Open system quantum dynamics and signatures of non-Markovianity

Dr. Sudha,
Department of Physics,
Kuvempu University,
Shankaraghatta-577451

<u>Collaborators:</u> Dr. A.R. Usha Devi, Bangalore University, Bangalore
Prof. A.K.Rajagopal, Inspire Institute Inc.,McLean, VA 22101, USA

Outline

- Open system quantum dynamics– concept of dynamical maps
- Equivalence of A, B dynamical maps
- An Example of Unitary dynamics
- Signatures of non-Markovianity
- Illustrative Examples
- Quantitative Sigantures of non-Markovianity
- Concluding remarks

Open system quantum dynamics

When a quantum system, chosen initially to be in a tensor product state with environment, undergoes dynamical evolution, the final state of the system is related to the initial system through a Completely Positive (CP) dynamical map.

G. Lindblad, Commun. Math. Phys. 48, 119 (1976)
R. Alicki and K.Lendl, *Quantum Dynamical Semigroups and Applications*, Lecture Notes in Physics, Vol. 286 (Springer, Berlin, 1987)

Open system quantum dynamics

$$\rho_{SE}(0) = \rho_S \otimes \rho_E$$ → Initial system environment state (uncorrelated)

$$\rho_S(0) = \mathrm{Tr}_E \rho_{SE}(0)$$ → Initial state of the system

$$\rho_{SE}(t) = \mathbf{U}(t)\rho_{SE}(0)\mathbf{U}^\dagger(t)$$ → Unitary dynamics

$$\rho_S(t) = \mathrm{Tr}_E \rho_{SE}(t)$$ → State of the system at time t

Completely Positive map

$$\Lambda(t) : \rho_S(0) \to \rho_S(t)$$

Open system quantum dynamics

Kraus decomposition of the dynamical map

--- possible only when the map is CP

$$\rho_S(t) = \sum_i K_i(t)\rho_S(0)K_i^\dagger(t); \quad \sum_i K_i^\dagger(t)K_i(t) = I$$

$K_i(t)$ are the Kraus operators associated with $\Lambda(t)$

K. Kraus, *States, Effects and Operations: Fundamental Notions of Quantum Theory*, Vol. 190 of Lecture Notes in Physics (Springer-Verlag, New York, 1983)

Completely Positive Maps

It is physically meaningful to define quantum operation as a map from the set of density matrices of the input Hilbert space to the set of density matrices of the output Hilbert space satisfying the following properties:

1) Linearity: $\Phi\,(a\rho_1 + b\rho_2) = a\,\Phi(\rho_1) + b\,\Phi(\rho_2)$

2) Trace–preservation (TP): $\operatorname{Tr}\Phi(\rho) = \operatorname{Tr}\rho$

3) Positivity: $\Phi(\rho) \geq 0$

4) **Complete Positivity (CP):**

If H_A denotes the Hilbert space of the system and if $H_A \otimes H_B$ is any extension of H_A, then the map Φ is Completely positive iff $\Phi \otimes \mathrm{id}_B$ is positive for all such extensions. Here $\mathrm{id_B}$ is the identity map on H_B.

A, B dynamical maps

The concept of A, B dynamical maps---initiated by E.C.G. Sudarshan and co-workers

A general open system dynamics relates the elements $[\rho_S(t)]_{r's'}$ with $[\rho_S(0)]_{rs}$ through a linear map

$$\boxed{[\rho_S(t)]_{r's'} = \sum_{r,s=1}^{n} A_{r's';rs}\,[\rho_S(0)]_{rs}\,; \quad r',\, s' = 1,\, 2, \ldots, n}$$

E.C.G. Sudarshan, P. Mathews and **J.Rau**, Phys.Rev. **121**, 920 (1961)

T.F.Jordan and **E.C.G. Sudarshan**, J.Math.Phys. **2**, 772 (1961)

A and B Dynamical maps

Preservation of hermiticity of $\rho_S(t)$ $\longrightarrow$ $A_{s'r';sr} = A^*_{r's';rs}$

Unit trace condition on $\rho_S(t)$ $\longrightarrow$ $\sum_{r'} A_{r'r';rs} = \delta_{rs}$

It has been found convenient to define a realigned matrix B

to bring out the properties of the A map

E.C.G. Sudarshan, P. Mathews and **J.Rau**, Phys.Rev. **121**, 920 (1961)

A and B dynamical maps

$$B_{r'r;s's} = A_{r's';rs}$$ Definition of the B-map

$$A_{s'r';sr} = A^*_{r's';rs} \Longrightarrow B_{s's;r'r} = B^*_{r'r;s's}$$ B is hermitian

The hermitian nature of B-map is exploited to identify the general features of dynamics

Eigenvalues of B matrix non-negative implies CP map

M.D.Choi, Can.J.Math. **24**, 520 (1972);

M.D.Choi, Linear Algebra and Appl. **10**, 285 (1975)

Canonical form of the A-map and equivalence of A- and B-maps

Consider an orthonormal set $\{T_\alpha,\ \alpha = 1, 2, \ldots, n^2\}$

of $n \times n$ basis matrices satisfying $\mathrm{Tr}\left[T_\alpha^\dagger T_\beta\right] = \delta_{\alpha\beta}$

