Nonclassicality and Entanglement

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Motivation

- Most of the work in entanglement has been done for systems of finite dimensional Hilbert space.
- Recently people have started working in the continuous variable case (Infinite dimensions).
- In the continuous variable case most of the work has been done on gaussian states.
- Present work deals with non gaussian states.

- Entanglement in two-mode gaussian states is primarily generated by Squeezing (a type of nonclassicality).

Present work aims to generate entanglement from other well known nonclassicalities

- Antibunching
- Nonclassical Photon Number Distributions

Basic idea

- Nonclassicality and entanglement are intimately connected.
- Demonstrate this aspect using states of form $\rho_{\rm in} = \rho_a \otimes |0\rangle_{bb}\langle 0|$ as input of a beamsplitter with the ouput $\rho_{\rm out}$ state being entangled.
  $\rho_a \otimes |0\rangle_{bb}\langle 0| \longrightarrow$ beamsplitter $\rightarrow \rho_{\rm out}$
  Nonclassical $\rho_a \Rightarrow$ Entangled $\rho_{\rm out}$.
- Extract useful(distillable) entanglement from the output state.

Outline

- Nonclassicality.
- Beamsplitter.
- Entanglement.
- Partial transpose.
- LOCC.
- Distillablity.
- Photon Number Distribution.
- Conversion of Nonclassicality into Entanglement.
- Results

Nonclassicality

For a two mode radiation field, any state can be represented in the Sudarshan's 'ϕ' representation as

$$\hat{\rho}^{(ab)} \;=\; \int_C \pi^{-1} d^2 z_a d^2 z_b \, \phi(z_a, z_b) |z_a\rangle\langle z_a| \otimes |z_b\rangle\langle z_b| \, ,$$

Classical states are identified as follows.

$$\hat{\rho}^{(ab)} \; \text{`classical}' \; \Leftrightarrow \; \phi(z_a, z_b) \geq 0 \text{ for all } z_a, \; z_b \; \in \mathcal{C}.$$

$\phi(z_a, z_b)$ is a classical probability.

- A classical state is separable by definition

Consider the operator

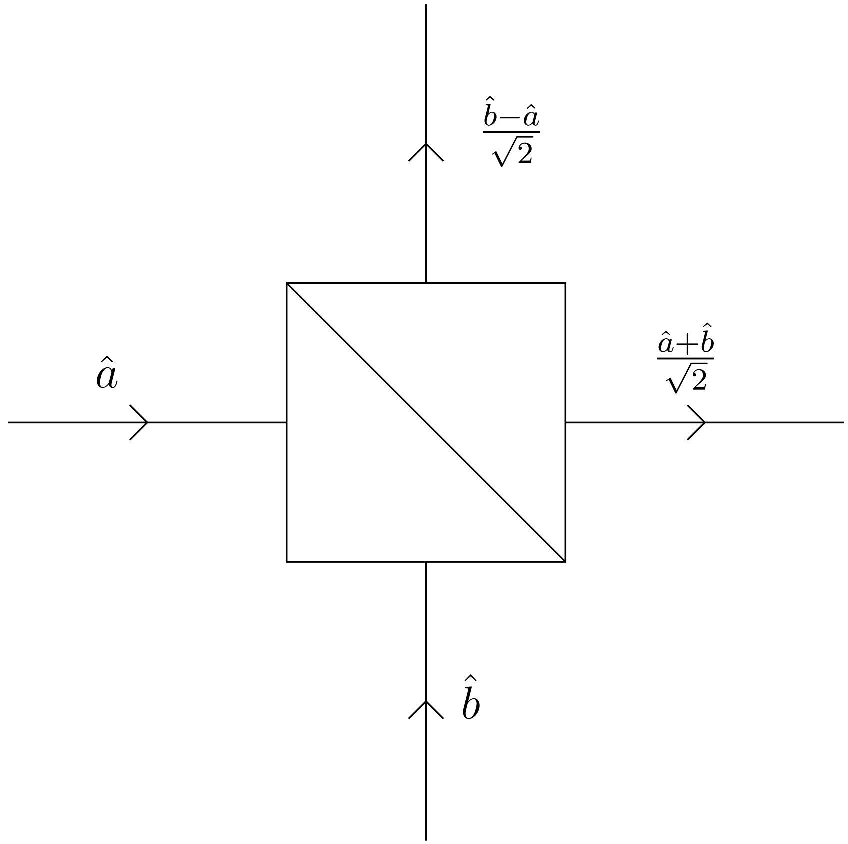
$$\eta(\hat{a},\hat{b}) = \sum_{jklm} c_{jklm}\hat{a}^{\dagger j}\hat{a}^{k}\hat{b}^{\dagger l}\hat{b}^{m}$$

