NON-LOCALITY AND CONTEXTUALITY IN QUANTUM MECHANICS

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Quantum mechanics

System $\Rightarrow$ Hilbert space

State $\Rightarrow$ Density operator

If ρ is a density operator, then

i) $\rho^{\dagger} = \rho$ (self adjoint)

ii) ρ is positive (eigen values are non-negetive)

iii) $\mathrm{Tr}\,[\rho] = 1$

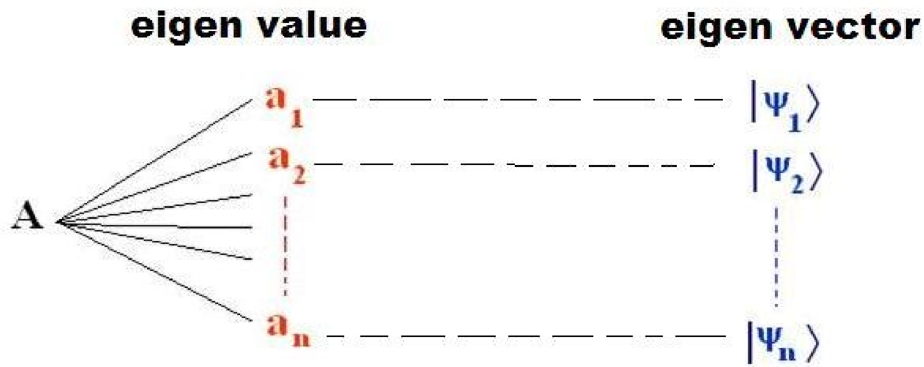
If $\rho^2 = \rho$, then there exists a vector $|\psi\rangle$ such that $\rho = |\psi\rangle\langle\psi|$

$|\psi\rangle\langle\psi|$ being one dimensional projection operator.

Collection of all density operators form a convex set, the extremal points being one dimensional projection operator.

Observable $\Rightarrow$ Self adjoint operator

A is a self adjoint operator



$$A|\psi_r\rangle = a_r|\psi_r\rangle$$

$|\psi_r\rangle\langle\psi_r| = P_r$ **is a projection operator**

$$P_r^2 = P_r$$

$$P_r P_s = 0 \quad \text{for} \quad r \neq s$$

Spectral representation

$$A = \sum a_r P_r$$

Resolution of identity

$$\sum P_r = I$$

– Measurement rules –

Spectral representation : $A = \sum a_r |\psi_r\rangle\langle\psi_r|$

- Measurement results is one of the eigen values.
- Probability : $P_\rho(A = a_r) = Tr[\rho\, |\psi_r\rangle\langle\psi_r|]$
- If the measurement result is a_r, then the final state is $|\psi_r\rangle$.

$|\psi_r\rangle\langle\psi_r|$ being projection operator, has two eigen values 1 and 0.

$$P_\rho(|\psi_r\rangle\langle\psi_r| = 1) = Tr[\rho\, |\psi_r\rangle\langle\psi_r|]$$

$$P_\rho(A = a_r) \Rightarrow P_\rho(|\psi_r\rangle\langle\psi_r| = 1)$$

$$A = a_r \Rightarrow |\psi_r\rangle\langle\psi_r| = 1$$

Measurement rules

Initial state = ρ

Measurement of A where $A = \sum a_r |\Psi_r\rangle\langle\Psi_r|$

Possible results	Probabilities	Final State
a_1 ($\|\Psi_1\rangle\langle\Psi_1\|$ = 1)	$Tr[\rho\|\Psi_1\rangle\langle\Psi_1\|]$	$\|\Psi_1\rangle\langle\Psi_1\|$
a_2 ($\|\Psi_2\rangle\langle\Psi_2\|$ = 1)	$Tr[\rho\|\Psi_2\rangle\langle\Psi_2\|]$	$\|\Psi_2\rangle\langle\Psi_2\|$
•	•	•
•	•	•
•	•	•
a_n ($\|\Psi_n\rangle\langle\Psi_n\|$ = 1)	$Tr[\rho\|\Psi_n\rangle\langle\Psi_n\|]$	$\|\Psi_n\rangle\langle\Psi_n\|$

For $\rho = |\Psi_r\rangle\langle\Psi_r|$,

$$Prob_\rho(A = a_1) = Prob_\rho(|\Psi_r\rangle\langle\Psi_r| = 1) = 1$$

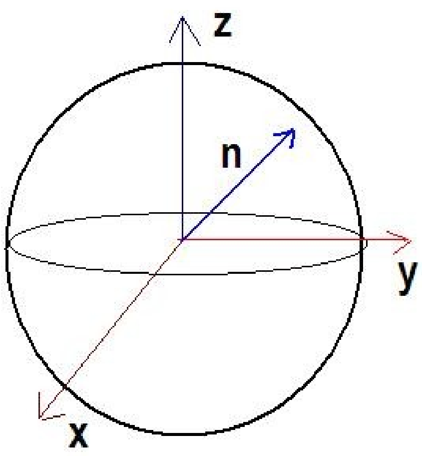
The property $A = a_1$ or equivalently $|\Psi_r\rangle\langle\Psi_r| = 1$ is real.

- A QUBIT -

Projection operator: $P = \frac{1}{2}[I + \hat{m}.\sigma]$; $\hat{m}$ is a unit vector.

Density operator : $\rho = \frac{1}{2}[I + n.\sigma]$; $|n| \leq 1$

Pure state : $|\Psi><\Psi| = \frac{1}{2}[I + \hat{n}.\sigma]$ $\hat{n}$ is a unit vector.



ρ can be represented by points in the unit sphere.

Points on the surface represents pure states.

Spin observable : $\sigma.\hat{\mathbf{r}} = (+1)\,\frac{1}{2}[I + \hat{\mathbf{r}}.\sigma] + (-1)\,\frac{1}{2}[I - \hat{\mathbf{r}}.\sigma]$

$$Prob_\rho(\text{Spin up}) = Tr[\rho\, \frac{1}{2}[I + \hat{\mathbf{r}}.\sigma]]$$

$$= Tr[\frac{1}{2}[I + n.\sigma]\,\frac{1}{2}[I + \hat{\mathbf{r}}.\sigma]] = \frac{1}{2}(1 + n.\hat{\mathbf{r}})$$

For pure state

$$Tr[|\Psi><\Psi|\,\frac{1}{2}[I + \hat{\mathbf{r}}.\sigma]] = <\Psi|\,\frac{1}{2}[I + \hat{\mathbf{r}}.\sigma]\,|\Psi> = \frac{1}{2}(1 + \hat{n}.\hat{\mathbf{r}})$$

PROBABILITY IN QUANTUM MECHANICS IS IRREDUCIBLE.