We now have, $A = \sum_{\alpha\beta} \mathcal{A}_{\alpha\beta} \left(T_\alpha \otimes T_\beta^*\right)$ where

$$\mathcal{A}_{\alpha\beta} = \mathrm{Tr}\left[A(T_\alpha^\dagger \otimes \tilde{T}_\beta)\right]$$

Thus, $A_{r's';rs} = \sum_{\alpha,\beta=1}^{n^2} \mathcal{A}_{\alpha\beta}[T_\alpha]_{r'r}[T_\alpha^*]_{s's}$

Now $A_{s'r';sr} = A^*_{r's';rs} \Longrightarrow \mathcal{A}_{\alpha\beta} = \mathcal{A}^*_{\beta\alpha}$

Canonical form of A-map

That is, the coefficients $\mathcal{A}_{\alpha\beta}$ form a $n^2 \times n^2$ hermitian matrix. We can therefore have,

$\sum_{\alpha\beta} \mathcal{U}_{\mu\alpha}\mathcal{A}_{\alpha\beta}\mathcal{U}^*_{\mu\beta} = \lambda_\mu;$ λ_μ are eigenvalues of $\mathcal{A} = (\mathcal{A}_{\alpha\beta})$

Finally,

$$
\begin{aligned}
A &= \sum_{\alpha,\beta,\mu} \lambda_\mu \mathcal{U}_{\mu\beta}\mathcal{U}^*_{\mu\alpha}(T_\alpha \otimes T^*_\beta) \\
&= \sum_{\mu} \lambda_\mu C_\mu \otimes C^*_\mu; \quad C_\mu = \sum_\alpha \mathcal{U}^*_{\mu\alpha} T_\alpha
\end{aligned}
$$

Canonical structure of the A-map

Canonical form of A-map

On substituting $A_{r's';rs} = \sum_\mu \lambda_\mu \, [\mathcal{C}_\mu]_{r'r} \, [\mathcal{C}^*_\mu]_{s's}$ in $[\rho_S(t)]_{r's'} = \sum_{r,s=1}^{n} A_{r's';rs} \, [\rho_S(0)]_{rs}$ we get

$$\rho_S(t) = \sum_\mu \lambda_\mu \, \mathcal{C}_\mu \, \rho_S(0) \, \mathcal{C}^\dagger_\mu$$

which gives the action of the A-map on the initial density matrix $\rho_S(0)$.

Equivalence of A and B maps

It can be readily seen that

$$B_{r'r;s's} = \sum_\mu \lambda_\mu \, [\mathcal{C}_\mu]_{r'r} \, [\mathcal{C}^*_\mu]_{s's}$$

which corresponds to the spectral decomposition of the B matrix. Thus, the eigenvalues λ_μ of the B-matrix and the coefficient matrix $\mathcal{A}$ are identical.

The dynamical map is CP if all the eigenvalues λ_μ are non-negative

It is NCP if at least one eigenvalue is negative

Not Completely Positive Maps

The emergence of the concept of Not Completely Positive Maps (NCP)

P. Pechukas, Phys.Rev. Lett. **73**, 1060 (1994)

The nature of dynamical maps for NCP dynamics is investigated by Sudarshan and co-workers

T.F.Jordan, A.Shaji and **E.C.G. Sudarshan**, Phys.Rev.A 70, 052110 (2004)
C.A.Rodriguez-Rosario et.al., J.Phys.A. 41, 205301 (2008)
K.Modi and **E.C.G. Sudarshan**, Phys.Rev.A 81, 052119 (2010)

Not Completely Positive Maps

When the initial system environment state is correlated, that is, when it is not of the form $\rho_{SE}(0) \neq \rho_S \otimes \rho_E$, the resultant dynamical map $\Lambda(t) : \rho_S(0) \rightarrow \rho_S(t)$ is Not Completely Positive (NCP).

If atleast one of the Eigenvalues of the B-map or equivalently the eigenvalues of the matrix $\mathcal{A} = (\mathcal{A}_{\alpha\beta})$ is negative, the map is NCP

An Example of 2-qubit unitary dynamics

Consider the unitary dynamical evolution on 2-qubit states

given by $U(t) = e^{-iHt/\hbar}$ where $H = \frac{1}{2}\hbar\omega\,\sigma_{1z}\sigma_{2x}$.

Explicitly,

$$U(t) = \begin{pmatrix} \cos\left(\frac{\omega t}{2}\right) & -i\sin\left(\frac{\omega t}{2}\right) & 0 & 0 \\ -i\sin\left(\frac{\omega t}{2}\right) & \cos\left(\frac{\omega t}{2}\right) & 0 & 0 \\ 0 & 0 & \cos\left(\frac{\omega t}{2}\right) & i\sin\left(\frac{\omega t}{2}\right) \\ 0 & 0 & i\sin\left(\frac{\omega t}{2}\right) & \cos\left(\frac{\omega t}{2}\right) \end{pmatrix}$$

