Combinatorial methods for studying LOCC incomparability

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- For multi-qubit systems, the joint state of the system is a tensor product of qubit states.

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- **I**f qubit A is with *Alice* and qubit B with *Bob* with their individual states,
  $|\Phi_A\rangle = \frac{1}{\sqrt{2}}|0_A\rangle + \frac{1}{\sqrt{2}}|1_A\rangle$
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- then the composite system AB is:
  $|\Phi_{AB}\rangle = |\Phi_A\rangle \otimes |\Phi_B\rangle$,
  $|\Phi_{AB}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$.

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- The state shared between parties A,B and C, $|\Phi_{ABC}\rangle = \frac{(|0_A 0_B 0_C\rangle + |1_A 1_B 1_C\rangle}{\sqrt{2}}$, is a *tri-partite entangled state* or $GHZ state$.

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- The above maximally entangled state is also called an $n - CAT$ *state*.

Local operations and classical communication

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- Other parties perform local operations conditioned on the results received.

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- If state $|\Phi\rangle$ can converted to the state $|\Psi\rangle$ using $LOCC$ transformations, then the two states are called $LOCC - comparable$, and this is denoted as $|\Phi\rangle \geq |\Psi\rangle$.

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- If neither $|\Phi\rangle \geq |\Psi\rangle$ nor $|\Psi\rangle \geq |\Phi\rangle$, then the two states are called $LOCC - incomparable$.
- Partial entropies cannot increase in the system due to LOCC.

Bicolor Merging Technique

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- *Bicolor merging* is a technique used to prove $LOCC - incomparability$ between two multipartite states using *partial entropic* criteria.
- It uses the fact that we cannot increase the *number of entanglements* shared between two parties under $LOCC$.
- In this technique we color vertices of the graph using two colors. We merge the adjacent vertices colored with same color.
- If number of EPR pairs in reduced graph of graph G_1 is greater than number of EPR pairs in reduced graph of graph G_2 then G_1 cannot be tranformed to G_2 using $LOCC$.

Bicolor merging v/s partial entropy

We can show that the *bicolor merging technique* is as powerful as the *partial entropic criteria*, when we restrict ourselves to only maximally entangled states.

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- The *degree of a vertex* cannot increase.
- Singh et al. [1] proved that two labeled EPR trees and r-uniform hypertrees with same number of vertices are $LOCC-incomparable$.
- As a generalization, we show that two EPR graphs with same number of edges and vertices are $LOCC$ comparable if and only if they are identical.

[1]S.K. Singh, S.P. Pal, Somesh Kumar, R. Srikant, *A combinatorial approach for studying local operations and classical communication transformation for multipartite states*, J. Math Phys 46, 122105(2005).

Results for hypergraphs

- We also show that two labeled r-uniform hypergraphs with the same vertex set and equal number of hyperedges are $LOCC$ comparable if and only if they are identical.

Result on connectivity of EPR graphs

- Using $LOCC$ transformations one cannot create an EPR pair between two disconnected vertices of an EPR graph.

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- Under this restricted model: Given two EPR graphs , Is $G \geq H$?
- Let $G(V, E_1)$ and $H = (V, E_2)$ be two EPR graphs on the same vertex set. Let $E_2 \setminus E_1 = \{\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_m, v_m\}\}$. Then, $H \leq G$ if and only if there are edge disjoint paths from u_i to v_i, $1 \leq i \leq m$, in $G(V, E_1 \setminus E_2)$.

G_{LOCC} and its properties

- We define a directed graph $G_{LOCC}(V, E)$ where (i) the nodes in the set V represent equivalence classes of such ensembles under $LOCC$ transformations, and (ii) the directed edge from node $X \in V$ to node $Y \in V$ implies that there is an ensemble x in the equivalence class X which can be transformed to an ensemble y in the equivalence class Y, using only $LOCC$ transformations.

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- G_{LOCC} is transitive graph, forms a partial order, and has no non-trivial cycles.

