mathcal{N}_{kl,ij} \rho_{ij} y_l \geq 0$ **if all** $\lambda_{k'i'}$**'s are non-negative, i.e., if** $\mathcal{T} \geq 0$**.**

Appendix: Completely positive maps (continued)

- **Assumption (1): $\mathcal{T}$ is a positive operator.**
- **Thus we see that all the three conditions (a), (b) and (c) will be simultaneously satisfied if the linear operator $\mathcal{T} : \mathbb{C}^{d'd} \to \mathbb{C}^{d'd}$ is Hermitian, $Tr_{S'}(\mathcal{T}) = I_{d\times d}$ and all the eigen values of $\mathcal{T}$ are non-negative, where dim$(H_{S'}) = d'$.**
- **We will now see that the Assumption (1) is stronger than what is needed to satisfy all the three conditions (a), (b) and (c).**
- **Consider an arbitrary quantum system T where dim$(H_T) = D$ can be any positive integer. Consider now the linear map $(\mathcal{N} \otimes I) : \mathcal{B}(H_S \otimes H_T) \to \mathcal{B}(H_{S'} \otimes H_T)$, where $I : \mathcal{B}(H_T) \to \mathcal{B}(H_T)$ is the identity map.**

Appendix: Completely positive maps (continued)

- **Any density matrix σ of the combined system $S+T$ can be written as: $\sigma = \sum_{\alpha=1}^{M} w_\alpha \rho_\alpha^{(S)} \otimes \tau_\alpha^{(T)}$, where $\rho_\alpha^{(S)} \in \mathcal{D}(H_S)$, $\tau_\alpha^{(T)} \in \mathcal{D}(H_T)$, and w_α's are real numbers such that $\sum_{\alpha=1}^{M} w_\alpha = 1$.**
- **So $(\mathcal{N} \otimes I)(\sigma) = \sum_{\alpha=1}^{M} w_\alpha (\mathcal{N}(\rho_\alpha^{(S)}) \otimes \tau_\alpha^{(T)})$.**
- **Then $\{(\mathcal{N} \otimes I)(\sigma)\}^\dagger = \sum_{\alpha=1}^{M} w_\alpha (\{\mathcal{N}(\rho_\alpha^{(S)})\}^\dagger \otimes \{\tau_\alpha^{(T)})\}^\dagger = \sum_{\alpha=1}^{M} w_\alpha (\mathcal{N}(\rho_\alpha^{(S)}) \otimes \tau_\alpha^{(T)})$, as $\mathcal{N}$ is a Hermiticity-preserving operator. So $(\mathcal{N} \otimes I)$ is a Hermiticy-preserving operator.**
- **$Tr[(\mathcal{N} \otimes I)(\sigma)] = \sum_{\alpha=1}^{M} w_\alpha (Tr[\mathcal{N}(\rho_\alpha^{(S)})] \times Tr[\tau_\alpha^{(T)}]) = \sum_{\alpha=1}^{M} w_\alpha = 1$, as $\mathcal{N}$ is a trace-preserving map. So $(\mathcal{N} \otimes I)$ is trace-preserving.**

Appendix: Completely positive maps (continued)