If for a quantum state ρ

$$p_\rho(A = a_i) = 1$$

Then there exists an observable B, such that

$$p_\rho\left(B = b_j\right) \neq 1, 0$$

If we consider two states $|\psi>$ and $|\varphi>$

with $<\psi|\varphi> \neq 0$,

then there is no projector P for which,

$$<\psi|\mathrm{P}|\psi> = 1 \text{ and } <\varphi|\mathrm{P}|\varphi> = 0$$

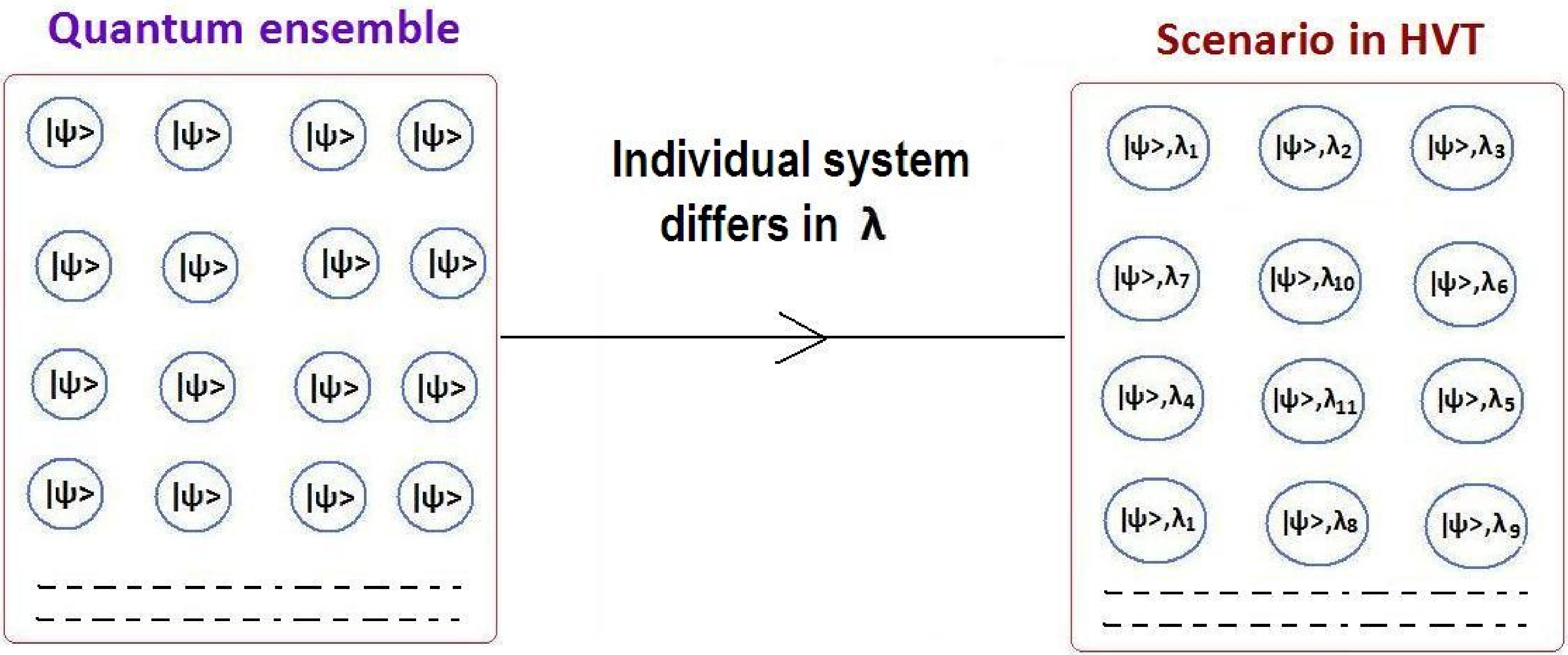
$$<\psi|\mathrm{P}|\psi> = 0 \text{ and } <\varphi|\mathrm{P}|\varphi> = 1$$

So they can not be reliably distinguished.

Going beyond QM, Can we costruct a theory where each state encodes definite values for all observables and still reproduce quantum probabilities ?

Then quantum state can be thought to arise due to subjective ignorance about those states.

– ψ Supplemented HVT –



Knowledge of $|\psi>, \lambda$ provides definite values for all possible observables.

$$v_{\psi,\lambda}(A) = \text{one of the eigenvalue of } A$$

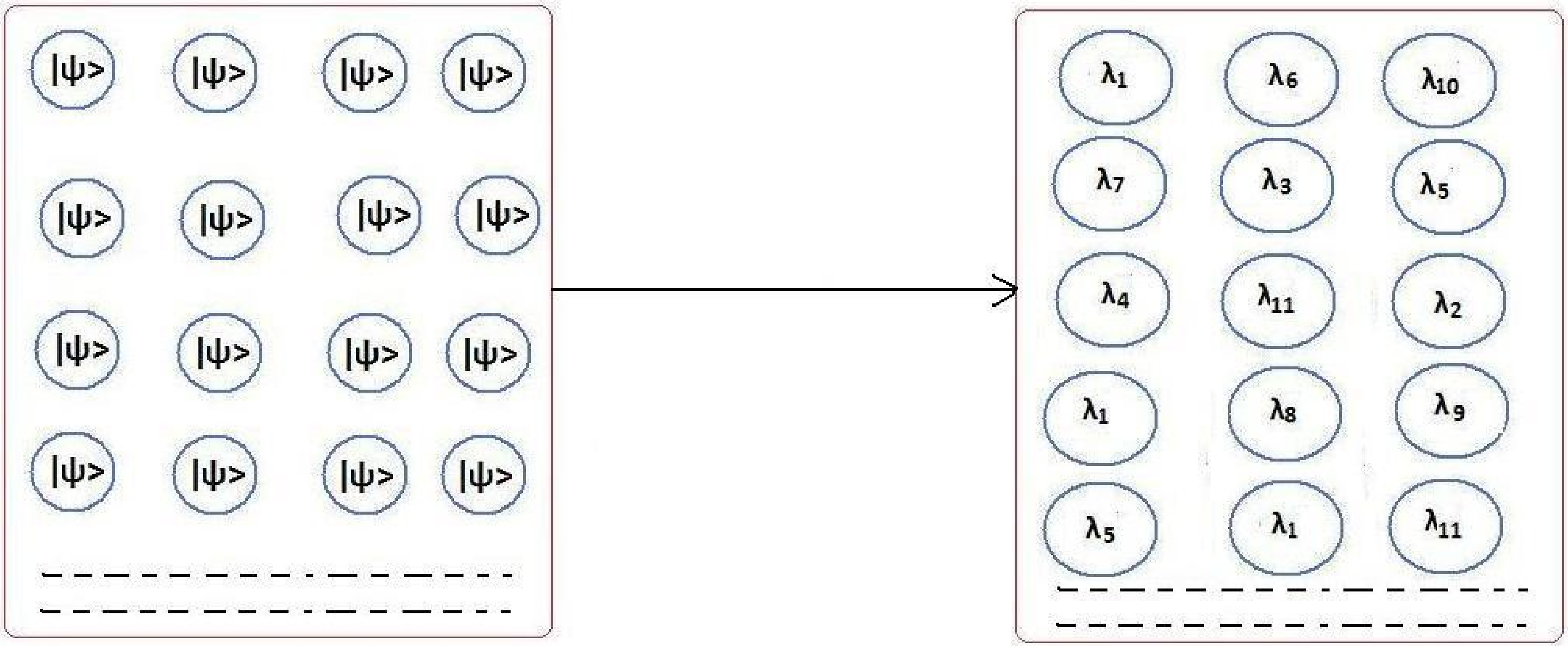
$$<\psi|A|\psi> = \int \theta_{\psi}(\lambda) v_{\psi,\lambda}(A) d\lambda \quad \text{with} \quad \int \theta_{\psi}(\lambda) d\lambda = 1$$

— ψ -Epistemic HVT model —

Knowledge of λ provides definite values for all possible observables.

ψ Correcponds to specific distribution of λ .

Different quantum states mean different distribution of λ .



$$v_{\lambda}(A) = \text{ one of the eigenvalue of } A$$

$$<\psi|A|\psi> = \int \theta_{\psi}(\lambda) v_{\lambda}(A) d\lambda \quad \text{with} \quad \int \theta_{\psi}(\lambda) d\lambda = 1$$

DOES UNCERTANTY PRINCIPLE PROHIBITS THE EXISTENCE OF SUCH THEORY?