T. F. Jordan, **A. Shaji**, and **E. C. G. Sudarshan**, Phys. Rev.A, **70**, 052110 (2004)

An Example of 2-qubit unitary dynamics

Time-evolution of the expectation values of the Pauli operators of the 1st qubit (in the Heisenberg picture)

$$
\begin{aligned}
\langle U^\dagger(t)\sigma_{1x}U(t)\rangle &= \langle\sigma_{1x}\rangle\,\cos(\omega\,t) - \langle\sigma_{1y}\sigma_{2x}\rangle\,\sin(\omega\,t) \\
&= \langle\sigma_{1x}\rangle\,\cos(\omega\,t) + a_1\,\sin(\omega\,t) \\
\langle U^\dagger(t)\sigma_{1y}U(t)\rangle &= \langle\sigma_{1y}\rangle\,\cos(\omega\,t) + \langle\sigma_{1x}\sigma_{2x}\rangle\,\sin(\omega\,t) \\
&= \langle\sigma_{1y}\rangle\,\cos(\omega\,t) + a_2\,\sin(\omega\,t) \\
\langle U^\dagger(t)\sigma_{1z}U(t)\rangle &= \langle\sigma_{1z}\rangle
\end{aligned}
$$

$\langle\sigma_{1x}\rangle,\ \langle\sigma_{1y}\rangle, \langle\sigma_{1z}\rangle$ are expectation values at time t=0

An Example of 2-qubit unitary dynamics

For fixed initial parameters $a_1 = -\langle \sigma_{1y}\sigma_{2x}\rangle,\ a_2 = \langle \sigma_{1x}\sigma_{2x}\rangle$

describing the evolution of the 1st qubit, the density matrix

$\rho_1(0) = \frac{1}{2}(I_1 + \sigma_{1x}\langle\sigma_{1x}\rangle + \sigma_{1y}\langle\sigma_{1y}\rangle + \sigma_{1z}\langle\sigma_{1z}\rangle)$ gets

mapped to

$$\begin{aligned}\rho_1(t) \quad = \quad & \frac{1}{2}\left[I_1 + (a_1\,\sigma_{1x} + a_2\,\sigma_{1y})\sin(\omega t) + \sigma_{1x}\langle\sigma_{1x}\rangle\cos(\omega t)\right. \\ & \left. + \sigma_{1y}\langle\sigma_{1y}\rangle\cos(\omega t) + \sigma_{1z}\langle\sigma_{1z}\rangle\right].\end{aligned}$$

An Example of 2-qubit unitary dynamics

A linear dynamical A-map $Q \rightarrow Q'$ for all 2X2 hermitian matrices, consistent with the given unitary dynamics is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2}\,S\,a^* & C & 0 & \frac{1}{2}\,S\,a^* \\ \frac{1}{2}\,S\,a & 0 & C & \frac{1}{2}\,S\,a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $a = a_1 + i\,a_2,\; C = \cos(\omega t),\; S = \sin(\omega t)$

An Example of 2-qubit unitary dynamics

Choosing $\left\{\frac{\sigma_{1\alpha}}{\sqrt{2}} \equiv \frac{I_1}{\sqrt{2}}, \frac{\sigma_{1x}}{\sqrt{2}}, \frac{\sigma_{1y}}{\sqrt{2}}, \frac{\sigma_{1z}}{\sqrt{2}}\right\}$ as the orthonormal set of basis matrices, we have

$A = \frac{1}{2}\sum_{\alpha\beta} \mathcal{A}_{\alpha\beta}\, \sigma_\alpha \otimes \sigma_\beta^*$ where

$$\begin{aligned} \mathcal{A} &= \frac{1}{2}\mathrm{Tr}[A(t)\sigma_\alpha \otimes \sigma_\beta^*] \\ &= \frac{1}{2}\begin{pmatrix} 2(1+C) & a_1\, S & a_2\, S & 0 \\ a_1\, S & 0 & 0 & i\, a_2\, S \\ a_2\, S & 0 & 0 & -i\, a_1\, S \\ 0 & -i\, a_2\, S & i\, a_1\, S & 2(1-C) \end{pmatrix}. \end{aligned}$$

An Example of 2-qubit unitary dynamics

The eigenvalues of the matrix $\mathcal{A}$ are given by

$$\begin{aligned}\lambda_{1\pm} &= \frac{1}{2}\left\{[1+\cos(\omega t)] \pm \sqrt{[1+\cos(\omega t)]^2 + |a|^2 \sin^2(\omega t)}\right\} \\ \lambda_{2\pm} &= \frac{1}{2}\left\{[1-\cos(\omega t)] \pm \sqrt{[1-\cos(\omega t)]^2 + |a|^2 \sin^2(\omega t)}\right\}.\end{aligned}$$

It can be seen that $\lambda_{1-}, \lambda_{2-}$ assume negative values

and thus the dynamical map is NCP.

Signatures of Non-Markovianity

An open system CP dynamical map is Markovian if it forms a one parameter semigroup which corresponds to

$$A(t+\tau) = A(t)A(\tau), \quad t, \tau \geq 0$$

for the A-map.

In other words, when CP dynamics is Markovian, A has an exponential structure $A = e^{\mathcal{L}\, t}$, $\mathcal{L}$ denoting the time-independent generator of the quantum dynamical semigroup.

