Structure of Quantum State Space

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Introduction
- Density Matrix
- Bloch Sphere

3-D System
- General Properties
- 3-D Quantum System
- 3-Section
- 4-Section

Definition

Density matrices are operators which acts on the Hilbert space of the given Quantum system. These are positive semi-definite operators with trace equals one. Density matrices represent the states of a quantum system.

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$$\rho = |\psi\rangle\langle\psi| \text{ For pure case and}$$
$$\rho = \sum p_i |\phi_i\rangle\langle\phi_i|$$

$$\rho \in \mathcal{B}(\mathcal{H}) : \mathcal{H} \to \mathcal{H}$$

Representation

In this talk we will represent density matrix ρ in one particular form. In this form one part contain the whole trace part and the other part is trace less.

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$$\mathrm{Tr}\Lambda = 0$$

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2-D Quantum System

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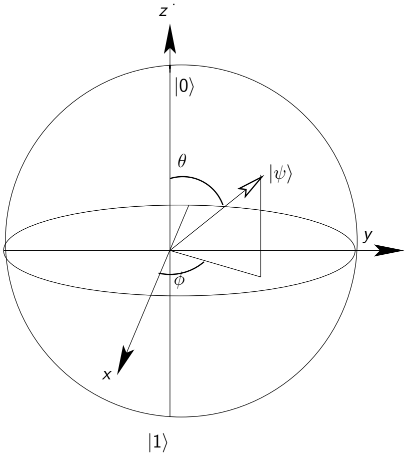
$$\mathrm{Tr}\rho^2 \leq 1 \implies |r_i|^2 \leq 1$$

Bloch Sphere



General Definitions

- Opposite points
- Hilbert Schmidt Norm

A line passing through a given fixed point will intersect a convex body at two poins. These point are called Opposite Points.

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$$||A||_2^2 = \mathrm{Tr}(A^\dagger A)$$

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- Points lying on the outer sphere are the pure state. So the structure on the outer sphere is only $2d-2$ dimensional.
- Point opposite to the pure state will be on the inner sphere.

3-D Quantum System

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3-D Quantum System

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- There is a 7 dimensional surface of this geometry.
- Four dimensional part of the surface is occupied by pure states.

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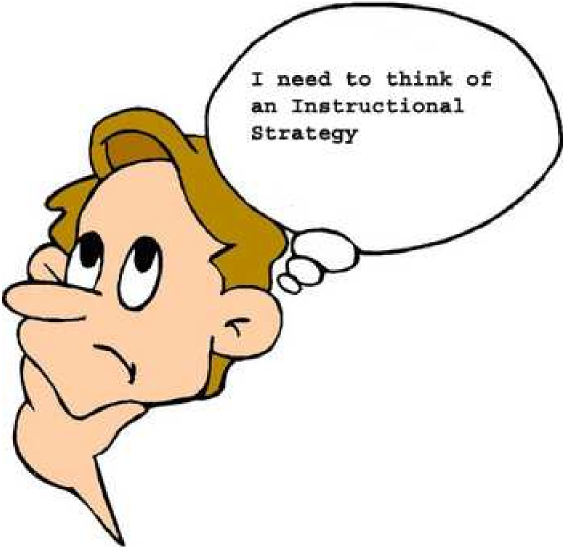
λ-Matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Now the problem is how to **visualize** the structure of quantum state space.


I need to think of an Instructional Strategy

Results

- People have studied 2-dimensional standard cross-sections.
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Gen Kimura and A. Kossakowski, Open Sys. Information Dyn. 12, 207 (2005)