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- $\geq$ forms a *partial order* on the set of labeled r-uniform entanglement hypergraph.
- **Sperner's theorem:** For a set A, the number of sets in any antichain in A under the partial order induced by set inclusion cannot exceed $\binom{|A|}{\lceil (|A|+1)/2 \rceil}$.
- The maximum number of mutually LOCC incomparable EPR graph with n nodes is $\binom{M}{\lceil (M+1)/2 \rceil}$ where $M = \binom{n}{2}$.
  This is also called the *Width* of the partial order induced by $\geq$.

Result for r-uniform hyper-graphs

The maximum number of mutually LOCC incomparable r-uniform hypergraph with n nodes is $\binom{M}{\lceil (M+1)/2 \rceil}$ where $M = \binom{n}{r}$.

Definitions

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- If one or both of $|\phi\rangle \geq |\psi\rangle$ and $|\psi\rangle \geq |\phi\rangle$ hold, then we say that the two ensembles or states are *LOCC comparable*.

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- The *degree* of a vertex subset in a hypergraph is the number of hyperedges containing all vertices of the vertex subset, in the hypergraph. We use $E(G)$ to denote the set of (hyper)edges of a (hyper)graph G.

Restricted LOCC transformations

- Given an *EPR graph* we restrict LOCC to EPR pair (edge) destruction and teleportation. If the EPR graphs G and H are such that H can be obtained from G using such *restricted LOCC*, then we say $G \geq_R H$.

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- **Lemma 1:** Let G and H be two EPR graphs defined on the same vertex set V. Then, $G \geq_R H$ if and only if there are *edge disjoint paths* from u to v in G, for each edge $\{u, v\} \in H$.

Good LOCC transformations

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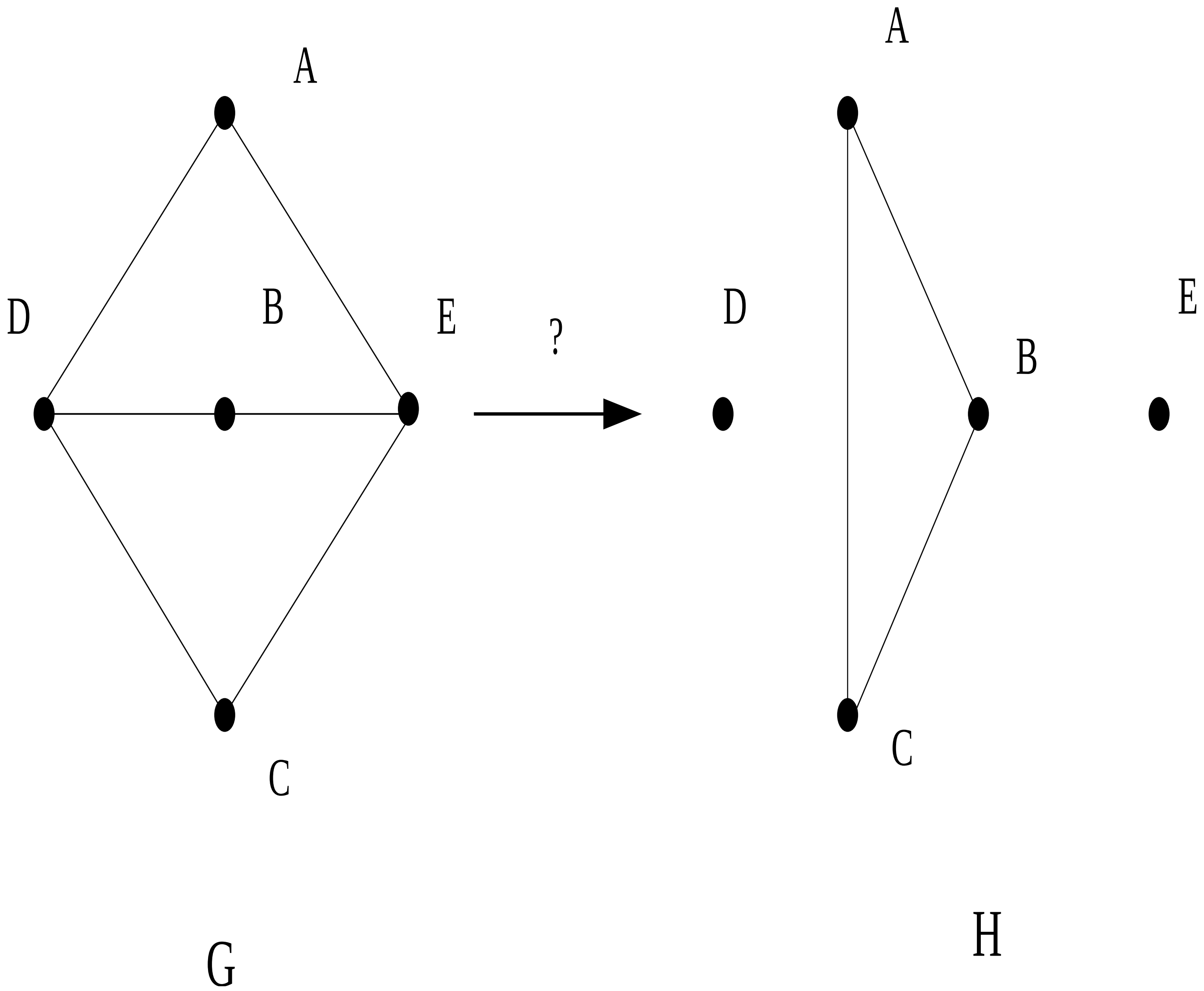
- **Lemma 2:** Suppose EPR graph G can be transformed to H via a sequence of good transformations. Then $G \geq_R H$.
- A sequence of good transformations can be simulated by a sequence of EPR pair destructions and teleportations.

