Spin chains and channels with memory

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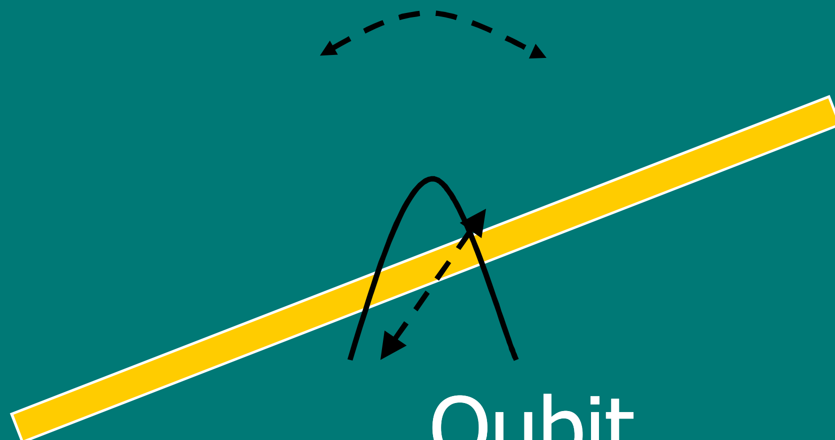
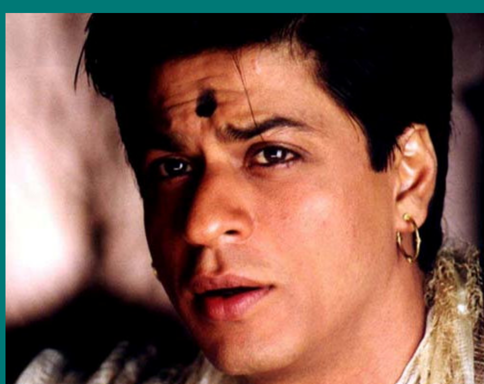
Outline

- Introduction to Channel Capacities.
- Motivation – correlations in error.
- Connections to many-body physics.
- Validity of assumptions.
- Conclusions.

Quantum Channel Capacities



Asoka



Kaurwaki

Quantum Channel Capacities

- Quantum Error Correction Codes reduce error – but at a cost: each *logical* qubit is encoded in a larger number of *physical* qubits.

- The *Communication rate, R,* of a code is:

$$Rate = \frac{\text{logical qubits}}{\text{physical qubits}}$$

Quantum Channel Capacities

- Quantum error correction very powerful, errors can be made to *vanish* for large data blocks, provided rates low enough.

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- Quantum error correction very powerful, errors can be made to *vanish* for large data blocks, provided rates low enough.
- The maximal allowed rate for a given channel Q(ε) is called the *quantum channel capacity*.
- If you try to communicate at a rate R > Q, then you will suffer *errors*.
- Communication at rates R < Q can be made essentially *error free* by choosing a clever code.

Quantum Channel Capacities

Q(ε) is the maximal rate at which quantum bits can be sent essentially error free over many uses of a quantum channel ε.

- *So how do we compute Q(ε) ?*
- *Unfortunately it is very difficult….!*

So how do we figure out *Q(ε)* ?

- The best known formula for *Q(ε)* for UNCORRELATED channels is:

$$Q(\varepsilon) = \lim_{n\to\infty}\left[\max_{\rho}\ \left[S(\varepsilon_n(\rho)) - S(\varepsilon_n \otimes \mathrm{I}(\psi))\right]\right]$$

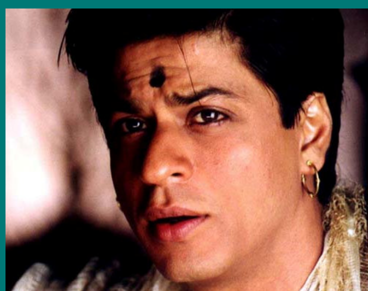
where: ψ is a purification of ρ

$$\varepsilon_n = \varepsilon \otimes \varepsilon \otimes \varepsilon \otimes .... \otimes \varepsilon \otimes \varepsilon$$

See e.g. Barnum et. al. '98, Devetak '05.