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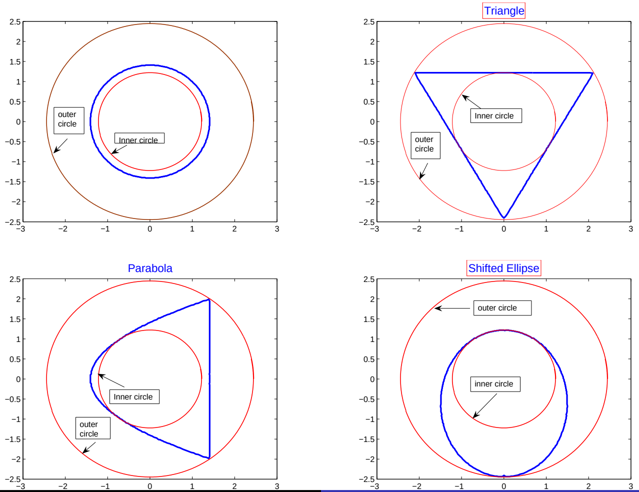
- People have studied 2-dimensional standard cross-sections.
- Higher dimensional Standard cross-sections has great importance as each three dimensional cross-section has infinitely many 2-sections.
- In case of three dimensional quantum system, studying 4-section will be enough as 5-sections are the complimentary to 3-sections and so on.

Standard 2-section



Circle	*Triangle*	*Parabola*	*Ellipse*
$area = 6.28$	$area = 7.8$	$area = 7.5$	$area = 8.6$
$12, 13, 23$ $14, 15, 16$ $17, 24, 25$ $26, 27, 45$ $46, 47, 56$ $57, 67$	18 28 38	34 35 36 37	48 58 68 78

3-Sections

- 3-Section has its own importance as each 3-Section include infinitely many 2-Section.
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- There are 56 standard 3-Sections.
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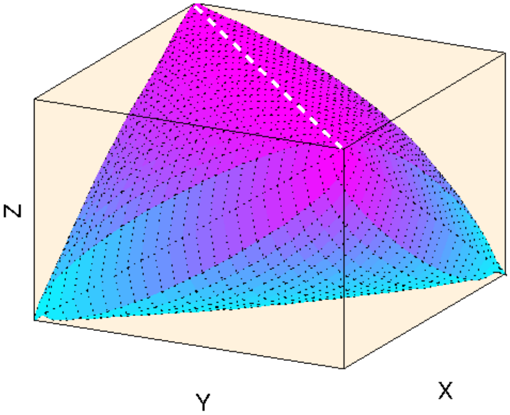
3-Sections

- 3-Section has its own importance as each 3-Section include infinitely many 2-Section.
- There are 56 standard 3-Sections.
- There are only 7 different standard 3-Sections.

Sphere	*Ellipsoid*	*cone*	*OT*	*RS1*	*RS2*	*Paraboloid*
123, 245	458	128	146	134	148	345
124, 246	468	138	157	135	158	367
125, 257	478	238	247	136	168	
126, 267	568	348	256	137	178	
127, 456	578	358	346	234	248	
145, 457	678	368	347	235	258	
147, 467		378	356	236	268	
156, 567			357	237	278	
167						

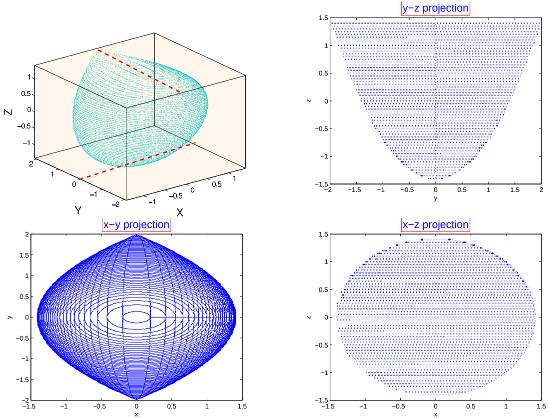
Obese Tetrahedron

$$x^2 + y^2 + z^2 - 2 - \sqrt{2}xyz = 0$$



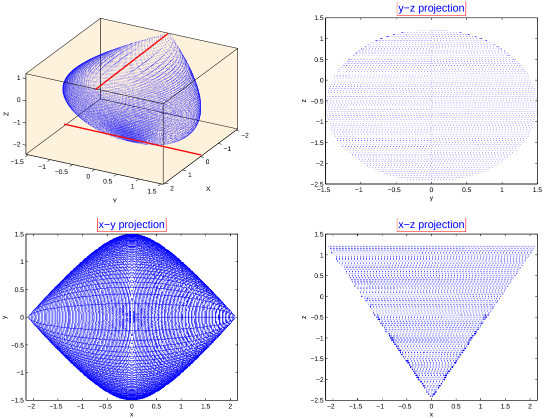
RS1

$$(x^2 + y^2 + z^2 - 2) - \frac{zy^2}{\sqrt{2}} = 0$$



RS2

$$\sqrt{2}(x^2+y^2+z^2-2)-\frac{z}{\sqrt{3}}\left(2x^2-y^2-\frac{2z^2}{3}\right)=0$$



4-Section

- There are seventy standard 4-sections.