Example...



Incomparability of the above example cannot be shown using partial entropy.

Cont...

- **Theorem 1:** Let G and H be EPR graphs defined on the same vertex set. Then, the following statements are equivalent.
  1. $G \geq_R H$
  2. There are edge disjoint paths in G from u to v for all edges $(u, v) \in H$.
  3. H can be obtained from G by a sequence of *good transformations*.

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  3. H can be obtained from G by a sequence of *good transformations*.
- **Conjecture:** *Good transformations* exhaust the set of all possible LOCC transformation between EPR graphs.

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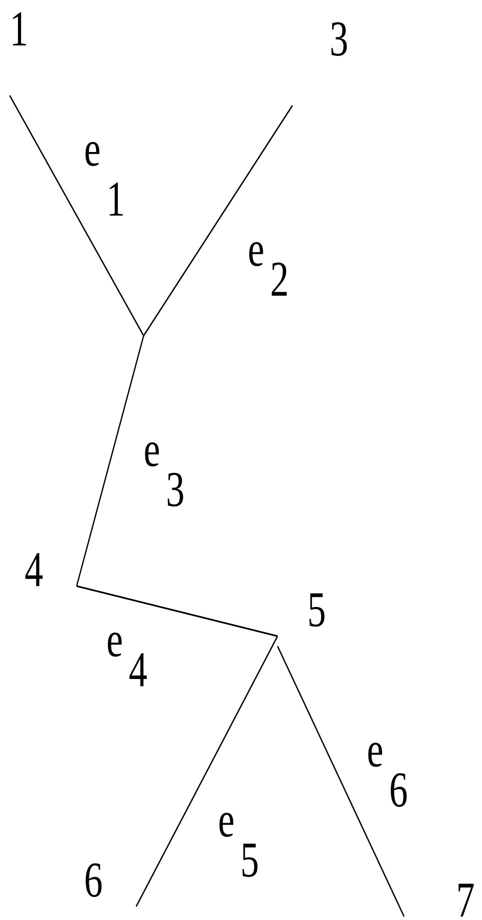
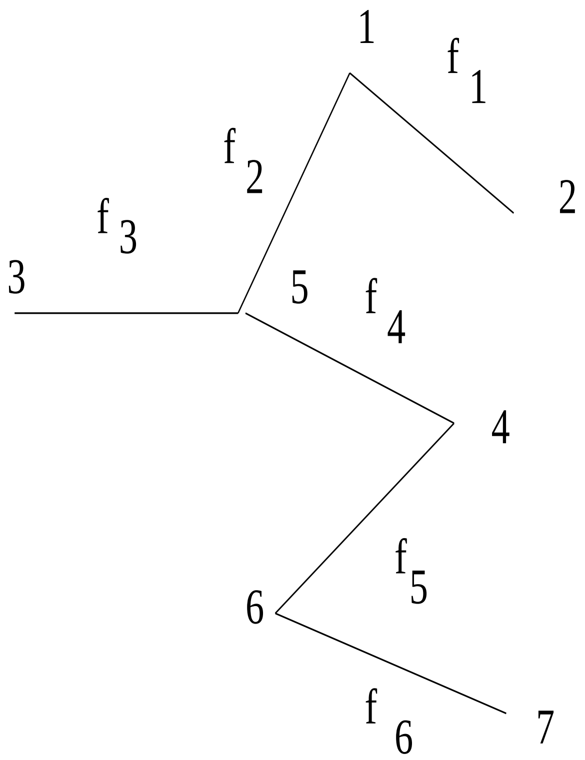
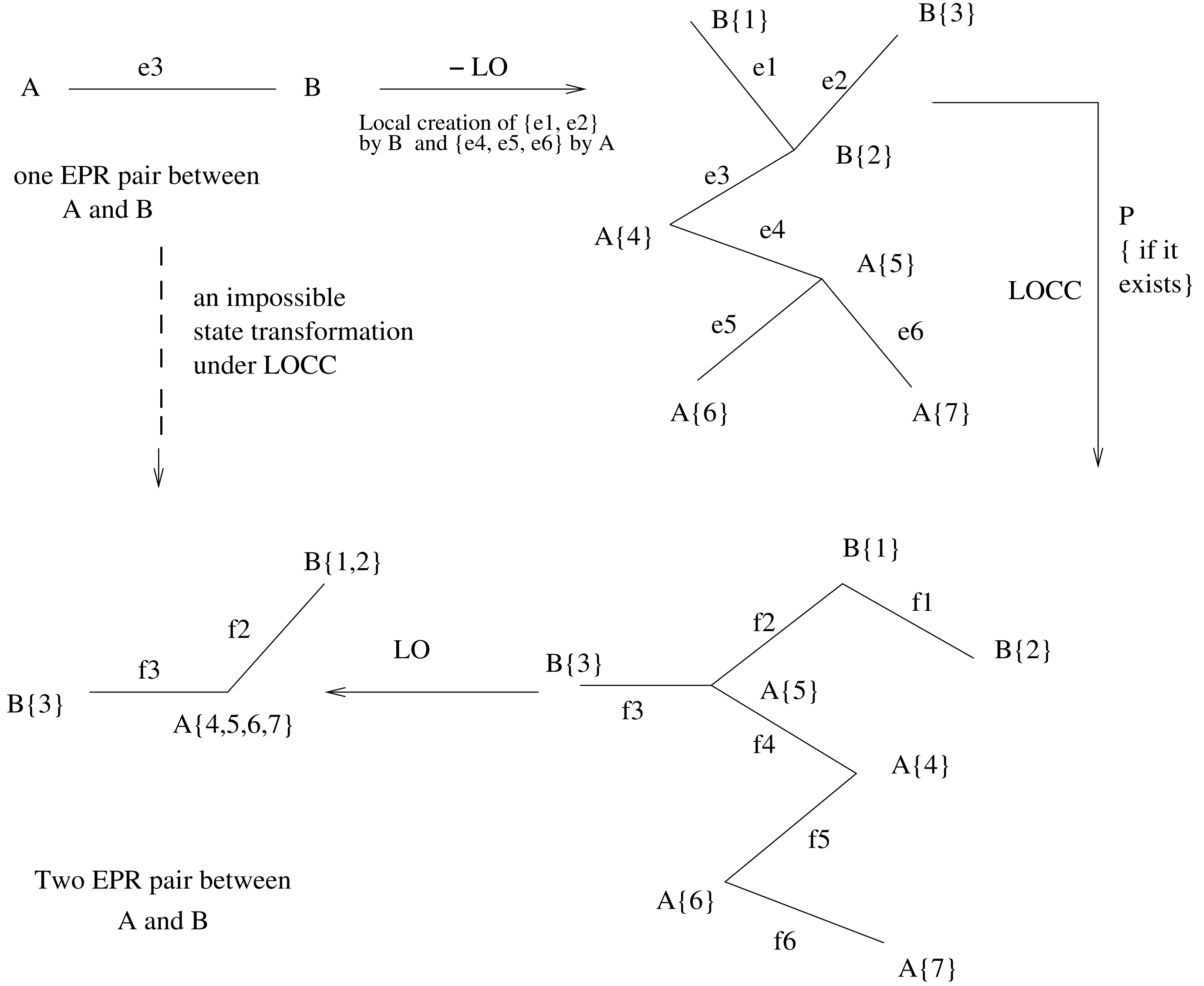