- **Finally, for any** $|\eta\rangle = \sum_{k=1}^{d'} \sum_{m=1}^{D} b_{km} |f_k g_m\rangle \in (H_{S'} \otimes H_T)$ **(where** $\{|g_m\rangle : m = 1, 2, \ldots, D\}$ **is an ONB for** H_T**), we have** $\langle\eta|(\mathcal{N} \otimes I)(\sigma)|\eta\rangle = \sum_{\alpha=1}^{M} w_\alpha \langle\eta|(\mathcal{N}(\rho_\alpha^{(S)}) \otimes \tau_\alpha^{(T)})|\eta\rangle =$ $\sum_{\alpha=1}^{M} \sum_{k,l=1}^{d}{}' \sum_{m,n=1}^{D} w_\alpha b_{km}^* b_{ln} \langle f_k|\mathcal{N}(\rho_\alpha^{(S)})|f_l\rangle\langle g_m|\tau_\alpha^{(T)}|g_n\rangle =$ $\sum_{\alpha=1}^{M} \sum_{i,j=1}^{d} \sum_{k,l=1}^{d'} \sum_{m,n=1}^{D} w_\alpha b_{km}^* b_{ln} \mathcal{T}_{ki,lj} \langle e_i|\rho_\alpha^{(S)}|e_j\rangle\langle g_m|\tau_\alpha^{(T)}|g_n\rangle$ **=** $\sum_{\alpha=1}^{M} \sum_{i,i',j=1}^{d} \sum_{k,k',l=1}^{d'} \sum_{m,n=1}^{D} w_\alpha b_{km}^* b_{ln} \lambda_{k'i'} a_{ki}^{(k'i')} (a_{lj}^{(k'i')})^* \times$ $\langle e_i|\rho_\alpha^{(S)}|e_j\rangle\langle g_m|\tau_\alpha^{(T)}|g_n\rangle =$ $\sum_{i,i',j=1}^{d} \sum_{k,k',l=1}^{d'} \sum_{m,n=1}^{D} b_{km}^* b_{ln} \lambda_{k'i'} a_{ki}^{(k'i')} (a_{lj}^{(k'i')})^* \times$ $\langle e_i g_m|(\sum_{\alpha=1}^{M} w_\alpha \rho_\alpha^{(S)} \otimes \tau_\alpha^{(T)})|e_j g_n\rangle =$ $\sum_{k'=1}^{d'} \sum_{i'=1}^{d} \lambda_{k'i'} \langle\Phi^{(k'i')}|\sigma|\Phi^{(k'i')}\rangle$**,**

Appendix: Completely positive maps (continued)

- **where** $|\Phi^{(k'i')}\rangle = \sum_{i=1}^{d} \sum_{m=1}^{D} (\sum_{k=1}^{d'} b_{km} (a_{ki}^{(k'i')})^*) |e_i g_m\rangle$.
- **Thus, under Assumption (1), it follows that** $(\mathcal{N} \otimes I)$ **is a positivity-preserving map, irrespective of the dimension** D **of** H_T.
- **So,** $\mathcal{N}$ **is CP map. What about the Kraus representation of** $\mathcal{N}$**?**
- **Here** $\langle f_k|\mathcal{N}(\rho)|f_l\rangle \equiv \sum_{i,j=1}^{d} \mathcal{N}_{kl,ij}\rho_{ij} = \sum_{i,j=1}^{d} \mathcal{T}_{ki,lj}\rho_{ij} = \sum_{i,j=1}^{d} \sum_{k'=1}^{d'} \sum_{i'=1}^{d} \lambda_{k'i'} a_{ki}^{(k'i')} (a_{lj}^{(k'i')})^* \langle e_i|\rho|e_j\rangle \equiv \sum_{i,j=1}^{d} \sum_{k'=1}^{d'} \sum_{i'=1}^{d} \langle f_k|A^{(k'i')}|e_i\rangle\langle e_i|\rho|e_j\rangle\langle e_j|(A^{(k'i')})^\dagger|f_l\rangle = \langle f_k|\{\sum_{k'=1}^{d'} \sum_{i'=1}^{d} A^{(k'i')}\rho(A^{(k'i')})^\dagger\}|f_l\rangle$, **where** $A^{(k'i')} = \sum_{k=1}^{d'} \sum_{i=1}^{d} \sqrt{\lambda_{k'i'}} a_{ki}^{(k'i')} |f_k\rangle\langle e_i|$.

Appendix: Completely positive maps (continued)

- **Note that** $\sum_{k'=1}^{d'}\sum_{i'=1}^{d}(A^{(k'i')})^{\dagger}A^{(k'i')} = \sum_{i,j=1}^{d}\{\sum_{k=1}^{d'}(\sum_{k'=1}^{d'}\sum_{i'=1}^{d}\lambda_{k'i'}a_{ki}^{(k'i')}(a_{kj}^{(k'i')})^{*})\}^{*}|e_i\rangle\langle e_j| = \sum_{i,j=1}^{d}\{\sum_{k'=1}^{d'}\mathcal{T}_{ki,kj}\}^{*}|e_i\rangle\langle e_j| = \sum_{i,j=1}^{d}\delta_{ij}|e_i\rangle\langle e_j| = I_{d\times d}$.
- **Thus a Kraus representation for the CP map** $\rho\mapsto\mathcal{N}(\rho)$ **is given by:** $\mathcal{N}(\rho)=\sum_{k'=1}^{d'}\sum_{i'=1}^{d}A^{(k'i')}\rho(A^{(k'i')})^{\dagger}$.