The answer is 'No'.

The uncertanty principle puts restriction on the ensemble that can be prepared.

An example:

If one prepares an ensemble of the quantum state $|\psi_z>$, then 50% of the system will have 'up spin' and 50% will have down spin along x-direction.

Does complementary principle prohibits such theory?

The answer is again 'No'.

In quantum mechanics some observables can not be measured jointly.

The arrangements to measure σ_z and σ_x are mutually exclusive.

But how can this prohibit the system to have definite value for both σ_z and σ_x .

But Von Neumann discarded the possibility of such theory.

In any theory expectation values of observables have to satisfy the following;

1) $E(I) = 1$

2) $E(aA + bB + \cdots) = aE(A) + bE(B) + \cdots$

$A, B, \ldots.$ are self adjoint operators And $a, b, \ldots$ are real numbers

3) $E(P) \geq 0$

For any projection operator P.

Then it is a simple exercise to show that

$$E(A) = Tr[\rho A]$$

Where ρ is a density operator.

What it would mean for HVT?

As λ (or ψ,λ) determines the value of every observables to be revealed in future measurement, those value have to be eigen values.

$$< A >_\lambda = \text{ one of the eigen value of } A$$

$$< A >_{\psi,\lambda} = \text{one of the eigen value of } A$$

$$A + B = C$$

$$A, B, C \text{ do not commute}$$

Von Neumann demands

$$< A >_\lambda + < B >_\lambda = < C >_\lambda$$

$$< A >_{\psi,\lambda} + < B >_{\psi,\lambda} = < C >_{\psi,\lambda}$$

Some eigen value of $A +$ some eigen value of B

$=$ some eigen value of C

- Bell's example -

$$A + B = C$$

$$A = \sigma_x$$

$$B = \sigma_y$$

$$C = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y)$$

$$C = \text{n.}\ \sigma$$

with $n = \frac{1}{\sqrt{2}}\,\hat{x} + \frac{1}{\sqrt{2}}\,\hat{y}$

Where C represents spin measurement in x-y plane along a direction which makes 45^0 with x –axis.

All the observables has eigen value ±1.

Satisfying Von Neumann demand implies

$$(\pm 1) + (\pm 1) = \frac{1}{\sqrt{2}}(\pm 1)$$

Which is never possible.

Von Neumann's demand is unjustified.

Why HVT has to satisfy a condition that can not be verified when the observables do not commute.

The HVT has only to reproduce

$$< A >_\psi + < B >_\psi = < C >_\psi$$

which means

$$\int < A >_\lambda \theta(\lambda) d\lambda + \int < B >_\lambda \theta(\lambda) d\lambda = \int < C >_\lambda \theta(\lambda) d\lambda$$

$$\int < A >_{\psi,\lambda} \theta(\lambda) d\lambda + \int < B >_{\psi,\lambda} \theta(\lambda) d\lambda = \int < C >_{\psi,\lambda} \theta(\lambda) d\lambda$$

for which

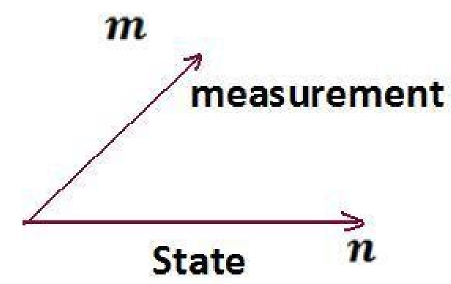
$$< A >_\lambda + < B >_\lambda = < C >_\lambda$$

$$< A >_{\psi,\lambda} + < B >_{\psi,\lambda} = < C >_{\psi,\lambda}$$

may be a sufficient condition,

but hardly a necessary condition.

— BELL MODEL —



$$\mathsf{p}_{n,\lambda}(P) = \frac{1}{2}[1 + sign(\lambda + \frac{1}{2}|n.m|sign(n.m))]$$

Where λ varies from $-\frac{1}{2}$ to $\frac{1}{2}$ and the distribution of λ is uniform.

$$Sign(x) = 1, \quad when\ x \geq 0$$
$$= -1 \quad when\ x < 0$$

This model correctly reproduces quantum probabilities.

For m = n, $\mathsf{p}_{n,\lambda}(P) = 1$

For m = -n, $\mathsf{p}_{n,\lambda}(P) = 0$

For m ⊥ n, $\mathsf{p}_{n,\lambda}(P) = \frac{1}{2}$

For $m.n = -ve$,

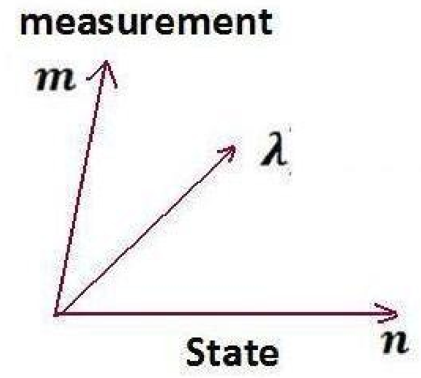
$$Prob_{\psi}(P) = \int_{-1/2}^{1/2} \rho(\lambda)\mathsf{p}_{n,\lambda}(P)d\lambda = \int_{-1/2}^{1/2} \frac{1}{2}\left[1 + sign\left(\lambda + \frac{1}{2}|n.m|sign(n.m)\right)\right]d\lambda$$

$$= \int_{-1/2}^{1/2|m.n|}(1-1)d\lambda + \int_{1/2|n.m|}^{1/2}(1+1)d\lambda = \frac{1}{2}(1 - |n.m|) = \frac{1}{2}(1 + n.m)$$

— Ψ-Epistemic model —

Actual states λ are unit vectors in the Poincare sphere.

For a given state λ, the probability that the projector P = 1 is given by $p_\lambda(P) = \Theta(m.\lambda)$



$$P = \frac{1}{2}[I + \hat{m}.\sigma] \qquad \Theta(x) = 1 \; for \; x > 0 \qquad = 0 \; for \; x \leq 0$$

Distribution of λ is determined by the quantum state ψ

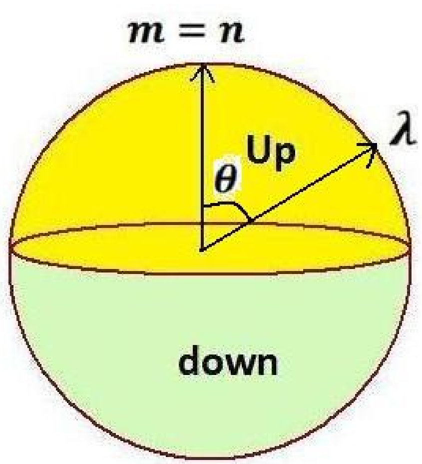
$$\rho_\Psi(\lambda) = \frac{1}{\pi}\Theta(n.\lambda)(n.\lambda)$$

Case: $m = n$

$$n.\lambda = cos\theta$$

$$d\lambda = sin\theta \, d\theta \; d\varphi$$

$$p_\psi(P) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} cos\theta \, sin\theta \, d\theta \int_0^{2\pi} d\varphi \;\; = 1$$