The spanning tree T_1



The spanning tree T_2

Proof of incomparability...



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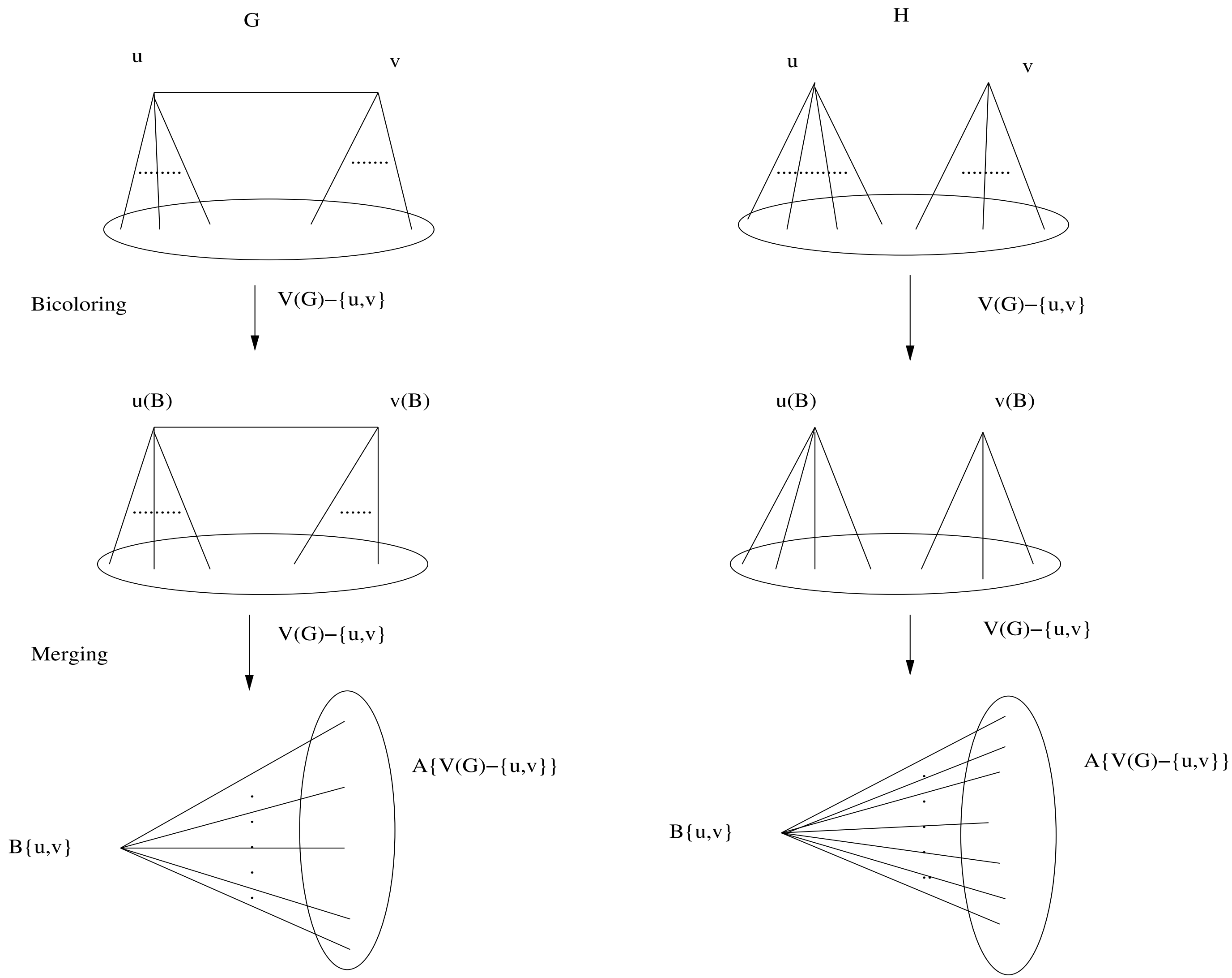
- **Lemma 1:** The degree of a vertex v in an *entanglement* hypergraph (or an *EPR graph*), cannot increase under *LOCC* transformations.
- **Theorem 2:** Any two distinct labeled EPR graphs with the same number of vertices and edges, are LOCC incomparable.
- **Theorem 3:** Any two distinct *labelled* r*-uniform entangled hypergraphs* defined on the same set V of vertices, with same number of hyperedges, are LOCC incomparable, $\forall$ even integer $r \geq 4$.