H.-P. Breuer and **F. Petruccione**, *The Theory of Open Quantum Systems* (Oxford Univ. Press, Oxford, 2007)
G.Lindblad, Comm.Math.Phy. 48, 119 (1976); **V.Gorini, A.Kossakowski** and **E.C.G. Sudarshan**, J.Math.Phys. **17**, 821 (1976)

Signatures of Non-Markovianity

In our quest for physical quantities that can capture the departure from CP Markovian semigroup property of evolution, we examine the relative entropy function

$$S(\rho||\gamma) = \mathrm{Tr}\left[\rho(\ln\rho - \ln\gamma)\right];$$

$$S(\rho||\gamma) \geq 0; \qquad S(\rho||\gamma) = 0 \quad \text{iff} \quad \rho = \gamma$$

Signatures of Non-Markovianity

Under CP, trace preserving maps $\Lambda,\ S(\rho||\gamma)$ obeys monotonicity property; That is, $S[\Lambda(\rho)||\Lambda(\gamma)] \leq S(\rho||\gamma)$

Thus, when $A:\ \rho(0) \rightarrow \rho(t)$ is a CP map, we have,

$$\begin{aligned} S[\rho(t)||\rho(t+\tau)] &\equiv S[A(t)\rho(0)||A(t)\rho(\tau)] \\ &\leq S[\rho(0)||\rho(\tau)]. \end{aligned}$$

Relative entropy difference $S(t,\ \tau)$;

$$S(t,\tau) = S[\rho(0)||\rho(\tau)] - S[\rho(t)||\rho(t+\tau)] > 0$$

for CP dynamical maps

Signatures of Non-Markovianity

$S(t,\,\tau) < 0 \Longrightarrow$ non-Markovian dynamics (both CP as well as NCP)

Sufficient condition for non-Markovianity

Another sufficient condition for Markovianity:

;

Fidelity Difference $G(t,\tau)$

A. K. Rajagopal, **A. R. Usha Devi**, **R. W. Rendell**, Physical Review A, **82** 042107 (2010)

Signatures of Non-Markovianity

The fidelity function

R. Jozsa, J. Mod. Optics, 41, 2315 (1994)

$$F[\rho(t),\rho(t+\tau)] = \left\{ \mathrm{Tr}\left[\sqrt{\sqrt{\rho(t)}\rho(t+\tau)\sqrt{\rho(t)}} \right] \right\}^2$$

never decreases from its initial value $F[\rho(0),\rho(\tau)]$ for the state undergoing Markovian CP evolution. Thus, for non-Markovian dynamics,

$$G(t,\tau) = \frac{F[\rho(t),\rho(t+\tau)] - F[\rho(0),\rho(\tau)]}{F[\rho(0),\rho(\tau)]} < 0$$

Sufficient condition for non-Markovianity

Signatures of Non-Markovianity

Both the relative entropy difference and fidelity difference offer operational advantages that they require only the specification of the initial density matrix $\rho(0)$ and the dynamically evolved state $\rho(t)$ for their evaluation without any apriori knowledge on the nature of the environment and/or the coupling between the system and environment.

A.R.Usha Devi, **A.K.Rajagopal, Sudha**, Physical Review A, **82,** 042107 (2011)

Illustrative Examples:

We investigate three different examples of 2-qubit initial states, jointly undergoing the unitary transformation $U(t)$ so that the joint evolution of the first qubit (system) is represented by the dynamical map

$$\rho_1(0) \to \rho_1(t) = A\,\rho_1(0) = \mathrm{Tr}_2\left[U(t)\rho_{12}(0)U(t)^\dagger\right]$$

Non-Markovianity is bound to emerge due to the NCP nature of the dynamical map.

Illustrative Examples: Example 1

Two-qubit pure entangled state:

$$|\Psi\rangle = \frac{1}{\sqrt{3}} \left(e^{-i\phi}|0_1, 1_2\rangle + e^{i\phi}|1_1, 0_2\rangle + |1_1, 1_2\rangle \right)$$

Initial density matrix of the system (1st qubit):

$$\rho_1(t=0) = \frac{1}{3} \begin{pmatrix} 1 & e^{-i\phi} \\ e^{i\phi} & 2 \end{pmatrix}$$

State of the system after unitary evolution:

$$\begin{aligned} \rho_1(t) &= \mathrm{Tr}_2[U(t)|\Psi_{EP}\rangle\langle\Psi_{EP}|U^\dagger(t)] \\ &= \frac{1}{3} \begin{pmatrix} 1 & C\,e^{-i\phi} - iS\,e^{-2i\phi} \\ C\,e^{i\phi} + iS\,e^{2i\phi} & 2 \end{pmatrix} \end{aligned}$$

Illustrative Examples: Example 1

We also have, $[\rho_1(t)]_{r's'} = \sum_{r,s=1}^{2} A_{r's';rs}\,[\rho_1(0)]_{r's'}$ i.e.,

$$\frac{1}{3}\begin{pmatrix} 1 \\ C\,e^{-i\phi} - iS\,e^{-2i\phi} \\ C\,e^{i\phi} + iS\,e^{2i\phi} \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2}\,S\,a^* & C & 0 & \frac{1}{2}\,S\,a^* \\ \frac{1}{2}\,S\,a & 0 & C & \frac{1}{2}\,S\,a \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ e^{-i\phi} \\ e^{i\phi} \\ 2 \end{pmatrix}$$