Case: $m \perp n$

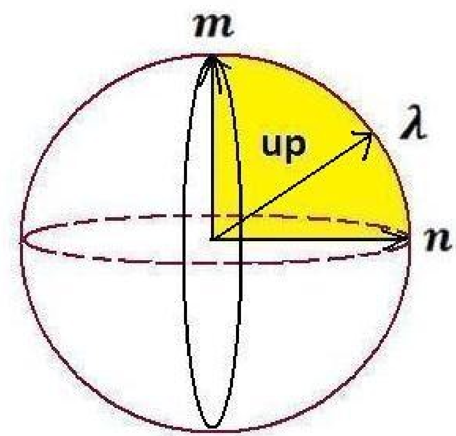
$$m = (0,0,1)$$

$$n = (1,0,0)$$

$$\lambda = (Sin\theta Cos\varphi, Sin\theta Sin\varphi, Cos\theta)$$

$$n.\lambda = sin\theta\ cos\varphi$$

$$d\lambda = sin\theta\ d\theta\ d\varphi$$



$$Prob_{\psi}(P) = \int \rho_{\Psi}(\lambda) p_{\lambda}(P)\, d\lambda$$

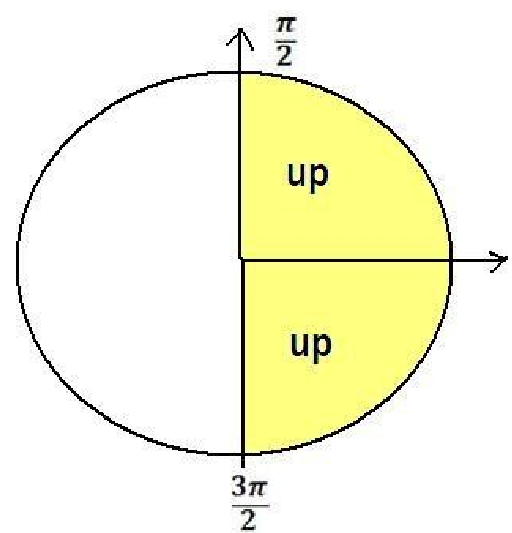
$$= \int \frac{1}{\pi} \Theta(n.\lambda)(n.\lambda)\,\Theta(m.\lambda)\, d\lambda$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} sin^2\theta\ d\theta \int_0^{\frac{\pi}{2}} cos\varphi\ d\varphi$$

$$+ \frac{1}{\pi}\int_0^{\frac{\pi}{2}} sin^2\theta\ d\theta \int_{\frac{3\pi}{2}}^{2\pi} cos\varphi\ d\varphi$$

$$= 2 \times \frac{1}{\pi}\int_0^{\frac{\pi}{2}} sin^2\theta\ d\theta \int_0^{\frac{\pi}{2}} cos\varphi\ d\varphi = \frac{1}{2}$$

Allowed region of φ

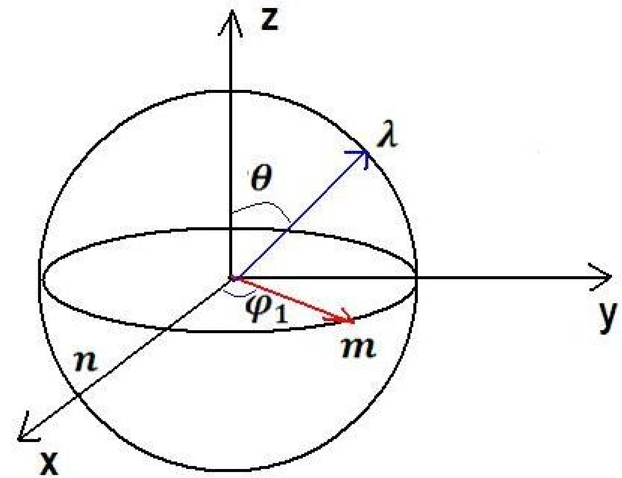


General case:

$$n = (1, 0, 0)$$

$$\lambda = (Sin\theta Cos\varphi, Sin\theta Sin\varphi, Cos\theta)$$

$$m = (Cos\varphi_1, Sin\varphi_1 . 0)$$



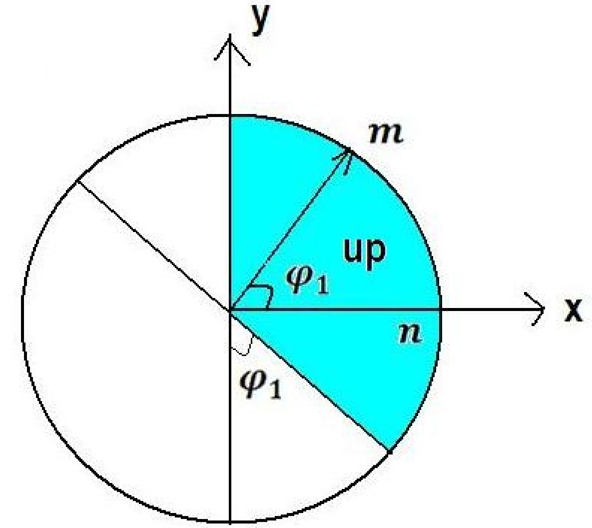
$$Prob_\psi(P) \ = \int \rho_\Psi(\lambda) p_\lambda(P)\, d\lambda$$

$$\int \frac{1}{\pi} \Theta(n.\lambda)(n.\lambda)\,\Theta(m.\lambda)\, d\lambda$$ up

$$= \ \frac{1}{\pi} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{\frac{\pi}{2}} Sin^2\theta\ Cos(\varphi - \varphi_1) d\theta d\varphi$$

$$+ \ \frac{1}{\pi} \int_{\theta=0}^{\pi} \int_{\varphi=\frac{3\pi}{2}+\varphi_1}^{2\pi} Sin^2\theta\ Cos(\varphi - \varphi_1) d\theta d\varphi$$

$$= \frac{1}{2}(1 + Cos\varphi_1) \ = \frac{1}{2}(1 + n.m)$$



Some meaningful constraint on HVT imposed by quantum theory

1) Values assigned ($v(A), v(B) \ldots\ldots$) by the HVT to the observables ($A, B \ldots\ldots$) can only be eigenvalues.

2) If a mutually commuting set $A, B. C \ldots$ satisfy the functional identity

$$f(A, B, C \ldots\ldots) = 0$$

Then the values assigned to them in an individual system must also satisfy

$$f(v(A), v(B), v(C) \ldots\ldots) = 0$$

Hint: If A and B are two commuting observables, then there exists a maximal observables C, such that

$$A = f(C) \quad and \quad B = g(C)$$

Verification of this constraint is meaningful as commuting observables can be measured simultaneously.

Gleason's Theorem

The set of all projection operators $P(H)$.

μ is a probability measure on $P(H)$.

1) $0 \leq \mu(P) \leq 1$

2) $\mu(I) = 1$

3) $\mu(\sum P_i) = \sum \mu(P_i)$

where P_i are orthogonal projectors.

If $Dim(H) \geq 3$, then there exists a density operator ρ , such that

$$\mu(P) = Tr[\rho P]$$

So there is no probability measure other than quantum state for Hilbert space of dimension three or more.

As any HVT has to satisfy all the three conditions of Gleason's theorem, there is no probability measure, such that

$$\mu(P) = \mathbf{1}\ \boldsymbol{or}\ \mathbf{0}$$

for all projection operators P.

Does this result discards the possibility of any HVT?

According to Bell it only discards a class of HVT.

HVT in higher dimensional Hilbert space

Projective measurement in 3 dimension : $\sum_{i=1}^{3} P_i = I$

$$P_i = |\varphi_i >< \varphi_i|$$

$\{|\varphi_1 >, |\varphi_2 >, |\varphi_3 >\}$ being an orthogonal basis B_1 .