General Theorem

- **Theorem 4:** Any two r-*uniform entangled hypertrees* are LOCC incomparable for all $r \geq 2$.

Outline of the proof



Number of edges after bicolor merging = deg(u)+deg(v) − 2

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Degree lemma: General

- **Lemma 1.** *The degree of a vertex v in an EC hypergraph (or in an EPR graph) cannot increase under $LOCC$ transformations.*

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- **Lemma 3.** *The degree of a vertex v in an EC hypergraph (or in an EPR graph) cannot increase under $LOCC$ transformations.*

- *Proof.* Let H_1 be a *EC hypergraph* which can be transformed into another EC hypergraph H_2 by $LOCC$. For a vertex $v \in H_1$, define a bipartition of H_1 by placing v in one set and the remaining vertices in the other.

Applications of the degree lemma

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- **Theorem 2.** *Any two distinct labeled EPR graphs with the same number of vertices and edges are $LOCC$ incomparable.*

- $$\sum_{v \in V} deg_G(v) = \sum_{v \in V} deg_H(v) \qquad (.2)$$

  where $deg_G(v)$ ($deg_H(v)$) is the degree of the vertex $v \in V$ in EPR graph G (H).

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- **Theorem 3.** *Any two distinct labeled EPR graphs with the same number of vertices and edges are $LOCC$ incomparable.*

- $$\sum_{v \in V} deg_G(v) = \sum_{v \in V} deg_H(v) \qquad (.3)$$

  where $deg_G(v)$ ($deg_H(v)$) is the degree of the vertex $v \in V$ in EPR graph G (H).

- Further, by Lemma 3, the degree of no vertex can increase under LOCC. Therefore, the degrees of all vertices remain unchanged.

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- Define a bipartition $(\{u, v\}, V \setminus \{u, v\})$ of the graph G by coloring vertices u and v with color 1, and the rest of the vertices with color 2. The number of edges across the cut in this partition is $deg_G(u) + deg_G(v) - 2$.

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- Since (u, v) is not present in H, the same cut due to the same bipartition of the vertices will have $deg_H(v) + deg_H(u)$ edges in H.

