NON-LOCALITY AND CONTEXTUALITY IN QUANTUM MECHANICS

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KOLKATA

$a_1 = ?$

or

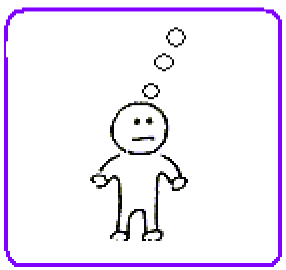
$a_2 = ?$

They are not allowed to communicate after the game starts.

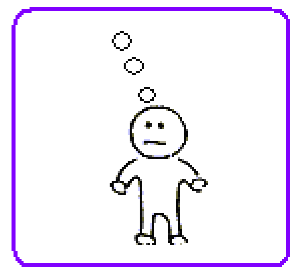
$b_1 = ?$

or

$b_2 = ?$



Alice

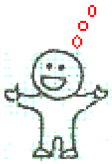


Bob

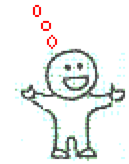
Answers can be 1 or -1

Winning condition

Alice



Bob



There answers have to satisfy

a_1	$V(a_1)\,V(b_1) = +1$	b_1
a_1	$V(a_1)\,V(b_2) = +1$	b_2
a_2	$V(a_2)\,V(b_1) = +1$	b_1
a_2	$V(a_2)\,V(b_2) = -1$	b_2

Obviously if they can win this game without communication, they can win it even if separated by space like distance.

Alice and Bob can not win this game by any strategy which decides the answers for both locally.

Question	Alices's answers	Question	Bob's answers
a_1	$V_{Alice}(a_1)$	b_1	$V_{Bob}(b_1)$
a_2	$V_{Alice}(a_2)$	b_2	$V_{Bob}(b_2)$

Now the answers have to satisfy all the winning conditions as pair of question in each turn are random.

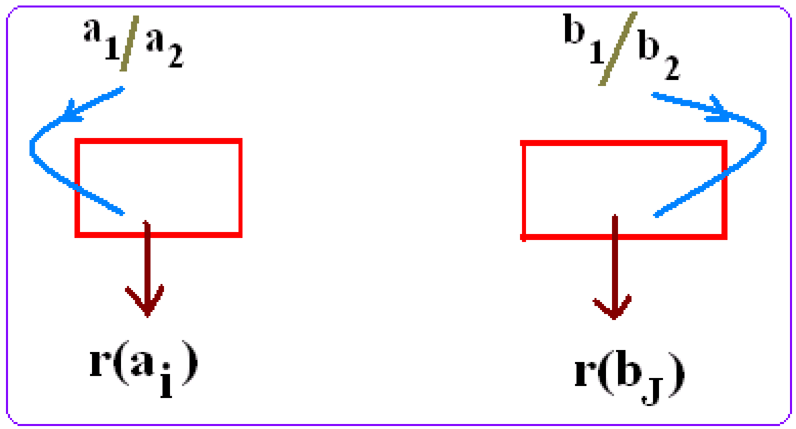
$$V_{Alice}(a_1)\,V_{Bob}(b_1) = +1$$

$$V_{Alice}(a_1)\,V_{Bob}(b_2) = +1$$

$$V_{Alice}(a_2)\,V_{Bob}(b_1) = +1$$

$$V_{Alice}(a_2)\,V_{Bob}(b_2) = -1$$

Existence of deterministic non-local correlation helping win this game would imply signaling (violation of special relativity).