Another projective measurement $P_1 + Q_2 + Q_3 = I$

Q_2 : Projector on $\frac{1}{\sqrt{2}} (|\varphi_2 > + |\varphi_3 >)$

Q_3 : Projector on $\frac{1}{\sqrt{2}} (|\varphi_2 > - |\varphi_3 >)$

$\left\{|\varphi_1 >, \frac{1}{\sqrt{2}} (|\varphi_2 > + |\varphi_3 >), \frac{1}{\sqrt{2}} (|\varphi_2 > - |\varphi_3 >)\right\}$ being an orthogonal basis B_2 .

Measuements in B_1 basis and B_2 basis are different.

A HVT is called non-contextual if it assigns value to observables in a context independent way i.e. independent of other observables along with which it is measured.

In this case non-contextuality implies,

$$v_{B_1}(P_1) = v_{B_2}(P_1)$$

– Non-contextual HVT –

18 vectors in 4-dimension:

$\varphi_1 = (0,0,0,1)$	$\varphi_2 = (0,0,1,0)$	$\varphi_3 = (1,1,0,0)$
$\varphi_4 = (1,-1,0,0)$	$\varphi_5 = (0,1,0,0)$	$\varphi_6 = (1,01,0)$
$\varphi_7 = (1,0,-1,0)$	$\varphi_8 = (1,-1,1,-1)$	$\varphi_9 = (0,0,1,1)$
$\varphi_{10} = (1,1,1,1)$	$\varphi_{11} = (0,1,0,-1)$	$\varphi_{12} = (1,0,0,1)$
$\varphi_{13} = (1,0,0,-1)$	$\varphi_{14} = (0,1,-1,0)$	$\varphi_{15} = (1,1,-1,1)$
$\varphi_{16} = (1,1,1,-1)$	$\varphi_{17} = (-1,1,1,1)$	$\varphi_{18} = (1,-1,-1,1)$

Rule of value assignment:

1) $v(\varphi_i) \equiv v(|\varphi_i><\varphi_i|)$ = 0 or 1

2) $\sum v(\varphi_i) = 1$, $\{\varphi_i\}$ form a orthogonal basis.

$$v(\varphi_1) + v(\varphi_2) + v(\varphi_3) + v(\varphi_4) = 1$$

$$v(\varphi_1) + v(\varphi_5) + v(\varphi_6) + v(\varphi_7) = 1$$

$$v(\varphi_8) + v(\varphi_{18}) + v(\varphi_3) + v(\varphi_9) = 1$$

$$v(\varphi_8) + v(\varphi_{10}) + v(\varphi_7) + v(\varphi_{11}) = 1$$

$$v(\varphi_2) + v(\varphi_5) + v(\varphi_{12}) + v(\varphi_{13}) = 1$$

$$v(\varphi_{18}) + v(\varphi_{10}) + v(\varphi_{13}) + v(\varphi_{14}) = 1$$

$$v(\varphi_{15}) + v(\varphi_{16}) + v(\varphi_4) + v(\varphi_9) = 1$$

$$v(\varphi_{15}) + v(\varphi_{17}) + v(\varphi_6) + v(\varphi_{11}) = 1$$

$$v(\varphi_{16}) + v(\varphi_{17}) + v(\varphi_{12}) + v(\varphi_{14}) = 1$$

If added, the L.H.S. is even as every vector has appeared twice and the R.H.S. is odd.

It shows that non-contextual HVT, in general can not reproduce quantum mechanics.

Another proof with spin operators for two qubits

$\sigma_x^1 \otimes I$	$I \otimes \sigma_x^2$	$\sigma_x^1 \otimes \sigma_x^2$	$= I \otimes I$
$I \otimes \sigma_y^2$	$\sigma_y^1 \otimes I$	$\sigma_y^1 \otimes \sigma_y^2$	$= I \otimes I$
$\sigma_x^1 \otimes \sigma_y^2$	$\sigma_y^1 \otimes \sigma_x^2$	$\sigma_z^1 \otimes \sigma_z^2$	$= I \otimes I$
$= I \otimes I$	$= I \otimes I$	$= - I \otimes I$	

For a non-contextual HVT, value for each observable is 1 or -1 as they are the eigen values.

$$v(\sigma_x^1 \otimes I)\, v(I \otimes \sigma_x^2)\, v(\sigma_x^1 \otimes \sigma_x^2) = 1$$

$$v(I \otimes \sigma_y^2)\, v(\sigma_y^1 \otimes I)\, v(\sigma_y^1 \otimes \sigma_y^2) = 1$$

$$v(\sigma_x^1 \otimes \sigma_y^2)\, v(\sigma_y^1 \otimes \sigma_x^2)\, v(\sigma_z^1 \otimes \sigma_z^2) = 1$$

$$v(\sigma_x^1 \otimes I)\, v(I \otimes \sigma_y^2)\, v(\sigma_x^1 \otimes \sigma_y^2) = 1$$

$$v(I \otimes \sigma_x^2)\, v(\sigma_y^1 \otimes I)\, v(\sigma_y^1 \otimes \sigma_x^2) = 1$$

$$v(\sigma_x^1 \otimes \sigma_x^2)\, v(\sigma_y^1 \otimes \sigma_y^2)\, v(\sigma_z^1 \otimes \sigma_z^2) = -1$$

Product on the right hand side is +1

Product on the left hand side is -1.

A proof of contextuality of HVT where quantum state is epistemic

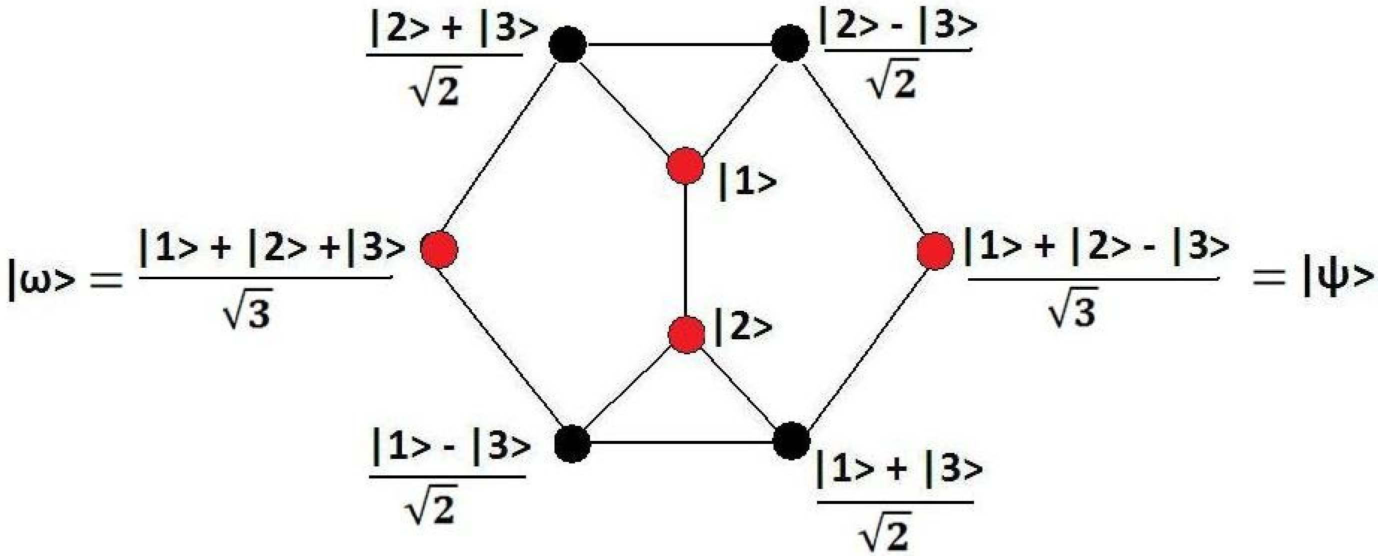
Consider two quantum state $|\psi\rangle$ and $|\omega\rangle$.