4-Section

- There are seventy standard 4-sections.
- There are twelve family of unitarily inequivalent 4-sections.

sphere	*cone*	*ellipsoid*	*OT*	*RS*11	*RS*12	*RS*21	*RS*22	*D*1
1245	1238	4568	3456	3126	3167	8125	8245	3146
1267	4538	4678	3457	3124	3145	8127	8267	3157
4567	6738	5678	3467	3125	3245	8126	8167	3256
		4578	3567	3127	3267	8124	8145	3247

*UN*1	*UN*1	*UN*1	*UN*1	*UN*2	*UN*2	*UN*2	*UN*2	*D*2
1246	1456	1456	3257	8146	3814	3824	3846	8257
1247	1457	1567	3246	8157	3815	3825	3856	8246
1256	2456	2467	3147	8256	3816	3826	3847	8147
1257	2457	2567	3156	8247	3817	3827	3857	8156

$$\textbf{Cone}: \quad \left(\frac{r_8}{\sqrt{3}}+\sqrt{2}\right)^2 - r_3^2 - r_1^2 - r_2^2 = 0$$

$$\textbf{Ellipsoid}: \quad \frac{2}{3}\left(r_8+\frac{\sqrt{3}}{2\sqrt{2}}\right)^2 + r_4^2 + r_5^2 + r_6^2 = \frac{9}{4}$$

$$\textbf{OT}: \quad \sqrt{2}[2 - r_3^2 - r_7^2 - (r_4^2 + r_5^2)] - r_3[r_7^2 - (r_4^2 + r_5^2)] = 0$$

$$\textbf{RS11}: \quad \sqrt{2}[2 - r_3^2 - r_1^2 - r_2^2 - r_4^2] + r_3 r_4^2 = 0$$

$$\textbf{RS12}: \quad \sqrt{2}[2 - r_1^2 - r_3^2 - r_4^4 - r_5^2] + r_3[r_4^2 + r_5^2] = 0$$

$$\textbf{RS21}: \quad \sqrt{2}[2 - r_8^2 - (r_1^2 + r_2^2) - r_4^2] - \frac{2r_8}{\sqrt{3}}\left[\frac{r_8^2}{3} - (r_1^2 + r_2^2) - \frac{r_4^2}{2}\right] = 0$$

$$\textbf{RS22}: \quad \sqrt{2}[2 - r_8^2 - (r_4^2 + r_5^2) - r_1^2] - \frac{2r_8}{\sqrt{3}}\left[\frac{r_8^2}{3} - r_1^2 - \frac{r_4^2 + r_5^2}{2}\right] = 0$$

$$\textbf{UN1}: \quad \sqrt{2}[2-r_1^2-r_2^2-r_4^2-r_6^2]+2r_1r_4r_6=0$$

$$\textbf{UN2}: \quad \sqrt{2}[2-r_3^2-r_1^2-r_4^2-r_8^2]-\frac{2r_8}{\sqrt{3}}\left[\frac{r_8^2}{3}-r_1^2-r_3^2-\frac{r_4^2}{2}\right]=0$$

$$\textbf{D1}: \quad \sqrt{2}[2-r_3^2-r_6^2-r_1^2-r_4^2]-r_3[r_6^2-r_4^2]+2r_1r_4r_6=0$$

$$\textbf{D2}: \quad \sqrt{2}[2-r_8^2-r_2^2-r_5^2-r_7^2]-\frac{2r_8}{\sqrt{3}}\left[\frac{r_8^2}{3}-\frac{(r_7^2+r_5^2)}{2}-r_2^2\right]=0$$

Your proposal is innovative. Unfortunately, we won't be able to use it because we've never tried something like that before.