Possible correlation 1

$r(a_1)$	$r(b_1)$
+1	+1
$r(a_1)$	$r(b_2)$
+1	+1
$r(a_2)$	$r(b_1)$
+1	+1
$r(a_2)$	$r(b_2)$
+1	−1

Possible correlation 2

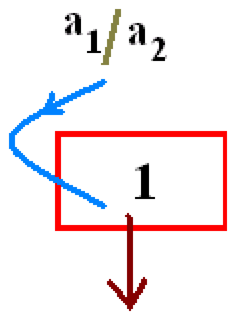
$r(a_1)$	$r(b_1)$
−1	−1
$r(a_1)$	$r(b_2)$
−1	−1
$r(a_2)$	$r(b_1)$
−1	−1
$r(a_2)$	$r(b_2)$
+1	−1

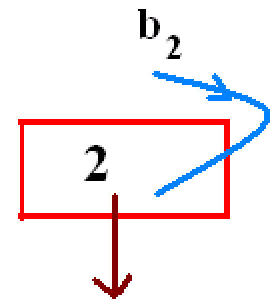
correlation 1

$r(a_1)$	$r(b_2)$
$+1$	$+1$
$r(a_2)$	$r(b_2)$
$+1$	-1

can be used for sending real information.

India vs Pakisthan
Cricket match





	Input in Box 1	Output in Box 2
India won	a_1	$+1$
India lost	a_2	-1

Possible correlation which does not imply signalling

$r(a_1)$	$r(b_1)$	Probability
+1	+1	$\frac{1}{2}$
-1	-1	$\frac{1}{2}$

$r(a_1)$	$r(b_2)$	Probability
+1	+1	$\frac{1}{2}$
-1	-1	$\frac{1}{2}$

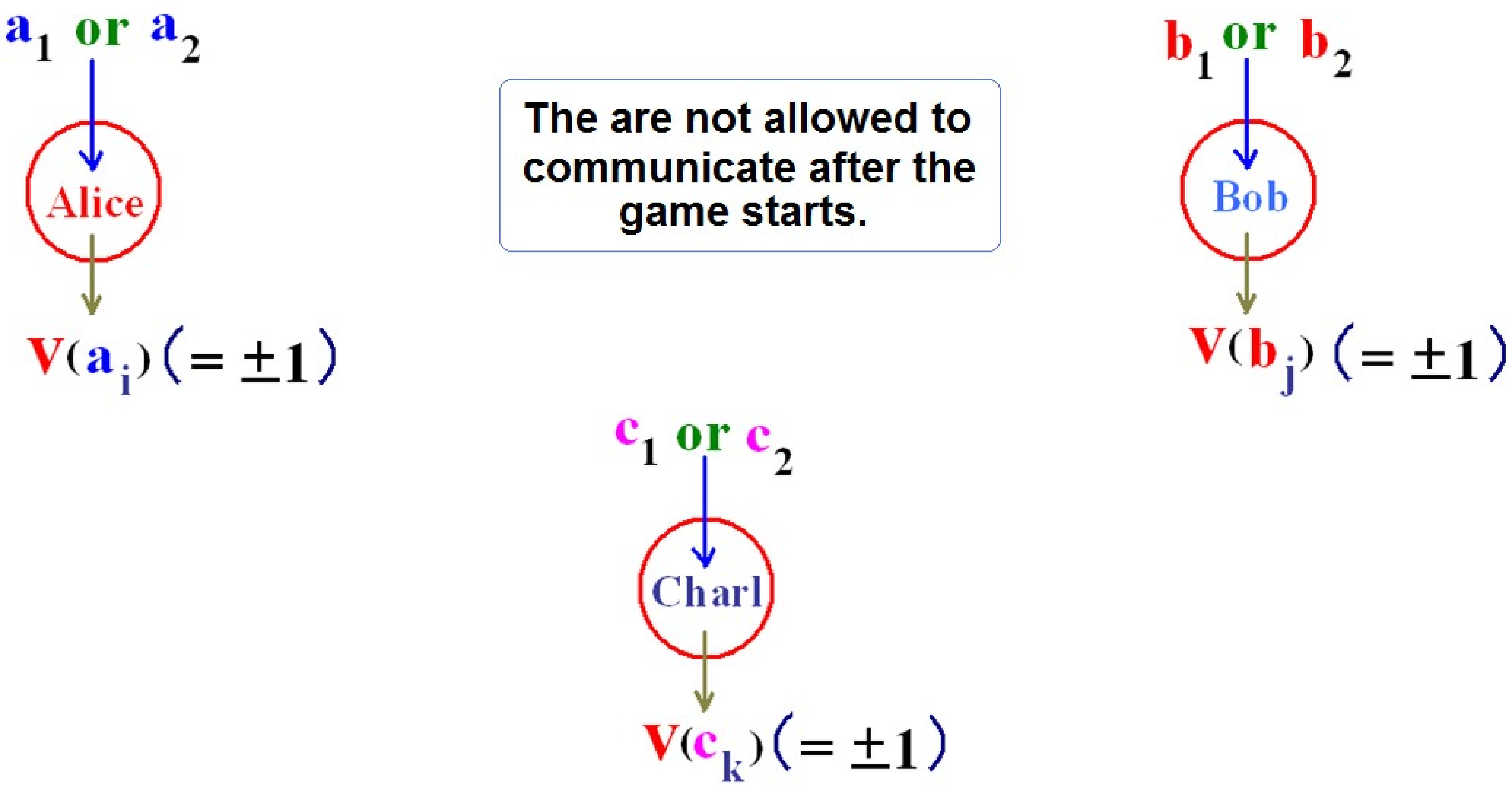
$r(a_2)$	$r(b_1)$	Probability
+1	+1	$\frac{1}{2}$
-1	-1	$\frac{1}{2}$