If $|\psi\rangle$ and $|\omega\rangle$ are non-orthogonal then there exists λ for which

$$\theta_{\psi}(\lambda) \neq 0 \text{ and } \theta_{\omega}(\lambda) \neq 0$$

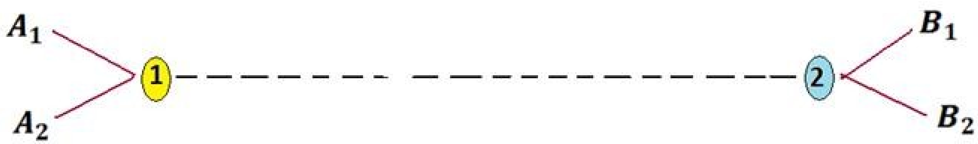
We consider such HVT state λ and observe the following

For this HVT state, $v_{\lambda}(P_{\psi}) = 1 \; and \; v_{\lambda}(P_{\varphi}) = 1$



Bell's inequality

v_λ



Possible values : ± 1

Values specified by a given HVT state λ : $v_\lambda(A_1), v_\lambda(B_1), v_\lambda(A_2), v_\lambda(B_2)$

Locality:

The Values specified for particle 1 by a given HVT state is independent of measurement on particle 2.

$$\mathrm{B}_\lambda = v_\lambda(A_1)\,[v_\lambda(B_1) + v_\lambda(B_2)] + v_\lambda(A_2)\,[v_\lambda(B_1) - v_\lambda(B_2)]$$

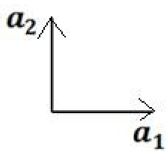
$$\mathrm{B}_\lambda = \pm 2$$

$$-2 \leq \int \rho_\lambda \mathrm{B}_\lambda d\lambda \leq 2$$

$$-2 \leq \int v_\lambda(A_1)\, v_\lambda(B_1)\, \rho_\lambda\, d\lambda + \int v_\lambda(A_1) v_\lambda(B_2)\, \rho_\lambda\, d\lambda + \int v_\lambda(A_2)\, v_\lambda(B_1)\, \rho_\lambda\, d\lambda + \int v_\lambda(A_2) v_\lambda(B_2)\, \rho_\lambda\, d\lambda \leq 2$$

$$-2 \leq \; < A_1 B_1 > + < A_1 B_2 > + < A_2 B_1 > - < A_2 B_2 > \; \leq 2$$

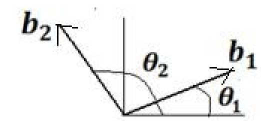
Measurement on Alice's side



$$a_1 = \hat{\imath}, \quad a_2 = \hat{k}$$

$$|\varphi>_{AB} = c_0|0>_A|0>_B + c_1|1>_A|1>_B$$

Measurement on Bob's side



$$b_1 = Cos\theta_1\hat{\imath} + Sin\theta_1\hat{k}$$

$$b_2 = Cos\theta_2\hat{\imath} + Sin\theta_2\hat{k}$$

$$Cos\theta_2 = -Cos\theta_1 = (1 + 4|c_0|^2|c_1|^2)^{-1/2}$$

$$B_{CHSH} = a_1.\sigma \otimes b_1.\sigma + a_1.\sigma \otimes b_2.\sigma + a_2.\sigma \otimes b_1.\sigma - a_2.\sigma \otimes b_2.\sigma$$

$$<\varphi|B_{CHSH}|\varphi> = 2\,(1 + 4|c_0|^2|c_1|^2)^{1/2}$$

So for any pure entangled state, one can choose observables such that BI is violated.

For maximally entangled state: $|c_0|^2 = |c_1|^2 = \frac{1}{\sqrt{2}}$

$$<\varphi|B_{CHSH}|\varphi> = 2\sqrt{2}$$

Stochastic HVT model

HVT state does not determine value of the observable to be revealed in measurement but probability.

$$p(a/A) = \int p_\lambda(a/A)\theta(\lambda)d\lambda$$

$$p(ab/AB) = \int p_\lambda(a/A)p_\lambda(b/B)\theta(\lambda)d\lambda$$

Bell locality condition:

$$p_\lambda(ab/AB) = p_\lambda(a/A)p_\lambda(b/B)$$

Outcome independence:

$$p_\lambda(a/AB = b) = p_\lambda(a/AB)$$

$$p_\lambda(b/A = a, B) = p_\lambda(b/AB)$$

Parameter independence:

$$p_\lambda(a/AB) = p_\lambda(a/A)$$

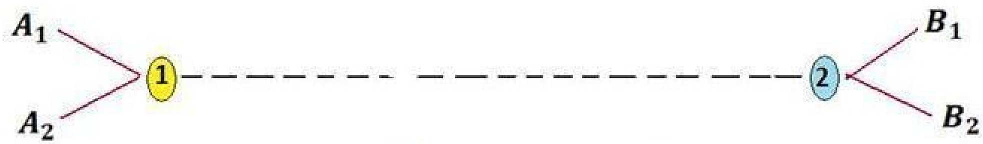
$$p_\lambda(b/AB) = p_\lambda(b/B)$$

BL = PI + OI

$$p_\lambda(ab/AB) = p_\lambda(a/A\ B = b)\, p_\lambda(b/AB)$$ (from conditional probability)

$$= p_\lambda(a/AB)\, p_\lambda(b/AB)$$ (from OI)

$$= p_\lambda(a/A)\, p_\lambda(b/B)$$ (from PI)



Possible values : ± 1

$$\alpha = p_\lambda(+1/A_1) \quad \overline{\alpha} = p_\lambda(+1/A_2) \quad \beta = p_\lambda(+1/B_1) \quad \overline{\beta} = p_\lambda(+1/B_2) \qquad 0 \le \alpha, \overline{\alpha}, \beta, \overline{\beta} \le 1$$

$$k(\lambda) = p_\lambda(+1/A_1) + p_\lambda(+1/B_1) + p_\lambda(+1/A_2)\, p_\lambda(+1/B_2) - p_\lambda(+1/A_1)\, p_\lambda(+1/B_1)$$

$$- p_\lambda(+1/A_2)\, p_\lambda(+1/B_1) - p_\lambda(+1/A_1)\, p_\lambda(+1/B_2)$$

$$= \alpha + \beta + \overline{\alpha}\,\overline{\beta} - \alpha\beta - \overline{\alpha}\beta - \alpha\overline{\beta}$$

$$= \alpha\,[\overline{\alpha}(1-\beta) + (1-\overline{\alpha})(1-\overline{\beta})] + (1-\alpha)[\overline{\alpha}\,\overline{\beta} + (1-\overline{\alpha})\beta]$$