Independence vs. Correlations

Independent error model: each transmission affected by noise independently of the others



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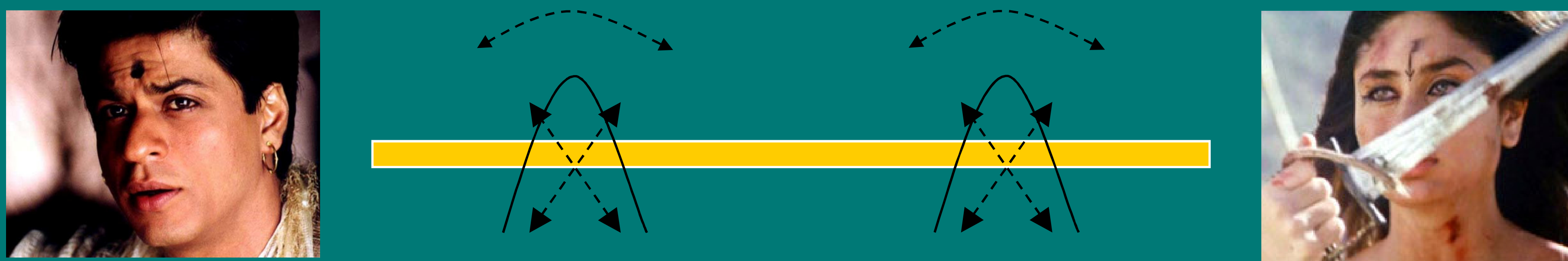

However realistic errors can often exhibit *correlations* :

E.g. scratches on a CD affect adjacent information pieces, birefringence in optical fibres (Banaszek experiments 04)

Correlated Errors.

- Independent errors: channel acts on *n* qubits as

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$$\varepsilon_n(\rho_n) = \varepsilon_1 \otimes \varepsilon_1 \otimes \ldots \otimes \varepsilon_1(\rho_n)$$

- *Family* of channels $\{\varepsilon_n\}$ – for each number of qubits *n* :

$$\varepsilon_n(\rho_n) \neq \varepsilon_1 \otimes \varepsilon_1 \otimes \ldots \otimes \varepsilon_1(\rho_n)$$

- *So how do correlations in noise affect our ability to communicate ?*

Motivating Example

- Consider an independent Pauli error channel:

$$\rho \rightarrow \sum_{\mathrm{i=0,x,y,z}} p(i,j,k....)[\sigma_{\mathrm{i}} \otimes \sigma_{\mathrm{j}} \otimes \sigma_{\mathrm{k}} ...]\, \rho\, [\sigma_{\mathrm{i}} \otimes \sigma_{\mathrm{j}} \otimes \sigma_{\mathrm{k}} ...]^{*}$$