$r(a_2)$	$r(b_2)$	Probability
-1	+1	$\frac{1}{2}$
+1	-1	$\frac{1}{2}$

There is no physical theory which provides this kind of correlation.

A three party game

The are not allowed to communicate after the game starts.



Pattern of questions

Alice	Bob	Charlie	Winning condition
a_1	b_2	c_2	Product of the answers = +1
a_2	b_1	c_2	Product of the answers = +1
a_2	b_2	c_1	Product of the answers = +1
a_1	b_1	c_1	Product of the answer = -1

Is it possible to win this game in the classcal world?

Let there is a classical strategy:

$V_{Alice}(a_1)$ $V_{Bob}(b_1)$ $V_{Charl}(c_1)$

$V_{Alice}(a_2)$ $V_{Bob}(b_2)$ $V_{Charl}(c_2)$

$$V_{Alice}(a_1)\ V_{Bob}(b_2)\ V_{Charl}(c_2) = +1$$

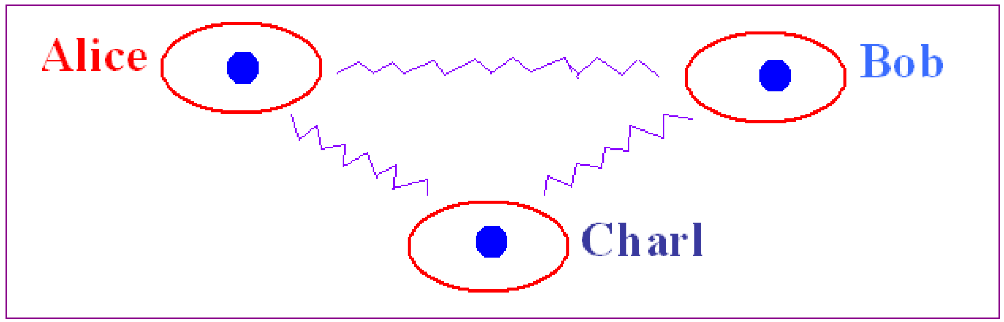
$$V_{Alice}(a_2)\ V_{Bob}(b_1)\ V_{Charl}(c_2) = +1$$

$$V_{Alice}(a_2)\ V_{Bob}(b_2)\ V_{Charl}(c_1) = +1$$

$$V_{Alice}(a_1)\ V_{Bob}(b_1)\ V_{Charl}(c_1) = -1$$

Impossible!