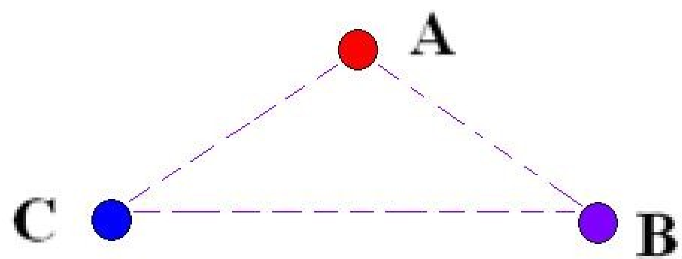
$$0 \le k(\lambda) \le 1$$

$$0 \le \int k(\lambda)\,\theta(\lambda)\,d\lambda \le 1$$

$$0 \le p(+1/A_1) + p(+1/B_1) + p(+1{+}1/A_2B_2) - p(+1{+}1/A_1B_1)$$

$$- p(+1{+}1/A_2B_1) - p(+1{+}1/A_1B_2) \le 1$$

Non-local arguments with three qubits



$$|\Psi\rangle_{GHZ} = \frac{1}{\sqrt{2}}\left[|0\rangle_A|0\rangle_B|0\rangle_C - |1\rangle_A|1\rangle_B|1\rangle_C\right]$$

satisfies the following eigen value equations

$$\sigma_x^A \otimes \sigma_y^B \otimes \sigma_y^C|\Psi\rangle_{GHZ} = |\Psi\rangle_{GHZ}$$

$$\sigma_y^A \otimes \sigma_x^B \otimes \sigma_y^C|\Psi\rangle_{GHZ} = |\Psi\rangle_{GHZ}$$

$$\sigma_y^A \otimes \sigma_y^B \otimes \sigma_x^C|\Psi\rangle_{GHZ} = |\Psi\rangle_{GHZ}$$

$$\sigma_x^A \otimes \sigma_x^B \otimes \sigma_x^C|\Psi\rangle_{GHZ} = -|\Psi\rangle_{GHZ}$$

What is meant by the following equation?

$$\sigma_x^A \otimes \sigma_y^B \otimes \sigma_y^C |\Psi>_{GHZ} = |\Psi>_{GHZ}$$

Alice measures σ_x^A

Bob measures σ_y^B

Charlie measures σ_y^C

$$p(\sigma_x^A = +1, \sigma_y^B = +1, \sigma_y^C = +1) = \frac{1}{4}$$

$$p(\sigma_x^A = -1, \sigma_y^B = -1, \sigma_y^C = +1) = \frac{1}{4}$$

$$p(\sigma_x^A = -1, \sigma_y^B = +1, \sigma_y^C = -1) = \frac{1}{4}$$

$$p(\sigma_x^A = +1, \sigma_y^B = -1, \sigma_y^C = -1) = \frac{1}{4}$$

Which means product of their results is always +1

This is an element of reality for the quantum state.

What is meant by the following equation?

$$\sigma_x^A \otimes \sigma_y^B \otimes \sigma_y^C |\Psi>_{GHZ} = |\Psi>_{GHZ}$$

Alice measures σ_x^A

Bob measures σ_y^B

Charlie measures σ_y^C

$$p(\sigma_x^A = +1, \sigma_y^B = +1, \sigma_y^C = +1) = \frac{1}{4}$$

$$p(\sigma_x^A = -1, \sigma_y^B = -1, \sigma_y^C = +1) = \frac{1}{4}$$

$$p(\sigma_x^A = -1, \sigma_y^B = +1, \sigma_y^C = -1) = \frac{1}{4}$$

$$p(\sigma_x^A = +1, \sigma_y^B = -1, \sigma_y^C = -1) = \frac{1}{4}$$

Which means product of their results is always +1

This is an element of reality for the quantum state.

Element of reality for the quantum state has to be satisfied for every HVT state λ which is a member of ensemble of GHZ state.

$$v_\lambda(\sigma_x^A)\, v_\lambda(\sigma_y^B)\, v_\lambda(\sigma_y^C) = 1$$

$$v_\lambda(\sigma_y^A)\, v_\lambda(\sigma_x^B)\, v_\lambda(\sigma_y^C) = 1$$

$$v_\lambda(\sigma_y^A)\, v_\lambda(\sigma_y^B)\, v_\lambda(\sigma_x^C) = 1$$

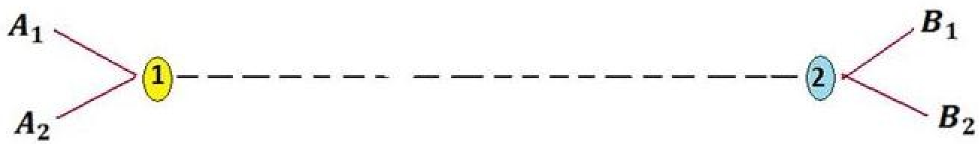
$$v_\lambda(\sigma_x^A)\, v_\lambda(\sigma_x^B)\, v_\lambda(\sigma_x^C) = -1$$

If we take product of these four equations,

the L.H.S. is positive

and R.H.S. is negetive.

Can this kind of argument (non-locality without inequality) be found in two qubits system?



Possible values : ± 1

$$p(A_1 = 1, B_1 = 1) = 0$$

$$p(A_2 = -1, B_1 = -1) = 0$$

$$p(A_1 = -1, B_2 = -1) = 0$$

$$p(A_2 = -1, B_2 = -1) = q$$

If some local HVT state reproduces this statistics, then there will be at least one HVT state λ which satisfies the following;

4th eqn. tells $v_\lambda(A_2) = -1, \quad v_\lambda(B_2) = -1$

From 3rd eqn. $v_\lambda(A_2) = -1 \Rightarrow v_\lambda(B_1) = 1$

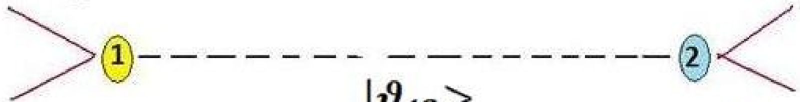
From 2nd eqn. $v_\lambda(B_2) = -1 \Rightarrow v_\lambda(A_1) = 1$

$\Rightarrow \quad v_\lambda(A_1) = 1, \quad v_\lambda(B_1) = 1$

1st eqn tells $v_\lambda(A_1) = 1 \quad v_\lambda(B_1) = 1$ never happens.

Is there a quantum state which shows this kind of non- locality?

$|\varphi_1\rangle\langle\varphi_1| - |\bar{\varphi}_1\rangle\langle\bar{\varphi}_1| = A_1$ $\qquad$ $B_1 = |\psi_1\rangle\langle\psi_1| - |\bar{\psi}_1\rangle\langle\bar{\psi}_1|$



$|\vartheta_{12}\rangle$

$|\varphi_2\rangle\langle\varphi_2| - |\bar{\varphi}_2\rangle\langle\bar{\varphi}_2| = A_2$ $\qquad$ $B_2 = |\psi_2\rangle\langle\psi_2| - |\bar{\psi}_2\rangle\langle\bar{\psi}_2|$

$$\langle\vartheta_{12}|\varphi_1\otimes\psi_1\rangle\langle\varphi_1\otimes\psi_1|\vartheta_{12}\rangle = |\langle\varphi_1\otimes\psi_1|\vartheta_{12}\rangle|^2 = 0$$

$$\langle\vartheta_{12}|\bar{\varphi}_2\otimes\bar{\psi}_1\rangle\langle\bar{\varphi}_2\otimes\bar{\psi}_1|\vartheta_{12}\rangle = |\langle\bar{\varphi}_2\otimes\bar{\psi}_1|\vartheta_{12}\rangle|^2 = 0$$

$$\langle\vartheta_{12}|\bar{\varphi}_1\otimes\bar{\psi}_2\rangle\langle\bar{\varphi}_1\otimes\bar{\psi}_2|\vartheta_{12}\rangle = |\langle\bar{\varphi}_1\otimes\bar{\psi}_2|\vartheta_{12}\rangle|^2 = 0$$

$$\langle\vartheta_{12}|\bar{\varphi}_2\otimes\bar{\psi}_2\rangle\langle\bar{\varphi}_2\otimes\bar{\psi}_2|\vartheta_{12}\rangle = |\langle\bar{\varphi}_2\otimes\bar{\psi}_2|\vartheta_{12}\rangle|^2 = q > 0$$

$|\varphi_1\otimes\psi_1\rangle$, $|\bar{\varphi}_2\otimes\bar{\psi}_1\rangle$, $|\bar{\varphi}_1\otimes\bar{\psi}_2\rangle$, $|\bar{\varphi}_2\otimes\bar{\psi}_2\rangle$ form an linearly independent set of vectors in 4 dimensianal Hilbert space.