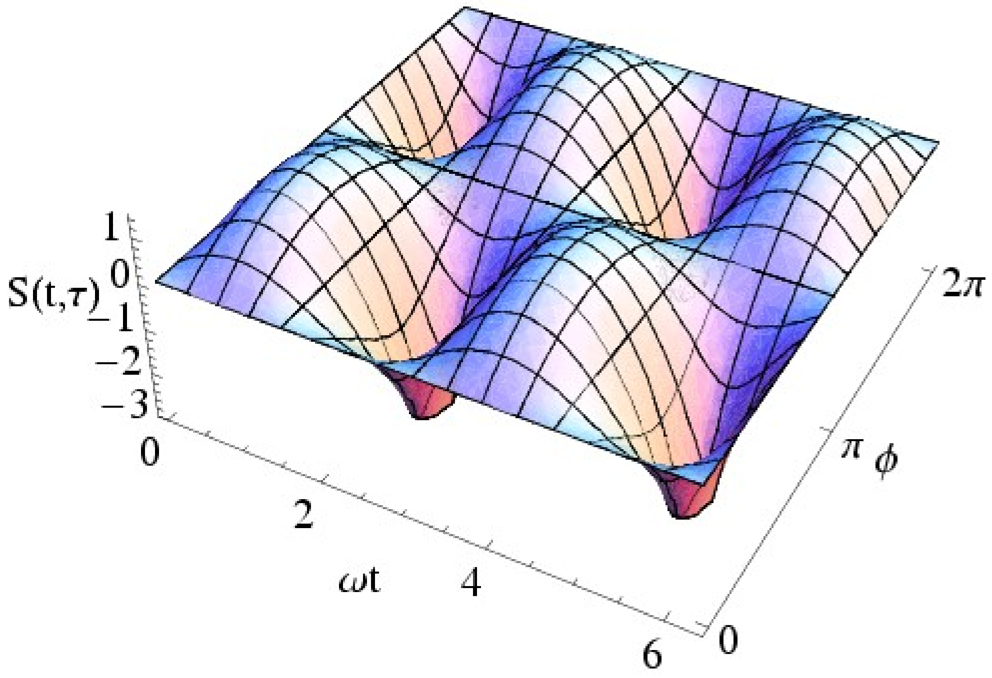
$$a_1 = -\langle\Psi|\sigma_{1y}\sigma_{2x}|\Psi\rangle = -\frac{2}{3}\sin(2\phi)$$

$$a_2 = \langle\Psi|\sigma_{1x}\sigma_{2x}|\Psi\rangle = \frac{2}{3}\cos(2\phi)$$

$\Rightarrow$ Initial state parameters

Illustrative examples: Example 1

A plot of Relative entropy difference as a function of ωt and ϕ.



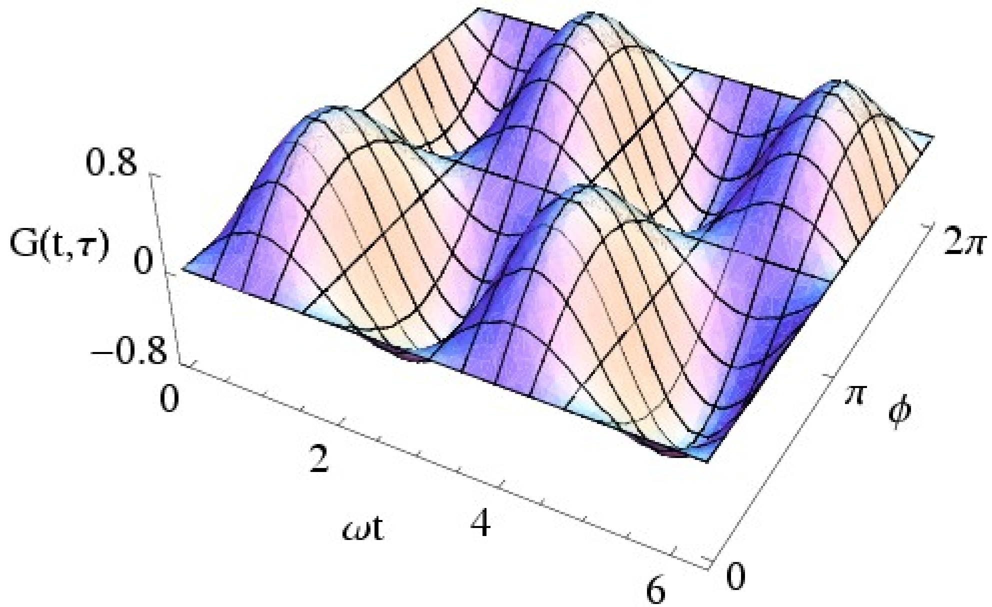
Negative values of

$S(t,\,\tau)$

indicate non-Markovianity in the evolution

Illustrative examples: Example 1

A plot of fidelity difference $G(t,\tau)$ as a function of ωt and ϕ.



Illustrative Examples: Example 2

Two-qubit Werner State:

$$\boxed{\rho_W(t=0) = \frac{x}{4} I_1 \otimes I_2 + (1-x)\,|\Psi_-\rangle\langle\Psi_-|;}$$

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}\left(|0_1, 1_2\rangle - |1_1, 0_2|\rangle\right)$$

Initial density matrix of the system (1st qubit): $\boxed{\rho_{W1}(0) = \frac{1}{2} I_1}$

State of the system after unitary evolution:

$$\begin{aligned}
\rho_{W1}(t) &= \mathrm{Tr}_2[U(t)\rho_W(0)U^\dagger(t)] \\
&= \frac{1}{2}\begin{pmatrix} 1 & -i\,(1-x)\,\sin(\omega t) \\ i\,(1-x)\,\sin(\omega t) & 1 \end{pmatrix},
\end{aligned}$$