- **Thus we see that Assumption (1) regarding the map** $\mathcal{T}$**, given by equation (5), not only makes the Hermiticity-preserving linear map** $\mathcal{N}$ **positive, it also makes it a CP map!**

Appendix: Completely positive maps (continued)

- **In the case when $d = d'$, Sudarshan et al. have <u>initially</u> taken the positivity condition for the map $\mathcal{N}$ as:**

$$\sum_{i,j,k,l=1}^{d} x_k^* y_l^* \mathcal{N}_{kl,ij} x_i y_j \geq 0 \text{ for all } x_i, y_j, x_k, y_l \in \mathbb{C}. \tag{6}$$

- **But after introducing the map $\mathcal{T}$, Sudarshan et al. have considered a <u>stronger</u> positivity condition, given as:**

$$\sum_{i,j,k,l=1}^{d} z_{ki}^* \mathcal{T}_{ki,lj} z_{lj} \geq 0 \text{ for all } z_{ki}, z_{lj} \in \mathbb{C}, \tag{7}$$

Appendix: Completely positive maps (continued)

- **instead of considering just the condition (6), i.e. the condition that** $\sum_{i,j,k,l=1}^{d} x_k^* x_i \mathcal{T}_{ki,lj} y_l^* y_j \geq 0$ for all $x_i, y_j, x_k, y_l \in \mathbb{C}$.
- **Note that condition (7) is nothing but Assumption (1)!**
- **Try to construct a linear operator** $U : (H_S \otimes H_T) \to (H_{S'} \otimes H_T)$ **and a fixed state** $|0\rangle_T \in H_T$**, where dim**$H_T = d'd$**, such that** $U^\dagger U = I_{(d'(d'd))\times(d'(d'd))}$ **and** ${}_T\langle e_{i'} f_{k'}|U|0\rangle_T \equiv A^{(k'i')}$ **for all** $k' = 1, 2, \ldots, d'$ **and for all** $i' = 1, 2, \ldots, d$**. This will give us:** $\mathcal{N}(\rho) = \sum_{k'=1}^{d'} \sum_{i'=1}^{d} {}_T\langle e_{i'} f_{k'}|U(\rho \otimes |0\rangle_T\langle 0|)U^\dagger|e_{i'} f_{k'}\rangle_T$ **for all states** ρ **of** S**.**

Appendix: Unitary realization of the bit-flip channel

- **Show that the unitary operator**

$U : H_S \otimes H_T \to H_S \otimes H_T$ **(where dim**$H_S =$ **dim**$H_T = 2$**), given by**

$U|00\rangle_{ST} = \sqrt{p}|00\rangle_{ST} + \sqrt{1-p}|11\rangle_{ST}$,

$U|10\rangle_{ST} = \sqrt{p}|10\rangle_{ST} + \sqrt{1-p}|01\rangle_{ST}$,

$U|01\rangle_{ST} = -\sqrt{p}|11\rangle_{ST} + \sqrt{1-p}|00\rangle_{ST}$,

$U|11\rangle_{ST} = -\sqrt{p}|01\rangle_{ST} + \sqrt{1-p}|10\rangle_{ST}$,

'realizes' the bit-flip channel, i.e., for any single-qubit density matrix ρ_S**, we have** $Tr_T[U(\rho_S \otimes |0\rangle_T\langle 0|)U^\dagger] = (\sqrt{p}I)\rho_S(\sqrt{p}I) + (\sqrt{1-p}\sigma_x)\rho_S(\sqrt{1-p}\sigma_x)$.

Appendix: Unitary realization of the phase-flip channel

- **Show that the unitary operator**

$U : H_S \otimes H_T \to H_S \otimes H_T$ **(where dim**$H_S = $ **dim**$H_T = 2$**),**
given by

$U|+0\rangle_{ST} = \sqrt{p}|+0\rangle_{ST} + \sqrt{1-p}|-1\rangle_{ST}$**,**
$U|-0\rangle_{ST} = \sqrt{p}|-0\rangle_{ST} + \sqrt{1-p}|+1\rangle_{ST}$**,**
$U|+1\rangle_{ST} = -\sqrt{p}|-1\rangle_{ST} + \sqrt{1-p}|+0\rangle_{ST}$**,**
$U|-1\rangle_{ST} = -\sqrt{p}|+1\rangle_{ST} + \sqrt{1-p}|-0\rangle_{ST}$**,**

'realizes' the phase-flip channel, i.e., for any single-qubit density matrix ρ_S**, we have**