— Strategy to win the three party game —



Question	Measurement	Outcome	Answer
a_1, b_1, c_1	σ_X	+1 (up)	+1
		-1 (down)	-1
a_2, b_2, c_2	σ_Y	+1 (up)	+1
		-1 (down)	-1

Kochen-Specker Game

The local but contextual model can not reproduce quantum correlation.

Rename the 18 vectors

$$S_1 = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \equiv \{S_1^i, i = 1,2,3,4\}$$

$$S_2 = \{\varphi_1, \varphi_5, \varphi_6, \varphi_7\} \equiv \{S_2^i, i = 1,2,3,4\}$$

$$S_3 = \{\varphi_8, \varphi_{18}, \varphi_3, \varphi_9\} \equiv \{S_3^i, i = 1,2,3,4\}$$

$$S_4 = \{\varphi_8, \varphi_{10}, \varphi_7, \varphi_{11}\} \equiv \{S_4^i, i = 1,2,3,4\}$$

$$S_5 = \{\varphi_2, \varphi_5, \varphi_{12}, \varphi_{13}\} \equiv \{S_5^i, i = 1,2,3,4\}$$

$$S_6 = \{\varphi_{18}, \varphi_{10}, \varphi_{13}, \varphi_{14}\} \equiv \{S_6^i, i = 1,2,3,4\}$$

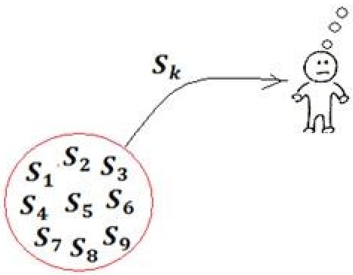
$$S_7 = \{\varphi_{15}, \varphi_{16}, \varphi_4, \varphi_9\} \equiv \{S_7^i, i = 1,2,3,4\}$$

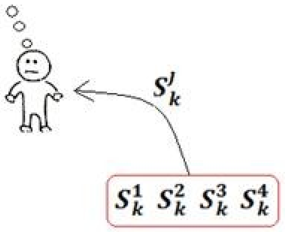
$$S_8 = \{\varphi_{15}, \varphi_{17}, \varphi_6, \varphi_{11}\} \equiv \{S_8^i, i = 1,2,3,4\}$$

$$S_9 = \{\varphi_{16}, \varphi_{17}, \varphi_{12}, \varphi_{14}\} \equiv \{S_9^i, i = 1,2,3,4\}$$

Consider a game;

They are not allowed to communicate after the game starts.





Alice has to assign 1 to one of the vector and 0 to other three vectors.

Bob has to assign 1 or 0 to his single vector.

winning condition:

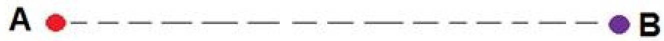
$$v_{Alice}(S_k^j) = v_{Bob}(S_k^j)$$

in each turn.

Values assigned to the 18 vectors can satisfy those 9 equations when value assignment is contextual at least for one vector.

But when this particular vector is given to Bob, he does not know what value to be assigned to win this game as he does not know which set it belongs to.

So without classical communication, they can not win this game.



$$|\varphi>_{AB} = \frac{1}{4}\left[|\varphi_1>_A|\varphi_1>_B + |\varphi_2>_A|\varphi_2>_B + |\varphi_3>_A|\varphi_3>_B + |\varphi_4>_A|\varphi_4>_B\right]$$

$$|\varphi^+> = \frac{1}{\sqrt{d}}\sum|i>_A|i>_B$$

$$U_A \otimes U_B^* |\varphi^+> = |\varphi^+>$$

For real vectors: $U_B^* = U_B$

$$|\varphi>_{AB} = \frac{1}{4}\sum_i |S_1^i>_A \otimes |S_1^i>_B$$

$$|\varphi>_{AB} = \frac{1}{4}\sum_i |S_k^i>_A \otimes |S_k^i>_B \qquad k = 1,2,3,4,5,6,7,8,9$$

let S_r be the set given to Alice.