Illustrative Examples: Example 2

We can obtain the evolved system state directly through

$$[\rho_{W1}(t)]_{r's'} = \sum_{r,s} A_{r's';rs}(t)\ [\rho_{W1}(0)]_{rs}$$

where the A-map has the following initial state parameters

$$a_1 = -\mathrm{Tr}[\rho_W(0)\,\sigma_{1y}\sigma_{2x}] = 0$$

$$a_2 = \mathrm{Tr}[\rho_W(0)\,\sigma_{1x}\sigma_{2x}] = (1-x)$$

Illustrative Examples: Example 2

$$S[\rho_{W1}(t)||\rho_{W1}(t+\tau)] = p_+(t)\,\ln\left[\frac{p_+(t)}{p_+(t+\tau)}\right] + p_-(t)\,\ln\left[\frac{p_-(t)}{p_-(t+\tau)}\right]$$

Relative entropy

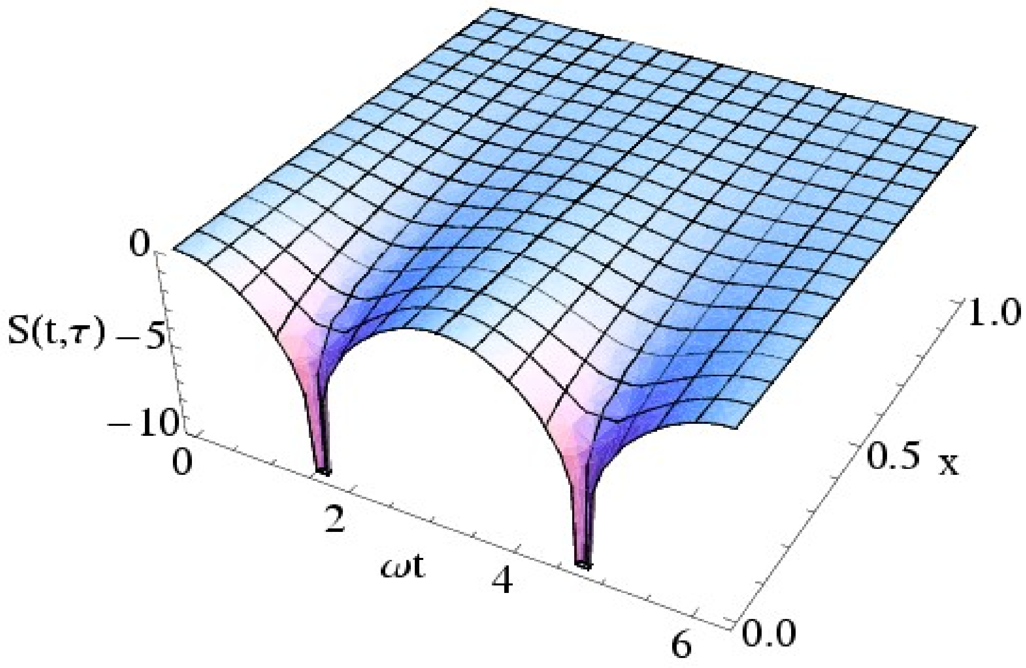
$$F[\rho_{W1}(t),\rho_{W1}(t+\tau)] = p_+(t)\,p_+(t+\tau) + p_-(t)\,p_-(t+\tau) + 2\sqrt{p_+(t)p_-(t)p_+(t+\tau)p_-(t+\tau)}$$

Fidelity

$$p_\pm(t) = \tfrac{1}{2}[1 \pm (1-x)\,\sin(\omega t)]$$

Illustrative examples: Example 2

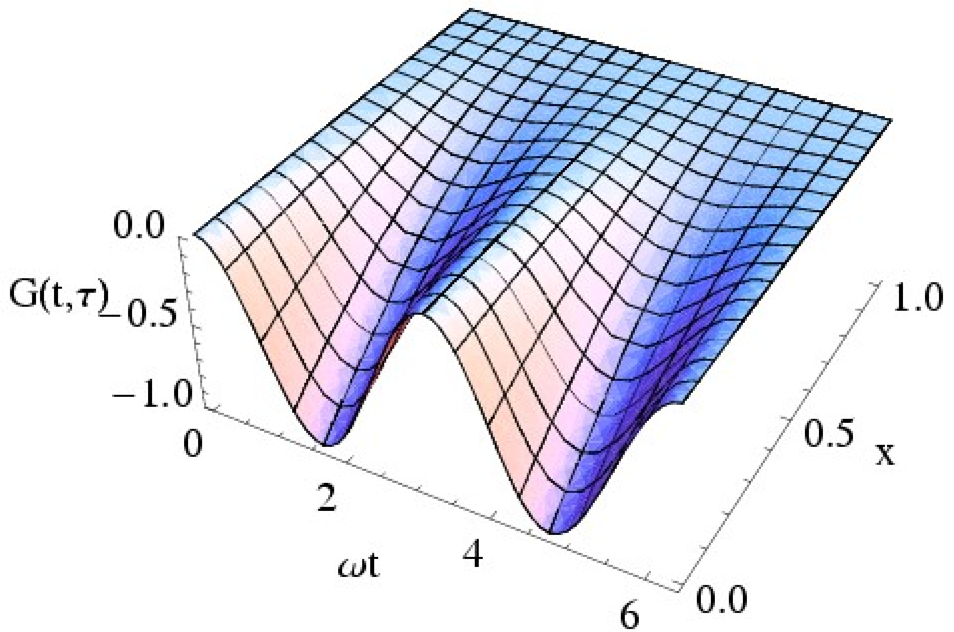
A plot of Relative entropy difference $S(t,\tau)$ as a function of ωt and the parameter x. Here, we have taken $\omega\tau = \pi$



Negative in almost the entire region except near x=1

Illustrative examples: Example 2

A plot of fidelity difference $G(t,\tau)$ as a function of ωt and the parameter x. Here too, $\omega\tau = \pi$.