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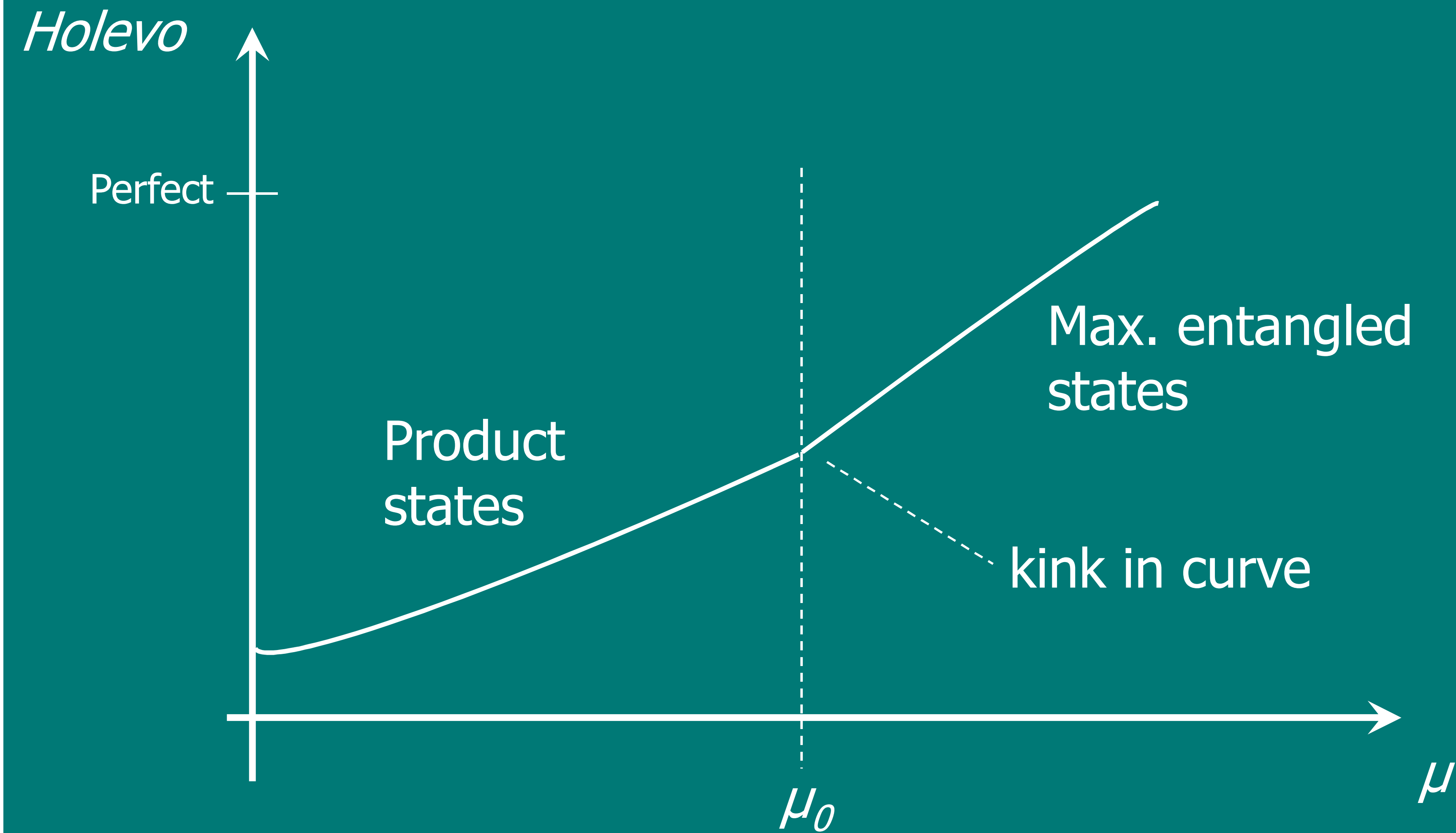
$$p(i,j,k....) = p(i)\,p(j)\,p(k).....$$

- Channel considered in Macchiavello & Palma ’02:

$$p(i,j,k....) = Q(i)\,p(j\,|\,i)\,p(k\,|\,j).....$$

$$p(j\,|\,i) = (1-\mu)Q(j) + \mu\delta(i,j)$$

Macchiavello-Palma channel:



Also see e.g. Macchiavello et. al. '04; Karpov et. al. '06; Banaszek et. al. '04, Daems 06

Hmmm……Statistical Physics?

- Non-analyticity in large *n*, *thermodynamic*, limit ?
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  That sounds just like Many-body physics!!