Alice measures in the basis $\{|S_r^i>\}$.

Let the state collapses to $|S_r^j>$

$$v(S_r^i) = 1, for\ i = j$$
$$= 0, for\ i \neq j$$

Bob is given the vector S_r^m .

He measures in a basis having a vector $|S_r^m>$.

If he collapses on $|S_r^m>$, he assigns;

$$v(S_r^m) = 1$$
$$= 0$$,otherwise.

Due to correlation of the state;

$$v_{Alice}(S_r^m) = v_{Bob}(S_r^m)$$

Magic square game

A 3 x 3 matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ with $a_{ij} = 0 \; or \; 1$

For row:

$a_{11} + a_{12} + a_{13} =$ even

$a_{21} + a_{22} + a_{23} =$ even

$a_{31} + a_{32} + a_{33} =$ even

Such matrix does not exist

For column:

$a_{11} + a_{21} + a_{31} =$ odd

$a_{12} + a_{22} + a_{32} =$ odd

$a_{13} + a_{23} + a_{33} =$ odd

- The game -

Alice is given a row and Bob is given a column. They are asked to give the entries.

The sum of Alice's entries should be even.

The sum of Bob's entries should be odd.

Winning condition:

They should assign same value to the common element in each turn.

A deterministic classical strategy can not exist.

A deterministic classical strategy would have to assign definite binary values to each nine entries of the magic square which is impossible.

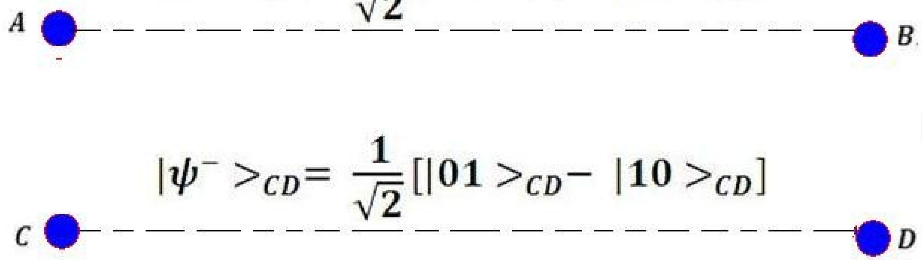
Quantum winning strategy

$$|\psi^- >_{AB} = \frac{1}{\sqrt{2}}[|01>_{AB} - |10>_{AB}]$$

Alice

Bob

$$|\psi^- >_{CD} = \frac{1}{\sqrt{2}}[|01>_{CD} - |10>_{CD}]$$



$$|\Phi>_{ACBD} = |\psi^->_{AB} \otimes |\psi^->_{CD}$$

Write the state in the AC:BD cut.

$$|\Phi>_{AC:BD} = \frac{1}{2}[|00>_{AC}|11>_{BD} + |01>_{AC}|10>_{BD} + |10>_{AC}|01>_{BD} + |11>_{AC}|00>_{BD}]$$

Alice: **Row** **Unitary operation**

1 $$U_1 = \begin{bmatrix} i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & i & 1 & 0 \\ 1 & 0 & 0 & i \end{bmatrix}$$

2 $$U_2 = \begin{bmatrix} i & 1 & 1 & i \\ -i & 1 & -1 & i \\ i & -1 & 1 & -i \\ -i & 1 & 1 & -i \end{bmatrix}$$

3 $$U_3 = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

Bob: **Column** **Unitary operation**

1 $$V_1 = \begin{bmatrix} i & -i & 1 & 1 \\ -i & -i & 1 & -1 \\ 1 & 1 & -i & i \\ -i & i & 1 & 1 \end{bmatrix}$$

2 $$V_2 = \begin{bmatrix} -1 & i & 1 & i \\ 1 & i & 1 & -i \\ 1 & -i & 1 & i \\ -1 & -i & 1 & -i \end{bmatrix}$$

3 $$V_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

After the unitary operation Alice and Bob measure their qubits in the basis |00>, |01>, |10>, |11>.