Negative in the entire region except at x=1

Illustrative Examples: Example 3

Mixed separable state of two qubits:

$$\rho_S(t=0) = \tfrac{1}{4}\left(I \otimes I + s_x\,\sigma_{1x} + s_y\,\sigma_{1y} + s_z\,\sigma_{1z} + d\,\sigma_{1y}\,\sigma_{2x}\right)$$

$s_x,\ s_y,\ s_z$ and d are parameters of the composite state

Initial density matrix of the system (1st qubit):

$$\rho_{S1}(0) = \mathrm{Tr}_2[\rho_S(0)] = \tfrac{1}{2}\begin{pmatrix} 1+s_z & s_x - i s_y \\ s_x + i s_y & 1 - s_z \end{pmatrix}$$

Illustrative Examples: Example 3

State of the system after unitary evolution:

$$\rho_{S1}(t) = \mathrm{Tr}_2[U(t)\rho_S U^\dagger(t)] = \frac{1}{2}\begin{pmatrix} 1+s_z & s(t) \\ s^*(t) & 1-s_z \end{pmatrix}$$

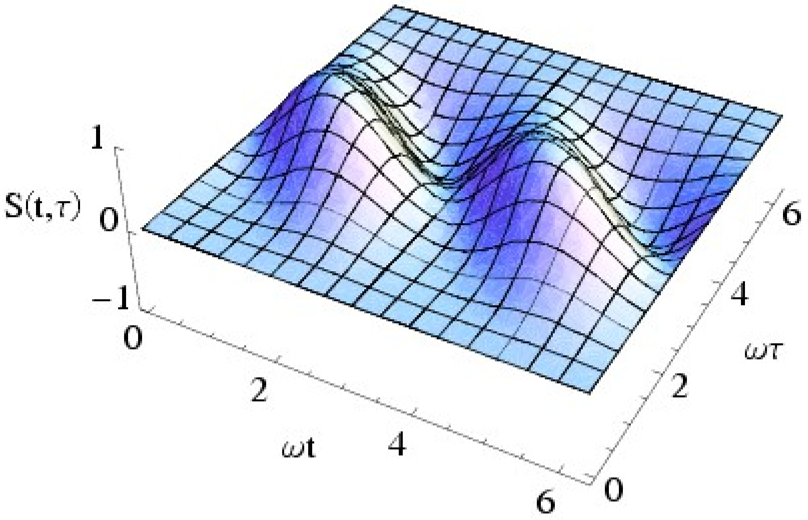
$$s(t) = (s_x - i s_y)\cos(\omega t) - d\,\sin(\omega\, t)$$

In the A-map, the initial parameters are given by

$$\begin{aligned} a_1 &= -\mathrm{Tr}[\rho_S(0)\,\sigma_1 y \sigma_{2x}] = -d \\ a_2 &= \mathrm{Tr}[\rho_S(0)\,\sigma_{1x}\sigma_{2x}] = 0 \end{aligned}$$

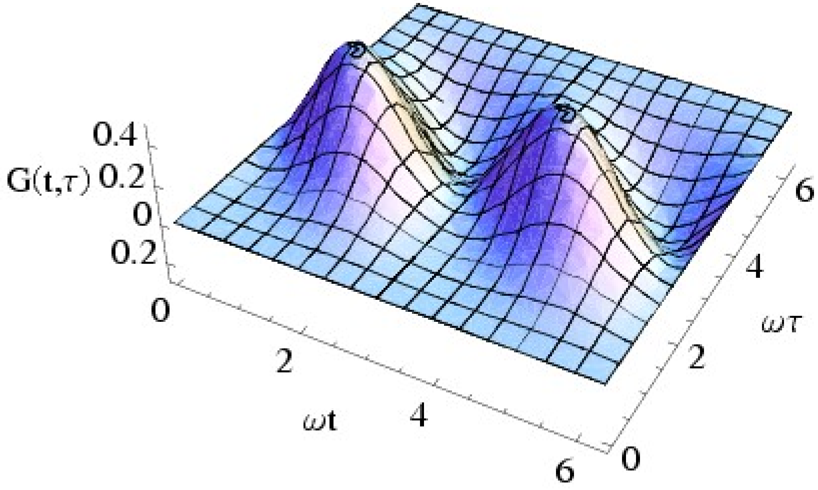
Illustrative examples: Example 3

A plot of Relative entropy difference $S(t,\tau)$ as a function of ωt and $\omega\tau$. We have chosen $s_x = s_y = s_z = d = \frac{1}{\sqrt{6}}$.



Illustrative examples: Example 3

A plot of fidelity difference $G(t,\tau)$ as a function of ωt and $\omega\tau$. Here too $s_x = s_y = s_z = d = \frac{1}{\sqrt{6}}$.