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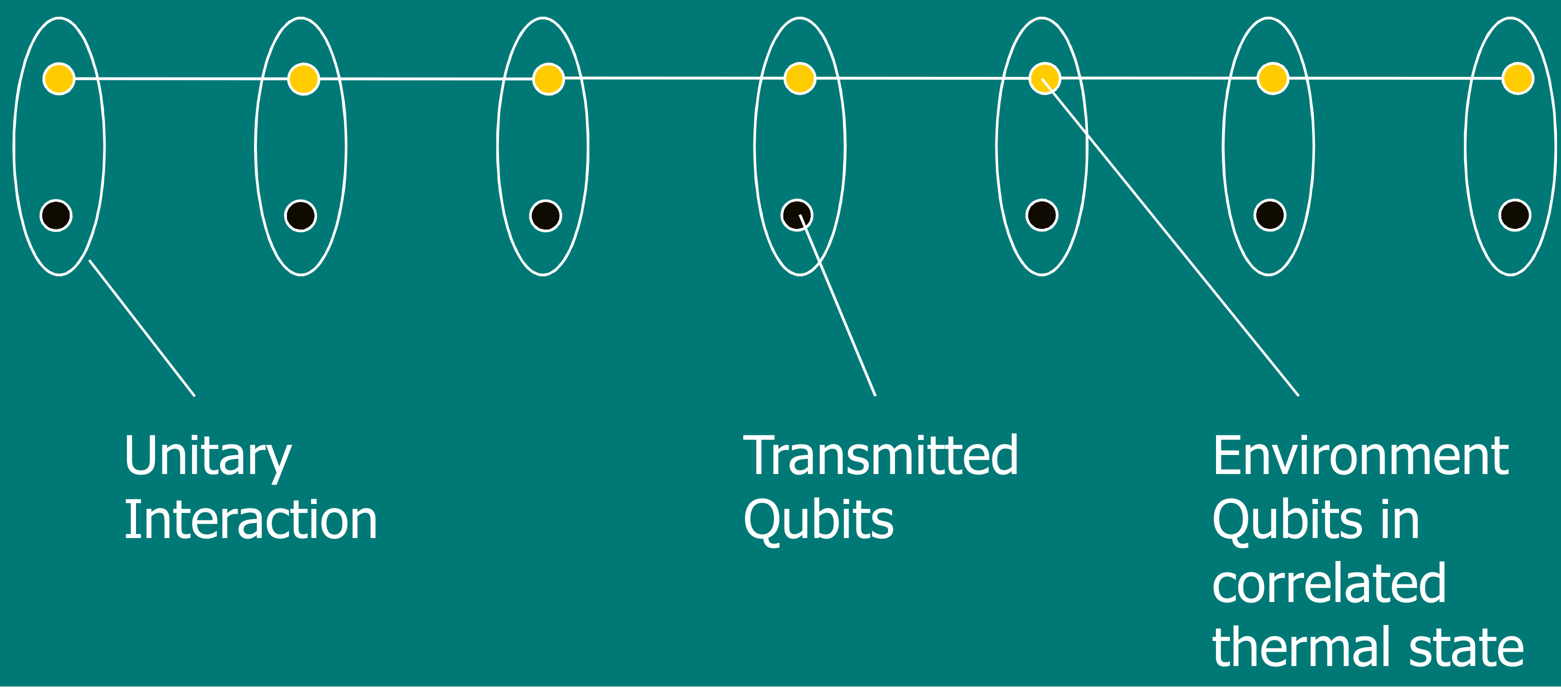
- Consider a many-body inspired model for correlated noise:



Capacity for *correlated* errors

- For our many body models we will compute:

$$Q(\{\varepsilon_n\}) = \lim_{n\to\infty}\left[\max_{\rho}\left[S(\varepsilon_n(\rho)) - S(\varepsilon_n \otimes \mathrm{I}(\psi))\right]\right]$$

where: ψ is a purification of ρ

in general: $\varepsilon_n \neq \varepsilon \otimes \varepsilon \otimes \varepsilon \otimes \ldots . \otimes \varepsilon \otimes \varepsilon$

This will NOT be the capacity in general, but for "sensible" models it will be the capacity

In general this expression is too difficult to calculate.

But for specific types of channel it can be simplified

Pick a simple interaction!

- Simple model:

  - Consider 2 level systems in environment – either classical or quantum particles

  - Let interaction be CNOT, environment controls

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- Such interaction gives some pleasant properties:

  - Essentially probabilistic application of Id or X
  - truncated Quantum Cap = Distillable ent.
  - Answer given by Hashing bound.

see Bennett et. al. ’96, Devetak & Winter ’04.

For such channels:

$$Q = 1 - \lim_{n \to \infty} \frac{H(\text{many - body system})}{\text{n}}$$

For classical environments *H* is just the entropy. Thermodynamic property!!

For quantum *H* is the entropy of computational basis diagonal.

This is very convenient! There are years of interesting examples, at least for classical environment.