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- Since (u, v) is not present in H, the same cut due to the same bipartition of the vertices will have $deg_H(v) + deg_H(u)$ edges in H.
- Since the degree of each labeled vertex is the same in both G and H, the number of edges in the reduced graph after bicolored merging

Degree lemma: Hypergraphs

Lemma 4. *Let* H_1 *and* H_2 *be two* r*-uniform EC hypergraphs defined on the same vertex set* V*. If* H_1 *and* H_2 *have the same number of hyperedges and* $H_1 \geq H_2$*, then the degrees of all vertices in* H_1 *and* H_2 *are the same.*

Proof. The degree of a vertex can not increase under $LOCC$ (see Lemma 3). Also, H_1 and H_2 have same number of hyperedges. Therefore the sum of degrees of all vertices in H_1 is equal to the sum of the degrees of all vertices in H_2. This enforces the degrees of all vertices to be same for hypergraphs H_1 to H_2. □

- We now exhibit two 3-uniform hypergraphs on 6 vertices having 4 edges each, such that all cut capacities are same in both the hypergraphs.

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$$H_2 = \{456\}, \{234\}, \{136\}, \{125\}$$

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- It is easy to verify that $deg_{H_1}(F) = deg_{H_2}(F)$ for $F \subset V$ and $|F| < 3$.
- From the above argument it follows that H_1 and H_2 cannot be shown to be incomparable by partial entropic characterizations.

- As $deg_{H_1}(F) = deg_{H_2}(F)$ for $F \subset V$ and $|F| < 3$ so they are isentropic. Also, marginally isentropic states are either LU (locally unitary) equivalent or incomparable.

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- Partition the vertices into three sets $A = \{1\}, B = \{2, 3\}, C = \{4, 5, 6\}$ and merge the vertices in the same sets.
- H_1 reduces to the EPR graph with edges $(A, B), (A, C)$ and two copies of (B, C).

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- Partition the vertices into three sets $A = \{1\}, B = \{2, 3\}, C = \{4, 5, 6\}$ and merge the vertices in the same sets.
- H_1 reduces to the EPR graph with edges $(A, B), (A, C)$ and two copies of (B, C).
- H_2 reduces to the entangled hypergraph with 2 GHZs shared between A, B, C and an EPR pair shared between B and C.

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- If H_1 and H_2 are LU-equivalent then so $R(H_1)$ and $R(H_2)$ must also be LU-equivalent.
- To prove that are not LU-equivalent, observe that the mixed state obtained by tracing out B from $R(H_2)$ i.e., $\rho_{AC}(R(H_2))$ is a maximally mixed, separable state of A and C.

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- This is not possible as one cannot create the entanglement (A, C) by $LOCC$.
- So, $R(H_2)$ and $R(H_2)$ are not LU equivalent, implying H_1 and H_2 are not LU equivalent. So, they are LOCC incomparable.

- We now state the main LOCC incomparability result for r-uniform EC hypergraphs using partial entropic criteria, for even integers $r \geq 4$.

- We now state the main LOCC incomparability result for r-uniform EC hypergraphs using partial entropic criteria, for even integers $r \geq 4$.
- **Theorem 5.** *Let H_1 and H_2 be any two labeled r-uniform EC hypergraphs defined on the set V of vertices. If H_1 and H_2 have the same number of hyperedges and either $H_1 \leq H_2$ or $H_2 \leq H_1$ then (i) $deg_{H_1}(S) = deg_{H_2}(S)$, $\forall S \subseteq V$ such that $|S| < r$, for all integers $r \geq 3$, and (ii) $H_1 = H_2$, for all even integers $r \geq 4$.*

- **Definition 1.** *For an EC hypergraph* H *with vertex set* V *and subset* $S \subseteq V$, $deg_H(S)$ *is defined as the number of hyperedges in* H *containing all the vertices of* S.

- **Definition 2.** *For an EC hypergraph H with vertex set V and subset $S \subseteq V$, $deg_H(S)$ is defined as the number of hyperedges in H containing all the vertices of S.*

- **Lemma 6.** *For a subset S of a r-uniform hypergraph H, the number of hyperedges across the cut $(S, V \setminus S)$ is given by*

$$\sum_{F \subseteq S} (-1)^{|F|-1} deg_H(F) - \sum_{F \subseteq S, |F|=r} deg_H(F)$$

- Let E be a hyperedge intersecting S in $t \leq r$ vertices. E contributes to the first part of the sum above through the terms $(-1)^{|F|-1} deg_{H_i}(F)$, $\forall F \subseteq E \cap S$, by virtue of the *inclusion-exclusion principle*.