Quantitative measures of non-Markovianity

Questions such as how can one rigorously define quantum non-Markovianity and how can one quantify the degree of non-Markovian behaviour in a way which does not refer to any specific representation e.g., to a master equation with a given structure are being raised recently.

H.P. Breuer, **E.-M Laine**, **J.Pilio**, Phys.Rev.Lett **103**, 210401 (2009)

The basis idea behind the measure constructed by Breuer et.al is that during a non-Markovian process the distinguishability between any pair of initial states increases due to the back-flow of information from environment to the system.

An important consequence of this process is that the dynamical map of the non-Markovian processes must necessarily be non-divisible.

Quantitative measures of non-Markovianity

If $D\left(\rho_1(t), \rho_2(t)\right)$ denotes the trace distance between any two pairs of states $\rho_1(t)$, $\rho_2(t)$, and Λ denotes the dynamical map,

$$\mathcal{N}(\Lambda) = \max \int_{\sigma>0} \sigma\left(t,\, \rho_1(0),\, \rho_2(0)\right)\, dt$$

where $\sigma\left(t,\, \rho_1(0),\, \rho_2(0)\right) = \frac{d}{dt} D\left(\rho_1(t), \rho_2(t)\right)$ is the quantitative measure of non-Markovianity proposed by Breuer et.al. Here the maximization is over a pair of initial states $\rho_1(0)$, $\rho_2(0)$.

$$\boxed{D(\rho_1,\, \rho_2) = \tfrac{1}{2}\mathrm{tr}\left|\rho_1 - \rho_2\right| \text{ where } |\rho| = \sqrt{\rho^\dagger \rho}}$$

Quantitative measures of non-Markovianity

While the measure proposed by Breuer et.al involves optimization procedure, a measure not involving optimization is proposed by Angel Rivas et.al

A. Rivas, S.F.Huelga, M.B.Plenio, Phys.Rev.Lett **105,** 050403 **(2010)**

A knowledge of the dynamical map $\Lambda(t) \equiv \Lambda(t,0)$ is essential for the evaluation of measure.

Knowing $\Lambda(t) \equiv \Lambda(t,0)$ one can split the dynamical map, owing to the continuity of time as, $\Lambda(t+\epsilon,0) = \Lambda(t+\epsilon,t)\Lambda(t,0)$ for any instance of time between t and ϵ. If the map $\Lambda(t+\epsilon,\, t)$ is CP for any intermediate times $(t+\epsilon) \geq t_2 \geq t_1 \geq 0,$ then the dynamics is Markovian provided $\Lambda(t,0)$ is also CP. That is, iff there exist times t and ϵ such that $\Lambda(t+\epsilon,\, t)$ is NCP the dynamics is non-Markovian.

A.Jamiolkowski, Rep. Math.Phys. **3,** 275 (1972);

M.D. Choi, Linear Algebra Appl**. 10, 285** (1975)

Quantitative measures of non-Markovianity

According to Choi-Jamiolkowski isomorphism, $\Lambda(t+\epsilon,\, t)$ is CP iff $(\Lambda(t+\epsilon,\, t)\otimes Id)|\Psi\rangle\langle\Psi| \geq 0$. Thus, a measure of the non-CP character of $\Lambda(t+\epsilon,\, t)$ is given by

$$f_{\mathrm{NCP}}(t+\epsilon, t) = ||(\Lambda(t+\epsilon,\, t)\otimes Id)(|\Psi\rangle\langle\Psi|)||.$$

In fact, $\Lambda(t+\epsilon,\, t)$ is CP iff $f_{\mathrm{NCP}}(t+\epsilon, t) = 1$ and otherwise $f_{\mathrm{NCP}}(t+\epsilon, t) > 1$. Rivas et.al define their measure of non-Markovianity on the basis of $f_{\mathrm{NCP}}(t+\epsilon, t)$: Defining

$$g(t) = \lim_{\epsilon\to 0}\frac{f_{\mathrm{NCP}}(t+\epsilon, t)-1}{\epsilon}$$

where $g(t)\geq 0$ iff $\Lambda(t+\epsilon,\, t)$ is CP, the integral $\mathcal{I} = \int_0^\infty g(t)dt$ is taken as the measure of non-Markoviantiy

Concluding remarks

While the theory of open system dynamics was formulated nearly five decades ago—with the introduction of dynamical A and B maps—by Sudarshan et.al., several interesting questions are being raised recently. Conceptual understanding of positive-but Not Completely Positive maps has gained increasing attention.

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H. A. Carteret, **D. R. Terno**, and **K. Zyczkowski**, Physical Review A **77**, 042113 (2008)
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Concluding remarks

- In this work, we have developed a canonical structure for the A-map and shown that this offers an elegant approach to investigate the CP/NCP dynamics.
- We have proposed a test of non-Markovianity based on the relative entropy of the dynamical state of the system
- Choosing a specific NCP dynamical map on initially correlated 2-qubit states we have identified the non-Markov features in three different examples.

Concluding remarks

Identification of features that can distinguish whether the initial correlations have caused the loss of memory (non-Markov) effects or whether the dynamics (CP/NCP) is responsible, is an interesting issue that needs to explored.

Our efforts in understanding open system dynamics continues. . .

THANK YOU ALL

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