Spin Squeezing in N-qubit Systems

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Heisenberg-Robertson inequality

$$(\Delta x)^2 (\Delta p_x)^2 \geq \frac{1}{4}, \qquad (1)$$

follows from

$$[x, p_x] = i \qquad (2)$$

where $(\Delta p_x)^2$ and $(\Delta x)^2$ are the variances of p_x and x. The state is squeezed in x if

$$(\Delta x)^2 < 1/2 \qquad (3)$$

or
squeezed in p_x if

$$(\Delta p_x)^2 < 1/2. \qquad (4)$$

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Similarly, a radiation field is said to be squeezed if one of the quadrature amplitudes satisfies the condition

$$(\Delta a_i)^2 < 1/2. \tag{5}$$

It is interesting to note that quantum mechanical correlations between photons established through non-liner interactions play a crucial role in the generation of squeezed states of light.

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As spin can be used to describe various physical systems, the problem of defining and achieving spin squeezing assumes a central role.
As before, the commutation relation,

$$[S_x, S_y] = iS_z \tag{6}$$

leads to the Heisenberg-Robertson uncertainty relation,

$$(\Delta S_x)^2(\Delta S_y)^2 \geq \frac{|\langle S_z \rangle|^2}{4}, \tag{7}$$

which would naturally suggest that a spin state would be regarded as squeezed if $(\Delta S_x)^2$ or $(\Delta S_y)^2$ is smaller than $\frac{|\langle S_z \rangle|}{2}$.

Here the expectation values and the variances are calculated in some arbitrary coordinate system.

Such a definition does not take into consideration the existence of quantum correlations and is coordinate dependent.

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A more general inequality

$$(\Delta S_x)^2(\Delta S_y)^2 - (\Delta S_x S_y)^2 \geq \frac{|\langle[S_x, S_y]\rangle|^2}{4} \tag{8}$$

where

$$(\Delta S_x S_y)^2 = \frac{\langle S_x S_y + S_y S_x\rangle}{2} - \langle S_x\rangle\langle S_y\rangle, \tag{9}$$

can be derived using Schwartz inequality.

The above relation, known as the Schrodinger inequality, is more precise and symmetric and provides a more stringent limitation (from below) to the product of two variances.

The vector nature of the spin operator leads one naturally to consider the mean spin direction $\hat{\mathbf{n}}$ as the preferred direction in space.
According to Kitagawa and Ueda the squeezing condition becomes,

$$(\Delta S_{n_\perp})^2 \langle \frac{|\langle S_n \rangle|}{2} \tag{10}$$

where $S_{n_\perp} = \vec{S}.\hat{n}_\perp$,$S_n = \vec{S}.\hat{n}$, $\hat{n} = \frac{\langle \vec{S} \rangle}{\sqrt{\langle \vec{S}.\vec{S} \rangle}}$ and $\hat{n}_\perp.\hat{n} = 0$.

Here $\hat{n}_\perp$ is arbitrary and can be along any one of the infinite directions in the plane perpendicular to $\hat{n}$.

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Therefore, we further make a rotation about the mean spin axis $\hat{n}$ such that the covariance (which also happens to be the correlation)

$$(\Delta S_{n_{1_\perp}} S_{n_{2\perp}})^2 = 0, \tag{11}$$

in the new coordinate system.

Here $\hat{n}, \hat{n}_{1_\perp}, \hat{n}_{2_\perp}$ form a triad of mutually perpendicular unit vectors.

This is possible since the correlation matrix in some arbitrary frame with mean spin direction $\hat{n}$ as the $\hat{z}-$ axis, can be brought to the form

$$\begin{bmatrix} \langle S^2_{n_{1_\perp}} \rangle & 0 & \langle S_{n_{1_\perp}} S_n + S_n S_{n_{1_\perp}} \rangle \\ 0 & \langle S^2_{n_{2_\perp}} \rangle & \langle S_{n_{2_\perp}} S_n + S_n S_{n_{2_\perp}} \rangle \\ \langle S_{n_{1_\perp}} S_n + S_n S_{n_{1_\perp}} \rangle & \langle S_{n_{2_\perp}} S_n + S_n S_{n_{2_\perp}} \rangle & \langle S^2_n \rangle - \langle S_n \rangle^2 \end{bmatrix} \tag{12}$$

by a planar rotation about the mean spin direction.

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It can be observed that in this frame, the Schroedinger inequality and the Heisenberg-Robertson inequality are of the same form and definition of squeezing becomes unambiguous and is given by

$$(\Delta S_{n_{1_\perp}})^2 \langle \frac{|\langle S_n \rangle|}{2} \tag{13}$$

or

$$(\Delta S_{n_{2_\perp}})^2 \langle \frac{|\langle S_n \rangle|}{2} \tag{14}$$

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Intrinsic spin associated with point particles such as electrons is described in terms of the *up* and *down* spinors which are well defined mathematically once the definition of spin

$$\vec{S} \times \vec{S} = i\vec{S} \tag{15}$$

is accepted.
Schwinger visualised any state $|sm\rangle$ as made up of $(s+m)$ *up* spinors and $(s-m)$ *down* spinors through ,

$$|sm\rangle = \frac{(a_+^\dagger)^{s+m}(a_-^\dagger)^{s-m}}{\sqrt{(s+m)!(s-m)!}}|00\rangle \tag{16}$$

where $a_+^\dagger, a_-^\dagger$ are the creation operators for the spin *up* and *down* states respectively.