$$\text{If } \operatorname{Tr}(\hat{\rho}^{(ab)} : \eta^{\dagger}\eta :) =$$
$$\int_C \pi^{-1} d^2 z_a d^2 z_b \, \phi(z_a, z_b) |\eta(z_a, z_b)|^2 < 0$$

Then the given state is nonclassical. The :: indicates normal ordering.

- $\eta^{\dagger}\eta$ is positive.

Beamsplitter



- Beamsplitter cannot take a classical state in the input to a nonclassical state at the output.

Entanglement

A bipartite system given by $A + B$ with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$.
A pure state $|\Psi\rangle$ is said to be entangled if

$$|\Psi\rangle \neq |\chi\rangle \otimes |\eta\rangle.$$

A mixed state $\hat{\rho}^{(ab)}$ is said to be entangled if it cannot be written as

$$\hat{\rho}^{(ab)} = \sum_k p_k \, \rho_{ak} \otimes \rho_{bk}$$

with

$$p_k \geq 0, \quad \sum p_k = 1$$

Partial Transpose

Given a bipartite system with Hilbert space $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ with basis states $|\Psi_{j\alpha}\rangle \equiv |\psi_j\rangle \otimes |\phi_\alpha\rangle$. Any $\hat{\rho}^{(ab)}$ can be expressed in this basis as

$$\hat{\rho}^{(ab)} = \sum_{j,k,\alpha,\beta} \rho_{j\alpha;k\beta} |\Psi_{j\alpha}\rangle\langle\Psi_{k\beta}|$$

Partial transpose is a map that takes $\hat{\rho}^{(ab)}$ to $\hat{\rho}^{(ab)PT}$ given by

$$\hat{\rho}^{(ab)PT} = \sum_{j,k,\alpha,\beta} \rho_{j\alpha;k\beta} |\Psi_{j\beta}\rangle\langle\Psi_{k\alpha}|$$

A separable state is taken to a physical state under partial transpose.

$$\mathrm{PT}: \ \hat{\rho}^{(ab)} = \sum p_k \, \rho_{ak} \otimes \rho_{bk}$$

$$\rightarrow \hat{\rho}^{(ab)PT} = \sum p_k \, \rho_{ak} \otimes \rho_{bk}^T$$

This need not be so for entangled states! The result of partial transpose can be a ‘nonstate’. Thus $\hat{\rho}^{(ab)}$ is entangled if $\hat{\rho}^{(ab)PT} \not\geq 0$. Such a state is said to be NPT (Negative under Partial Transpose).

- Not all entangled states are NPT

If

$$\mathrm{Tr}(\hat{\rho}^{(ab)PT}(\eta^\dagger\eta)) = \mathrm{Tr}(\hat{\rho}^{(ab)}(\eta^\dagger\eta)^{PT}) < 0,$$

then the given state is entangled, where

$$\eta = \sum_{jklm} c_{jklm}\hat{a}^{\dagger j}\hat{a}^k\hat{b}^{\dagger l}\hat{b}^m,$$

$$(\hat{a}^{\dagger j}\hat{a}^k\hat{a}^{\dagger p}\hat{a}^q\hat{b}^{\dagger l}\hat{b}^m\hat{b}^{\dagger r}\hat{b}^s)^{PT} = (\hat{a}^{\dagger j}\hat{a}^k\hat{a}^{\dagger p}\hat{a}^q\hat{b}^{\dagger s}\hat{b}^r\hat{b}^{\dagger m}\hat{b}^l)$$

- The operator $\eta^\dagger\eta$ is always positive.