Quantum example: Rank-1 MPS

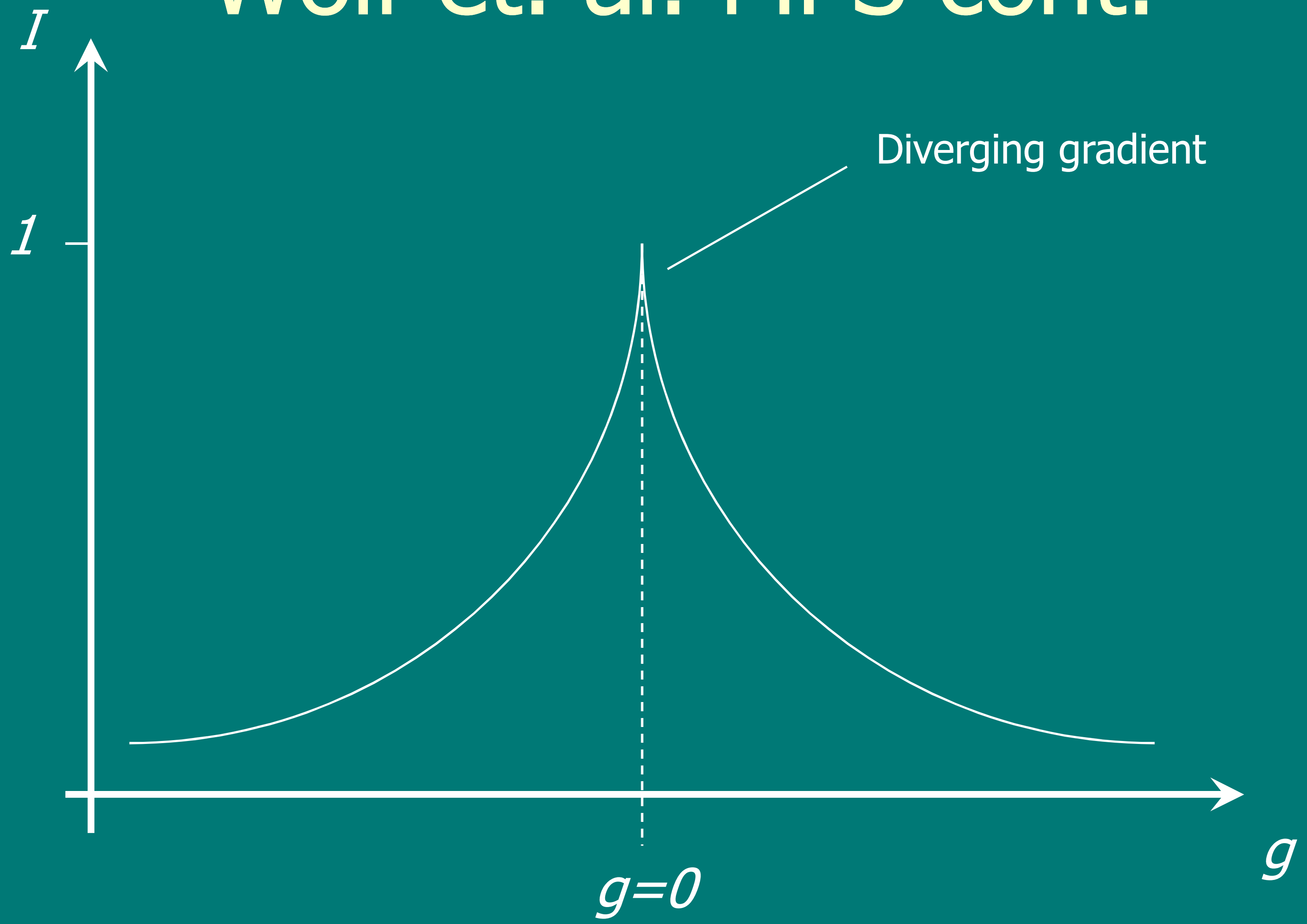
Matrix Product States (e.g. Perez-Garcia et al`06) are interesting class of states with efficient classical description.

Convenient result: If matrices are rank-1, *H* reduces to entropy of a classical Ising chain.

E.g. ground state of following Hamiltonian (Wolf et. al. '05):