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It must be noted here that spin *up* and spin *down* states as well as $|sm\rangle$ are all referred to the same axis of quantization. Recently it was shown that any general spin-s pure state $|\psi\rangle$ can be constructed using $2s$ spin -1/2 states which are specified with respect to $2s$ different directions $\hat{Q}_1(\theta_1\phi_1), \hat{Q}_2(\theta_2\phi_2)....., \hat{Q}_{2S}(\theta_{2s}\phi_{2s})$ such that

$$|\phi\rangle = \sum_m c_m |sm\rangle \tag{17}$$

where,

$$c_m = N_s d_m, N_s^{-1} = \{\sum_{m=-s}^{+s} |d_m|^2\}^{\frac{1}{2}} \tag{18}$$

and

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$$d_m = \sum_{m_1 m_2 \ldots m_{2s-1}} C(\tfrac{1}{2}\tfrac{1}{2}1; m_1 m_2 \mu_1) C(1\tfrac{1}{2}\tfrac{3}{2}; \mu_1 m_3 \mu_2) \ldots \tag{19}$$

$$C(s-\tfrac{1}{2}\,\tfrac{1}{2}S; \mu_{2s-2} m_{2s} m) D^{\frac{1}{2}}_{m_1 \frac{1}{2}}(\phi_1 \theta_1 0) \ldots D^{\frac{1}{2}}_{m_{2s} \frac{1}{2}}(\phi_{2s} \theta_{2s} 0). \tag{20}$$

In particular, if $\hat{Q}_1 = \pm\hat{Q}_2 = \ldots.. = \pm\hat{Q}_{2s}$, then this construction specialises to the realization suggested by Schwinger.

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Then $|\phi\rangle$ can be constructed using two spinors specified with respect to $\hat{Q}_1(\theta_1\phi_1)$ and $\hat{Q}_2(\theta_2\phi_2)$ as

$$|\phi\rangle = N_2 \sum_{m_1 m} D^{\frac{1}{2}}_{m_1 \frac{1}{2}}(\phi_1\theta_1 0|\phi\rangle) D^{\frac{1}{2}}_{m_2 \frac{1}{2}}(\phi_2\theta_2 0) C(\frac{1}{2}\frac{1}{2}1; m_1 m_2 m)|(\frac{1}{2}\frac{1}{2})1m \tag{21}$$

It can be shown that the mean spin direction $\hat{n}$ for a spin-1 pure state like $|\phi\rangle$ happens to be along the bisector of the two directions $\hat{Q}_1$ and $\hat{Q}_2$.

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Employing the frame $\hat{n}_{1_\perp}\hat{n}_{2_\perp}\hat{n}$ with $\hat{n}_{1_\perp}$ lying in the plane of $\hat{Q}_1$ and $\hat{Q}_2$, the state $|\phi\rangle$ has the explicit form

$$|\phi\rangle = \frac{-i\sqrt{2}}{\sqrt{Tr}(C^{12})1+\cos^2\theta}[\cos^2\frac{\theta}{2}|\frac{1}{2}\frac{1}{2}\rangle_{\hat{n}} - \sin^2\frac{\theta}{2}|-\frac{1}{2}-\frac{1}{2}\rangle_{\hat{n}}] \quad (22)$$

or

$$|\phi\rangle = \frac{-i\sqrt{2}}{\sqrt{1+\cos^2\theta}}[\cos^2\frac{\theta}{2}|11\rangle_{\hat{n}} - \sin^2\frac{\theta}{2}|-1-1\rangle_{\hat{n}}]. \quad (23)$$

For $\theta \neq 0, \pi$, $|\phi\rangle$ cannot be written as a simple product of the spin-1/2 states implying that it is indeed entangled.

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The squeezing condition for $S_{n_{1_\perp}}$ takes the form

$$\cos 2\theta \leq |\cos\theta|. \tag{24}$$

which is satisfied for all θ except when $\theta = 0, \pi/2, \pi$.

Thus for all other values of θ, the state ϕ is squeezed in either of the spin components $S_{n_{1_\perp}}$ or $S_{n_{2_\perp}}$.

Spin-spin correlation is defined through the relation

$$C^{12}_{\mu\nu} = \langle S_{1\mu} S_{2\nu} \rangle - \langle S_{1\mu} \rangle \langle S_{2\nu} \rangle \tag{25}$$

The eigen correlations satisfy the inequality

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$$0 \leq |C_{ii}^{12}| \leq 1/4 \; ; \quad i = \hat{n}, \hat{n}_{1_\perp}, \hat{n}_{2_\perp} \tag{26}$$

for all values of θ except when $\theta = 0, \pi/2, \pi$.