LOCC

- LOCC: Local Operation and Classical Communication.
- The two parties are not allowed to perform any joint operations, but only local operations.
- An LOCC can be represented by a separable superoperator

$$\rho \rightarrow \rho^{'} = \frac{1}{p} \sum_{i} (A_i \otimes B_i) \rho (A_i^{\dagger} \otimes B_i^{\dagger})$$

where p is a normalisation constant.

Distillation

Distillation is the procedure by which we extract pure $2 \otimes 2$ singlets from multiple copies of a given $m \otimes n$ mixed state using local operation and classical communication (LOCC). When this is possible, the given entangled state is said to be *distillable*.

- Distillable entanglement is useful in processes such as teleportation.
- Not all entangled states are distillable.

ρ is distillable if there exists some finite n such that

$$\langle\psi|[\rho^{\otimes n}]^{T_B}|\psi\rangle < 0$$

where $|\psi\rangle$ is a Schmidt rank-2 state, or equivalently,

$$[(P_A \otimes P_B)\rho^{\otimes n}(P_A \otimes P_B)]$$

is NPT, where P_A and P_B are two rank-2 projectors acting on $\mathcal{H}_A^{\otimes n}$ and $\mathcal{H}_B^{\otimes n}$ respectively.

- A NPT state in $2 \otimes 2$ dimensions is distillable.

Photon Number Distribution

We consider a restricted class of states given by

$$\rho\left(\{p(n)\}\right) \equiv \sum_{n=0}^{\infty} p(n)|n\rangle\langle n|\,.$$

$$p(n) = \langle n|\rho|n\rangle \geq 0\,,$$

$$\sum_{n=0}^{\infty} p(n) = 1$$

The $(\{p(n)\})$ specify a Photon Number Distribution or PND.

Not all information in $\phi(z)$ is required to describe a PND. A simple angle averaged auxillary distribution $P(I)$ would do

$$P(I) = \int_0^{2\pi} \frac{d\theta}{2\pi} \phi(I^{1/2} e^{i\theta})$$

$$p(n) = \langle n|\rho|n\rangle = \int_0^{\infty} dI P(I) e^{-I} I^n/n!, \;\; n = 0, 1, 2, \cdots$$

A well known signature of nonclassicality is the antibunching condition given by

$$\langle \Delta n^2 \rangle - \langle\ n \rangle \equiv \langle\ n^2 \rangle - \langle\ n \rangle^2 - \langle\ n \rangle < 0$$

$$\equiv \int_0^{\infty} dI P(I)(I - \langle\ I \rangle)^2 \equiv (\Delta I)^2 < 0,$$

which would imply that $P(I)$ was not pointwise positive.

The pointwise positivity of $P(I)$ would imply

$$L^{(N)}, \quad \tilde{L}^{(N)} \geq 0 \quad N = 1, 2, \ldots.$$

where

$$L^{(N)} = \begin{pmatrix} q_0 & q_1 & q_2 & \cdot & \cdot & q_N \\ q_1 & q_2 & q_3 & \cdot & \cdot & q_{N+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_N & q_{N+1} & q_{N+2} & \cdot & \cdot & q_{2N} \end{pmatrix},$$

$$q_n = n! p(n) = \int_0^\infty dI P(I) e^{-I} I^n$$

and

$$\tilde{L}^{(N)} = \begin{pmatrix} q_1 & q_2 & q_3 & \cdot & \cdot & q_{N+1} \\ q_2 & q_3 & q_4 & \cdot & \cdot & q_{N+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_{N+1} & q_{N+2} & q_{N+3} & \cdot & \cdot & q_{2N+1} \end{pmatrix}.$$

$$q_n = n!p(n) = \int_0^\infty dIP(I)e^{-I}I^n$$

The positivity of $L^{(N)}$ and $\tilde{L}^{(N)}$ for all N is a necessary and sufficient test for classicality of $\rho = \sum_{n=0}^{\infty} p(n)|n\rangle\langle n|$.