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- The contribution equals (i) 1, for the t singleton subsets $F \subseteq E \cap S$ with $|F| = 1$, (ii) -1, for the $\binom{t}{2}$ subsets $F \subseteq E \cap S$ with $|F| = 2$, and so on, ending with $(-1)^{t-1}$, for the subset $F = E \cap S$.

- Let E be a hyperedge intersecting S in $t \leq r$ vertices. E contributes to the first part of the sum above through the terms $(-1)^{|F|-1} deg_{H_i}(F)$, $\forall F \subseteq E \cap S$, by virtue of the *inclusion-exclusion principle*.
- The contribution equals (i) 1, for the t singleton subsets $F \subseteq E \cap S$ with $|F| = 1$, (ii) -1, for the $\binom{t}{2}$ subsets $F \subseteq E \cap S$ with $|F| = 2$, and so on, ending with $(-1)^{t-1}$, for the subset $F = E \cap S$.
- The total contribution of E to the first part of the sum is $\sum_{i=1}^{t} (-1)^{i-1} \binom{t}{i} = 1$. The second part of the sum counts the number of hyperedges having

- Hyperedge E belongs to the cut $(S, V \setminus S)$ if and only if $0 < t < r$. In this case E contributes $+1$ to the first part and 0 to the second part of the sum, making a net contribution of 1.

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- For $t = r$ (and $t = 0$) the contribution of E to both parts of the above sum is 1 (and 0) respectively, making a net contribution of 0.
- Therefore,
$$\sum_{F \subseteq S} (-1)^{|F|-1} deg_H(F) - \sum_{F \subseteq S, |F| = r} deg_H(F)$$
equals the number of hyperedges across the cut $(S, V \setminus S)$.

- Let the two hypertrees defined on the vertex set $\{1, 2, \cdots, n\}$ be T_1 and T_2.

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- For $r = 2$ the result follows as all trees on n vertices have the same number of $n - 1$ edges.
- For $r = 3$, we assume without loss of generality that the hyperedge $\{1, 2, 3\}$ is in $T_1 \setminus T_2$. If T_1 and T_2 are not LOCC incomparable, then by part (i) in the last Theorem, we have $deg_{T_1}(\{1, 2\}) = deg_{T_2}(\{1, 2\})$.

- So, there should be a hyperedge $E1 = \{1, 2, x\}$ in $T_2 \setminus T_1$, where x is not in $\{1, 2, 3\}$; this hyperedge E_1 cannot be in T_1 because no hypertree has two hyperedges with two common vertices. Similarly,

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- T_2 must have hyperedges $E2 = \{1, 3, y\}$ and $E3 = \{2, 3, z\}$, where y is not in $\{1, 2, 3, x\}$ and z is not in $\{1, 2, 3, x, y\}$. This implies that the cycle $\{3, E2, 1, E1, 2, E3, 3\}$ is present in T_2, a contradiction.

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- For $r > 3$, we assume without loss of generality that the hyperedge $\{1, 2, \ldots r\}$ is in $T_1 \setminus T_2$.

- If T_1 and T_2 are not LOCC incomparable, then by part (i) of the last Theorem, we have $deg_{T_1}(\{1, 2, \ldots, r-1\}) = deg_{T_2}(\{1, 2, \ldots, r-1\})$ and $deg_{T_1}(\{2, 3, \ldots, r\}) = deg_{T_2}(\{2, 3, \ldots, r\})$.

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- So, there must be hyperedges in T_2 containing $\{1, 2, \ldots, r-1\}$ and $\{2, 3, \ldots, r\}$. As $r > 3$, these two hyperedges intersect in at least 2 vertices, a contradiction.

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- So, there must be hyperedges in T_2 containing $\{1, 2, \ldots, r-1\}$ and $\{2, 3, \ldots, r\}$. As $r > 3$, these two hyperedges intersect in at least 2 vertices, a contradiction.
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