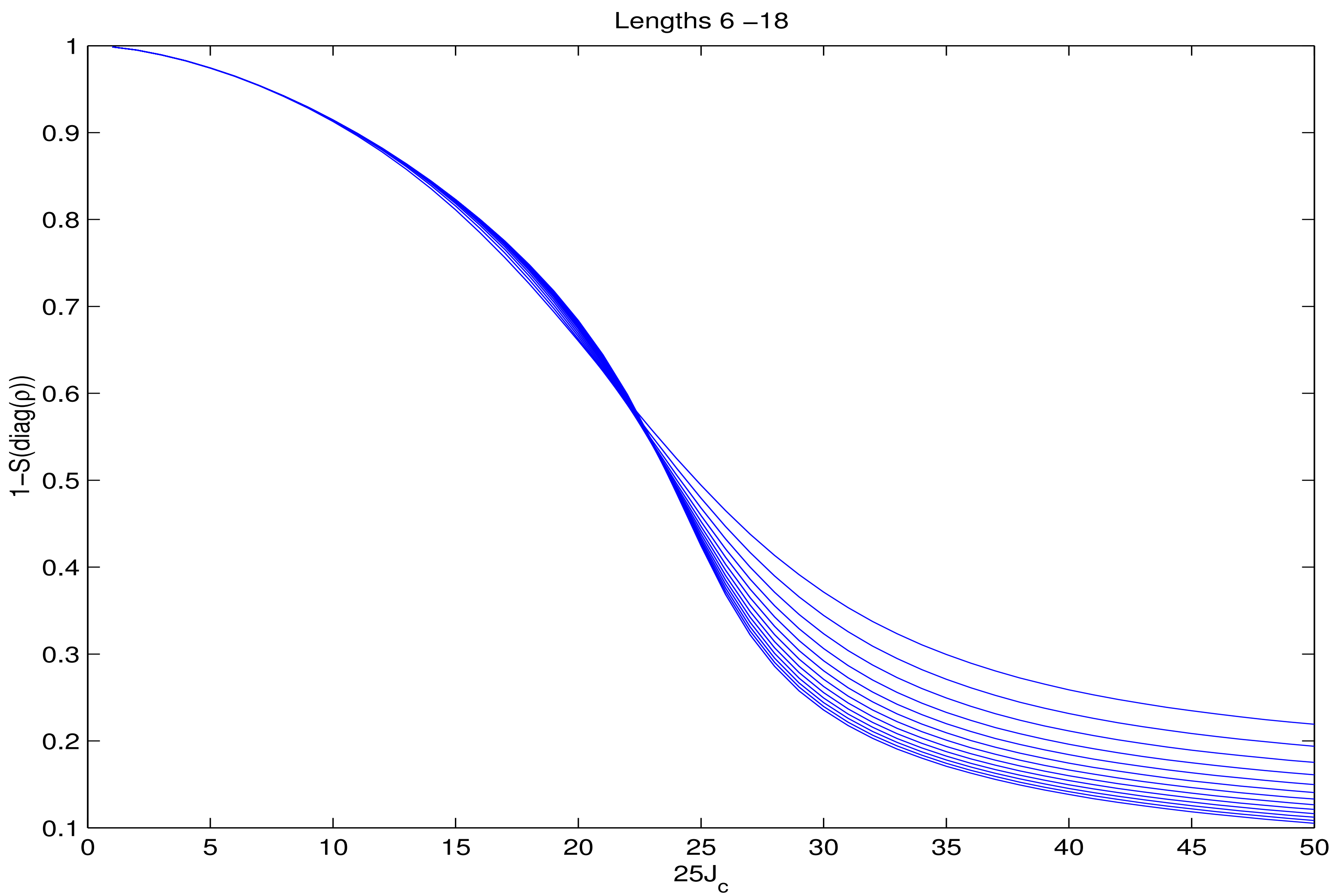
$$\mathrm{H} = \sum_i 2(g^2 - 1) Z_i Z_{i+1} - (1+g)^2 X_i + (g-1)^2 Z_i X_{i+1} Z_{i+2}$$

Wolf et. al. MPS cont.



- Slight Cheat : left-right symmetry as channel identical for g, $-g$

Quantum Ising (Numerics)



The Assumptions.

We have calculated is actually the coherent information:

$$I_C(\{\varepsilon_n\}) \;=\; \lim_{n\to\infty} \frac{1}{n}\left[\max_{\rho}\; \left[S(\varepsilon_n(\rho)) - S(\varepsilon_n \otimes \mathrm{I}(\psi))\right]\right]$$

For correlated errors this is NOT the capacity in general.

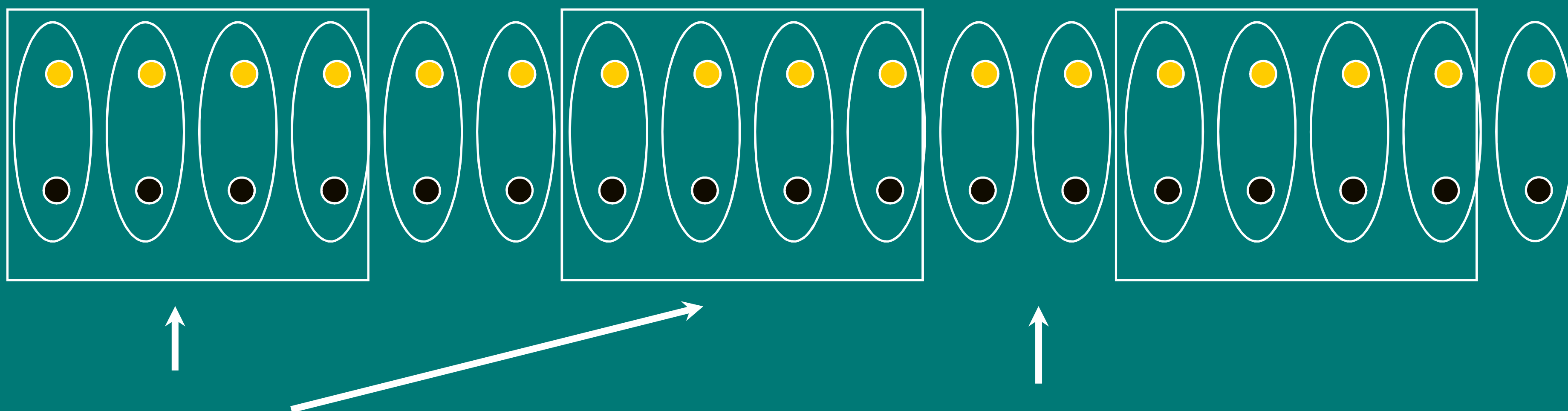
Is this *the* capacity for all many-body environments?