An important consequence of the above result is that, even if one of the following infinite set of conditions given by

$$\begin{pmatrix} q_{n-1} & q_n \\ q_n & q_{n+1} \end{pmatrix} \geq 0 \qquad i.e,\ q_{n-1}q_{n+1} - {q_n}^2 \geq 0$$

is violated for any n we have nonclassicality. We call such a condition as a Three Term Classicality Condition (TTCC).

- A poissonian distribution saturates every TTCC.

Conversion of Nonclassicality into Entanglement

Take the input state $\rho_{\rm in} = \rho_a \otimes |0\rangle_{bb}\langle 0|$ where

$$\rho_a\left(\{p(n_a)\}\right) \equiv \sum_{n_a=0}^{\infty} p(n_a)|n_a\rangle\langle n_a| \, .$$

pass it through a beamsplitter to get

$$\rho_{\rm out} = U_{BS}\rho_{\rm in}U_{BS}^{-1}$$

The output state is

$$
\begin{aligned}
\rho_{\mathrm{out}} &= U\rho_{\mathrm{in}}U^{-1} \\
&= U\sum_{n_a=0}^{\infty}\frac{p(n_a)}{n_a!}(\hat{a}^\dagger)^{n_a}|0,0\rangle\langle 0,0|(\hat{a})^{n_a}U^{-1} \\
&= \sum_{n_a=0}^{\infty}\frac{p(n_a)}{2^{n_a}n_a!}(\hat{a}^\dagger+\hat{b}^\dagger)^{n_a}|0,0\rangle\langle 0,0|(\hat{a}+\hat{b})^{n_a} \\
&= \sum_{n_a=0}^{\infty}\frac{p(n_a)n_a!}{2^{n_a}}\sum_{r,s=0}^{n_a}\frac{|r,n_a-r\rangle\langle s,n_a-s|}{\sqrt{r!(n_a-r)!s!(n_a-s)!}}\,.
\end{aligned}
$$

The general matrix element of this density matrix is

$$\langle {n_a}', {n_b}' | \rho_{\rm out} | n_a, n_b \rangle$$
$$= \delta_{{n_a}'+{n_b}', n_a+n_b} \frac{(n_a+n_b)! p(n_a+n_b)}{2^{n_a+n_b}\sqrt{{n_a}'!{n_b}'!n_a!n_b!}} \,.$$

The matrix elements of the partial transpose $\tilde{\rho}_{\rm out}$ are obtained simply by interchanging n_b and ${n_b}'$:

$$\langle {n_a}', {n_b}' | \tilde{\rho}_{\rm out} | n_a, n_b \rangle$$
$$= \delta_{{n_a}'+n_b, n_a+{n_b}'} \frac{q_{n_a+{n_b}'}}{2^{n_a+{n_b}'}\sqrt{{n_a}'!{n_b}'!n_a!n_b!}} \,.$$

Results

- $\rho_a \otimes |0\rangle_{bb}\langle 0| \rightarrow$ beamsplitter $\rightarrow \rho_{\text{out}}$ with $\rho_a\left(\{p(n_a)\}\right) \equiv \sum_{n_a=0}^{\infty} p(n_a)|n_a\rangle\langle n_a|$. Nonclassical $\rho_a \Leftrightarrow L^{(N)}, \;\; \tilde{L}^{(N)} \ngeq 0 \Leftrightarrow \rho_{\text{out}}$ is NPT, hence entangled.
- NPT is a necessary and sufficient test of entanglement of ρ_{out}.
- If any TTCC is violated, we have distillable entanglement.
- If ρ_a is antibunched ρ_{out} is distillable.

THANK YOU