The values of c_{ii}^{12} for $\theta = 0, \pi/2, \pi$ are either 0 or $\pm 1/4$.

Thus this construction quantitatively demonstrates the existence of quantum correlations and indicates as to how they lead to squeezing behaviour in the case of pure spin-1 state.

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Trace of the correlation matrix

$$Tr(C^{12}) = \left[\frac{\sin^2\theta}{2(1+\cos^2\theta)}\right]^2 \tag{27}$$

is invariant. $|\phi\rangle$ is squeezed if

$$0\langle Tr(C^{12})\langle\frac{1}{4}$$

This is a necessary and sufficient condition for $|\phi\rangle$ to be squeezed.

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Given the correlation matrix, the value of θ can be found out through

$$\cos\theta = \pm\left[\frac{1 - 2Tr(C^{12})^{1/2}}{1 + 2Tr(C^{12})^{1/2}}\right]^{1/2} \tag{28}$$

which identifies the structure of the state in terms of two spinors.

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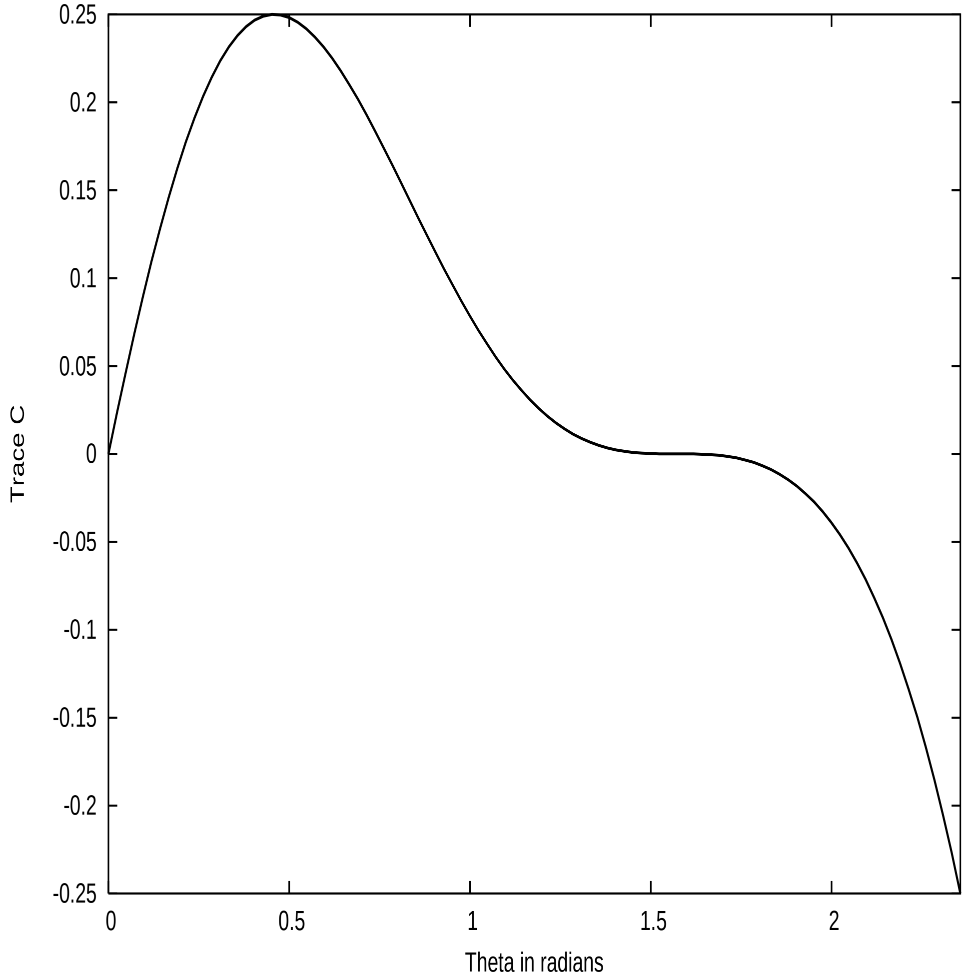
Experimentally squeezing can be achieved with a two-axis counter-twisting Hamiltonian of the form

$$H = \alpha(S_Y^2 - S_X^2) \tag{29}$$

Such a Hamiltonian can be implemented with two-laser fields of orthogonal polarisations and opposite de-tunings from an atomic resonance. By adjusting the liqht intensity and de-tuning one can obtain the Hamiltonian to be

$$H = H_o + \beta\frac{\hbar}{2}s(s+1) + \frac{\beta}{2}(S_Y^2 - S_X^2) \tag{30}$$

such an interaction leads to squeezing in an initial state of the form $|11\rangle$.


Trace C
Theta in radians
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
0
0.5
1
1.5
2

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