Monogamy of Bell Inequality violations

Ravishankar Ramanathan

P. Kurzynski, T. Paterek, W. Laskowski, and

D. Kaszlikowski; PRL 106, 180402 (2011)

Motivation

- Bell inequality violation gives quantum advantage:
  - Device independent Cryptography,
  - Communication Complexity,
  - Randomness amplification, etc.
- Multiple inequalities - Quantum correlations are 'monogamous'!
- Causes of monogamy: no-signaling, complementarity.
- Implications:
  - Security of key distribution,
  - Emergence of macroscopic local realism,
  - Properties of condensed matter.

Outline

- Qubit Bell inequalities - formalism
- Correlation complementarity.
- Derivation of Tsirelson bounds, Bell monogamies from
  - Correlation complementarity,
  - No-signaling.
- Bipartite and multipartite scenarios – graph formalism.
- Conclusions and Outlook

Qubit Bell inequalities

- Complete two-setting correlation inequalities for N qubits – Zukowski, Brukner (PRL (2002))
- Sufficient condition for existence of a local hidden variable model for N qubit correlations:

$$\sum_{x_1,\ldots,x_N=1}^{2} T^2_{x_1\ldots x_N} \leq 1,$$

- Quantum value has upper bound: $\mathbf{L}^2 \leq \sum_{x_1,\ldots,x_N=1,2} \mathbf{T}^2_{x_1,\ldots,x_N}$, where $T_{\mu_1\ldots\mu_N} = \mathrm{Tr}[\rho(\sigma_{\mu_1}\otimes\cdots\otimes\sigma_{\mu_N})]$
- $x_i = 1,2$: orthogonal local directions, sum and difference of vectors parametrising local settings.
- **$L \leq 1$: LHV**
- **$L \leq 2^{(N-1)/2}$: QM**
- **$L \leq 2^{N/2}$: NS**
- **Note:**

$$\rho = \frac{1}{2^N} \sum_{x_1,\ldots,x_N=0}^{3} T_{x_1\ldots x_N}\, \sigma^1_{x_1} \otimes \ldots \otimes \sigma^N_{x_N},$$

Correlation Complementarity

- Complementarity: If expectation value of one measurement is ± 1, then that for complementary measurement is zero.
- Operators corresponding to dichotomic complementary observables anti-commute.
- Proof: For two dichotomic complementary observables A and B

$$\langle A\rangle = 1 \Rightarrow A|a\rangle = |a\rangle \Rightarrow \langle a|B|a\rangle = 0 \Rightarrow B|a\rangle = |a_\perp\rangle$$
$$B^2 = \mathbf{1} \Rightarrow B|a_\perp\rangle = |a\rangle \Rightarrow |b\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |a_\perp\rangle)$$
$$\langle b|A|b\rangle = 0 \Rightarrow A|a_\perp\rangle = -1|a_\perp\rangle$$

- Argument applies to all +1 eigenstates , the two eigenspaces have equal dimension.

$$A = \sum_a (|a\rangle\langle a| - |a_\perp\rangle\langle a_\perp|)$$
$$B = \sum_a (|a_\perp\rangle\langle a| + |a\rangle\langle a_\perp|)$$
$$\{A, B\} = 0$$

Correlation Complementarity

- Lemma: Consider a set of traceless and trace orthogonal dichotomic Hermitian operators A_k that obey $\{A_k, A_j\} = 2\,\delta_{kj}$. Their expectation values for any state ρ obey $\Sigma\langle A_k\rangle^2 \leq 1$.
- Proof:

$$A \equiv \sum_k \langle A_k \rangle A_k. \qquad \langle A^2 \rangle - \langle A \rangle^2 \geq 0.$$

$$\langle A^2 \rangle = \sum_k \langle A_k \rangle^2. \qquad \langle A \rangle^2 = \langle A^2 \rangle^2.$$

$$\sum_k \langle A_k \rangle^2 \leq 1.$$

- Tight: There exists a state having these nos. as expectation values for anti-commuting observables - Wehner, Winter (J. Math. Phys. (2008)).
- Method: Find quantum bounds for Bell violations using Correlation Complementarity. Identify sets of anti-commuting operators for Bell parameters, bound from Lemma.
- Remember: $\mathbf{L}^2 \leq \sum_{x_1,\ldots,x_N=1,2} \mathbf{T}^2_{x_1,\ldots,x_N}$

Tsirelson bounds

- Application: CHSH Tsirelson bound.
- For two qubits and two setting inequalities, single Bell parameter is upper bounded $\mathbf{L}^2 \leq \mathbf{T_{xx}}^2 + \mathbf{T_{xy}}^2 + \mathbf{T_{yx}}^2 + \mathbf{T_{yy}}^2$.