So there is a unique vector $|\vartheta_{12}\rangle$ which is orthogonal to first three and non- orthogonal to 4^{th} one.

1) Every non-maximally entangled state show this property.

2) There is no set of observables and state for which q = 1.

Does all entangled state violates Bell's inequality?

Werner class: $W_p = p|\psi^- >< \psi^-| + \frac{1-p}{4} I \otimes I$

Entangled: $\frac{1}{3} < p \leq 1$

$$Tr[W_p B_{CHSH}]_{\max} = p < \Psi^-|B_{CHSH}|\Psi^- >_{\max} = 2\sqrt{2}\,p$$

Violates no BI: $\frac{1}{3} \leq p \leq \frac{1}{\sqrt{2}}$

Can this classbe simulated by local HVT?

For $\frac{1}{3} \leq p \leq \frac{1}{2}$ there is an local HVT model.

For $p = \frac{1}{2}$

$$W_{\frac{1}{2}} = \frac{1}{2}|\psi^- >< \psi^-| + \frac{1}{8} I \otimes I$$

$$p_{w_{\frac{1}{2}}}(\sigma_m = 1, \sigma_n = 1)$$

$$= Tr[W_{\frac{1}{2}} \frac{1}{2}(I + \sigma_m) \otimes \frac{1}{2}(I + \sigma_n) = \frac{1}{4}(1 - \frac{1}{2} cos\alpha)$$

LHV Model

λ is shared random variable between parties and they are unit vectors over unit sphere with uniform distributed distribution.

Alice

λ_1
λ_2
λ_3
λ_4
λ_1
λ_5
⋮

$$\rho(\lambda)d\lambda = \frac{1}{4\pi} Sin\Theta \, d\Theta \, d\varphi$$

Bob

λ_1
λ_2
λ_3
λ_4
λ_1
λ_5
⋮

Strаregy:

Alice outputs up with probability

$$p_\lambda(\sigma_m = +1) = cos^2(\frac{\alpha_m}{2})$$

α_m is the angle m between m and λ

Bob otputs up with probability

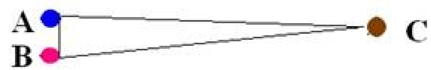
$$p_\lambda(\sigma_n = +1) = \begin{cases} 1 \; if \; 2cos^2\left(\frac{\alpha_n}{2}\right) < 1 \\ 0 \; if \; 2cos^2\left(\frac{\alpha_n}{2}\right) > 1 \end{cases}$$

α_n is the angle n between m and λ

$$p_{lhv}(\sigma_m = 1, \sigma_n = 1) = \int \rho(\lambda)d\lambda \;\; p_\lambda(\sigma_m = +1) \, p_\lambda(\sigma_n = +1)$$

$$= \frac{1}{4}(1 - \frac{1}{2} cos\alpha) = p_{quantum}(\sigma_m = 1, \sigma_n = 1)$$

Does the non-locality argument still hold?



λ_2

Alice

λ_1	λ_2
+1	+1
+1	-1
-1	-1
+1	+1
-1	+1
+1	+1
- - -	- - -
- - -	- -

In this case, non-local argument does not run as the result can be simulated by local theory.

Alice and Bob share two shared random variables λ_1 and λ_2 which take value +1 or -1.

Bob

λ_1	λ_2
+1	+1
+1	-1
-1	-1
+1	+1
-1	+1
+1	+1
- - -	- - -
- - -	- -

Alice's strategy:

$v(\sigma_y^A) = \lambda_1, \ v(\sigma_x^B) = \lambda_2$

$v(\sigma_y^A) = \lambda_1, \ v(\sigma_y^B) = \lambda_2$

$v(\sigma_x^A) = \lambda_1, \ v(\sigma_x^B) = -\lambda_2$

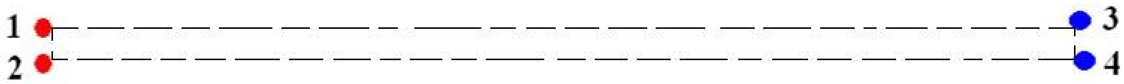
Bob's strategy:

$v(\sigma_x^C) = \lambda_1\lambda_2$

$v(\sigma_y^C) = \lambda_1\lambda_2$

So Local but contextual theory exactly reproduces the GHZ correlation

Cabello's non-locality argument with separable measurement on each particle



$$|\Psi>_{1234} = \frac{1}{2}[|0>_1|0>_2|0>_3\ 0>_4 + |0>_1|1>_2|0>_3|1>_4 + |1>_1|0>_2|1>_3\ 0>_4 - |1>_1|1>_2|1>_3|1>_4]$$

Eigen value equations:

$$X_1 I_2 X_3 Z_4 |\Psi>_{1234} = |\Psi>_{1234}$$
$$Y_1 I_2 Y_3 Z_4 |\Psi>_{1234} = -|\Psi>_{1234}$$
$$X_1 X_2 Y_3 Y_4 |\Psi>_{1234} = |\Psi>_{1234}$$
$$Y_1 X_2 X_3 Y_4 |\Psi>_{1234} = |\Psi>_{1234}$$

The locally assigned value has to satisfy;

$$v(X_1) = v(X_3)v(Z_4)$$
$$v(Y_1) = -v(Y_3)v(Z_4)$$
$$v(X_1)v(X_2) = v(Y_3)v(Y_4)$$
$$v(Y_1)v(X_2) = v(X_3)v(Y_4)$$

Contradiction!

But there is something wrong.

Alice and Bob share two random variables λ_1 and λ_2, and Bob has another random variable η, all of them taking values +1 or -1

Alice

λ_1	λ_2
+1	-1
+1	+1
-1	+1
+1	+1
-1	-1
+1	+1
- - -	- -
- - -	- -

Alice's strategy:

$$v(X_1) = v(Y_1) = \lambda_1$$
$$v(X_2) = \lambda_2$$

Bob

λ_1	λ_2	η
+1	-1	+1
+1	+1	-1
-1	+1	-1
+1	+1	+1
-1	-1	-1
+1	+1	+1
- - -	- - -	.
- - -	- - -	.

Bob's strategy:

X_3, Y_4:	$v(X_3) = \eta,\ v(Y_4) = \eta\lambda_1\lambda_2$
X_3, Z_4:	$v(X_3) = \eta,\ v(Z_4) = \eta\lambda_1$
Y_3, Y_4:	$v(Y_3) = \eta,\ v(Y_4) = \eta\lambda_1\lambda_2$
Y_3, Z_4:	$v(Y_3) = \eta,\ v(Z_4) = -\eta\lambda_1$

This strategy exactly reproduces the following crrelation.

$$v(X_1) = v(X_3)v(Z_4)$$
$$v(Y_1) = -v(Y_3)v(Z_4)$$
$$v(X_1)v(X_2) = v(Y_3)v(Y_4)$$
$$v(Y_1)v(X_2) = v(X_3)v(Y_4)$$

It shows that a local but contextual theory can reproduce the correlation.