If Alice collapses on $|a_1 a_2\rangle_{AC}$

she outputs

$$(a_1, a_2, a_1 \oplus a_2)$$

If Bob collapses on $|b_1 b_2\rangle_{BD}$

he outputs

$$(b_1, b_2, b_1 \oplus b_2 \oplus 1)^T$$

Quantum correlation reduces communication

$$x = (x_1, x_2, x_2 \dots \dots x_n), x_i \in \{0, 1\}$$

Similarly for y and z

There is a constraint on the inputs:

$$x_i + y_i + z_i = 1$$

for all i.

The task for Alice is to compute the function

$$f(x, y, z) = x_1 . y_1 . z_1 + x_2 . y_2 . z_2 + \cdots + x_n . y_n . z_n$$

In classical world, it has been shown that more than 2 bits of communication are necessary.
3 bits of communication is sufficient.

Let $n = 3$.

If $x_i . y_i . z_i = 1$, then none of them can be zero.

If $x_i . y_i . z_i = 0$, then two of them have to be zero. ($x_i + y_i + z_i = 1$ for all i.)

r_A, r_B, r_C be the no. of zeros for Alice's, Bob's and Charie's input respectively.

Total no. of zeros among all their inputs is even and let it be equal to 2k. $r_A + r_B + r_C = 2k$

k no. of terms in $x_1 . y_1 . z_1 + x_2 . y_2 . z_2 + \cdots + x_n . y_n . z_n$ are zero.

$$f(x, y, z) = (n - k) mod 2$$

To compute k, Alice has to learn r_B and r_C.

Possible values of r_B and r_C are 0,1,2,3 which can be communicated by 2 bits 00, 01, 10, 11.

But one of them (say Charlie) can communicate just one bit (first bit) as Alice can determine r_C as

$$r_A + r_B + r_C = even$$

Quantum protocol needs two bits of communication

They share n copies of the following 3 qubits state.

$$|\psi >_{ABC}^{k}= \frac{1}{2}[|001> +|010> +|001> -|111>] \quad k = 1,2 \ldots . n$$

- If the ith bit $x_i = 1$, Alice measures in the $\{|0>,|1>\} basis$ and notes down the output s_i^A
- If the ith bit $x_i = 0$, Alice first applies Hadamard transform on the respective qubit and then follows the same procedure.

Bob and Charlie do the same.

Alice computes $S_A = \sum s_i^A$

Bob computes $S_B = \sum s_i^B$ and communicate to Alice by 1 bit.

Charlie computes $S_C = \sum s_i^C$ and communicate to Alice by 1 bit.

Alice outputs $S_A + S_B + S_C$ as $f(x,y,z)$.

The protocol works as follows;

First observe that $S_i^A + S_i^B + S_i^C = x_i.y_i.z_i$ for all i.

Possible values of $x_i\ y_i\ z_i$ are (100, 010, 001, 111)

Case: $x_i\ y_i\ z_i$ = 111

then all possible measurement results ($S_i^A\ S_i^B\ S_i^C$) satisfiy

$$S_i^A + S_i^B + S_i^C = x_i.y_i.z_i$$

Case: $x_i\ y_i\ z_i$ = 001 $\quad H \otimes H \otimes I\ |\psi\overset{i}{>}_{ABC} = \frac{1}{2}[|011> + |101> + |000> - |110>]$

In this case also measurements results ($S_i^A\ S_i^B\ S_i^C$) satisfiy

$$S_i^A + S_i^B + S_i^C = x_i.y_i.z_i$$

This thing works in other two cases ($x_i\ y_i\ z_i$ = 100, 010)
due to symmetry of the entangled state.

$$S_A + S_B + S_C = \sum S_i^A + \sum S_i^B + \sum S_i^C = \sum (S_i^A + S_i^B + S_i^C) = \sum x_i.y_i.z_i = f(x,y,z).$$