Certainly the Hamiltonian must satisfy some constraints.

What are they ?

Cheat’s guide to correlated coding

Consider the whole system over many uses:



large LIVE blocks, l spins each block

small SPACER blocks, s spins each block

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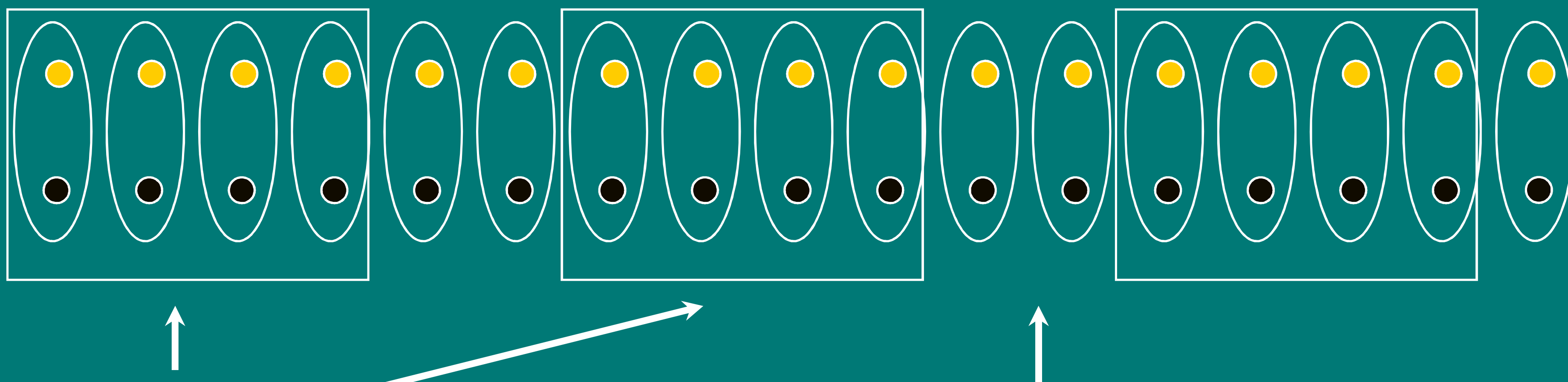

large LIVE blocks, l spins each block

small SPACER blocks, s spins each block

If correlations in the environment decay sufficiently, reduced state of LIVE blocks will be approximately a *product*

See e.g. Kretschmann & Werner '05

Cheat's guide to correlated coding II

So if correlations decay sufficiently fast, can apply known results on *uncorrelated* errors.

How fast is sufficiently fast ? Sufficient conditions are:

$$\| \ \rho_{L_1 L_2 L_3 .. L_v} - (\rho_{L_1})^{\otimes v} \ \|_1 \ \leq \ C v l^E \exp(-Fs)$$

We also require a similar condition, demonstrating that the bulk properties are sufficiently independent of boundary conditions.

These conditions can be proven for MPS and certain bosonic and fermionic system.

Conclusions and Further work:

- Results from many-body theory can give interesting insight into the coherent information of correlated channels.
- What about more complicated interactions? Methods give LOWER bounds to capacity for all random unitary channels.
- For which many body systems can decay be proven?
- How about other capacities of quantum channels?
- A step towards physically motivated models of correlated error. 2d, 3d…..?
- Is there a more direct connection to quantum coding.

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