$Tr_T[U(\rho_S \otimes |0\rangle_T\langle 0|)U^\dagger] = (\sqrt{1-p}\sigma_z)\rho_S(\sqrt{1-p}\sigma_z) +$
$(\sqrt{p}I)\rho_S(\sqrt{p}I)$**, where** $|\pm\rangle = (1/\sqrt{2})(|0\rangle \pm |1\rangle)$**.**

Appendix: Unitary realization of the bit-phase flip channel

- **Show that the unitary operator**

$U : H_S \otimes H_T \to H_S \otimes H_T$ **(where dim**$H_S =$ **dim**$H_T = 2$**), given by**

$$U|+y0\rangle_{ST} = \sqrt{p}|+y0\rangle_{ST} + \sqrt{1-p}|-y1\rangle_{ST},$$
$$U|-y0\rangle_{ST} = \sqrt{p}|-y0\rangle_{ST} + \sqrt{1-p}|+y1\rangle_{ST},$$
$$U|+y1\rangle_{ST} = -\sqrt{p}|-y1\rangle_{ST} + \sqrt{1-p}|+y0\rangle_{ST},$$
$$U|-y1\rangle_{ST} = -\sqrt{p}|+y1\rangle_{ST} + \sqrt{1-p}|-y0\rangle_{ST},$$

'realizes' the phase-flip channel, i.e., for any single-qubit density matrix ρ_S**, we have**

$$Tr_T[U(\rho_S \otimes |0\rangle_T\langle 0|)U^\dagger] = (\sqrt{1-p}\sigma_y)\rho_S(\sqrt{1-p}\sigma_y) + (\sqrt{p}I)\rho_S(\sqrt{p}I),$$ **where** $|\pm y\rangle = (1/\sqrt{2})(|0\rangle \pm i|1\rangle)$.

Appendix: Unitary realization of the amplitude damping channel

- **Show that the unitary operator**

$U : H_S \otimes H_T \to H_S \otimes H_T$ **(where dim**$H_S =$ **dim**$H_T = 2$**), given by**

$U|00\rangle_{ST} = |00\rangle_{ST}$,

$U|10\rangle_{ST} = \cos\theta|10\rangle_{ST} + \sin\theta|01\rangle_{ST}$,

$U|01\rangle_{ST} = |11\rangle_{ST}$,

$U|11\rangle_{ST} = \sin\theta|10\rangle_{ST} - \cos\theta|01\rangle_{ST}$,

'realizes' the amplitude damping channel, i.e., for any single-qubit density matrix ρ_S**, we have**

$Tr_T[U(\rho_S \otimes |0\rangle_T\langle 0|)U^\dagger] = E_0\rho_S E_0^\dagger + E_1\rho_S E_1^\dagger$**, where**

$0 \le \theta \le \pi/2$ **and** $E_0 = |0\rangle\langle 0| + \cos\theta|1\rangle\langle 1|$, $E_1 = \sin\theta|0\rangle\langle 1|$.

Appendix: Unitary realization of the depolarizing channel

- **Show that the unitary operator** $U : H_S \otimes H_T \to H_S \otimes H_T$ **(where dim**$H_S = 2$ **and dim**$H_T = 4$**), given by** $U|00\rangle_{ST} = \sqrt{1-3p/4}|00\rangle_{ST} + (\sqrt{p}/2)|01\rangle_{ST} + i(\sqrt{p}/2)|12\rangle_{ST} + (\sqrt{p}/2)|13\rangle_{ST}$**,** $U|10\rangle_{ST} = \sqrt{1-3p/4}|10\rangle_{ST} - (\sqrt{p}/2)|11\rangle_{ST} - i(\sqrt{p}/2)|02\rangle_{ST} + (\sqrt{p}/2)|03\rangle_{ST}$ **(actions of** U **on other basis elements defined suitably), 'realizes' the depolarizing channel, i.e., for any single-qubit density matrix** ρ_S**, we have** $Tr_T[U(\rho_S \otimes |0\rangle_T\langle 0|)U^\dagger] = (\sqrt{1-3p/4}I)\rho_S(\sqrt{1-3p/4}I) + ((\sqrt{p}/2)\sigma_x)\rho_S((\sqrt{p}/2)\sigma_x) + ((\sqrt{p}/2)\sigma_y)\rho_S((\sqrt{p}/2)\sigma_y) + ((\sqrt{p}/2)\sigma_z)\rho_S((\sqrt{p}/2)\sigma_z)$**.**