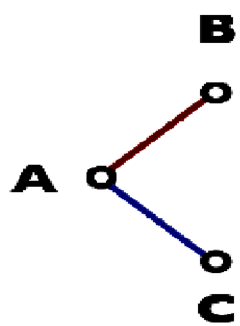
$$\mathcal{L}^2 \equiv \frac{1}{4} S^2 \leq T_{xx}^2 + T_{xy}^2 + T_{yx}^2 + T_{yy}^2$$

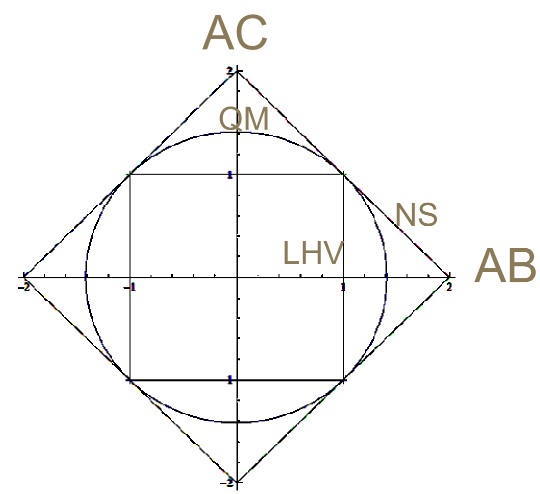
$$\leq 1 + 1$$

- Identifying two sets of anti-commuting observables (T_{xx}, T_{xy}) and (T_{yx}, T_{yy}) $\mathbf{L} \leq \sqrt{\mathbf{2}}$, Tsirelson bound.
- Get Tsirelson bounds of multi-setting inequalities: Laskowski et al. (PRL, (2004)) and many-qubit two-setting inequalities.
- Tsirelson bound born out of complementarity! ☺

Bipartite Bell monogamies

- Three qubits A, B, C. If AB violate two-qubit BI, then correlations AC admit LHV description.

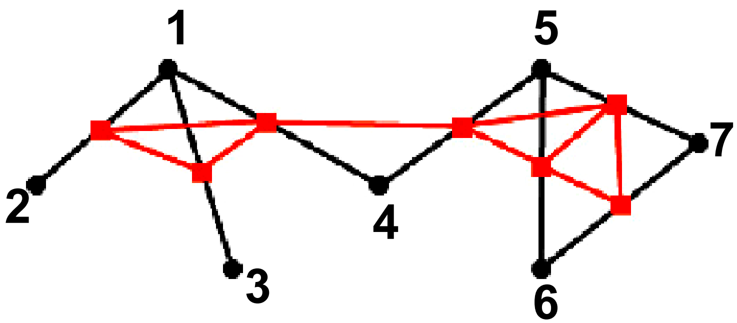




- Vertices represent observers violating Bell inequalities which are represented by edges.
- The upper bound reads $\mathbf{L^2_{AB} + L^2_{AC} \leq \Sigma_{k,l=x,y} T^2_{kl0} + \Sigma_{k,m=x,y} T^2_{k0m}}$. Settings of A are same in both inequalities.
- Identifying two sets of mutually anti-commuting operators: {XX0, XY0, Y0X, Y0Y} and {YX0, YY0, X0X, X0Y}; X = σ_x and Y = σ_y gives $\mathbf{L^2_{AB} + L^2_{AC} \leq 2 \rightarrow B^2_{AB} + B^2_{AC} \leq 8!}$

Bipartite B.I.'s – complete monogamies

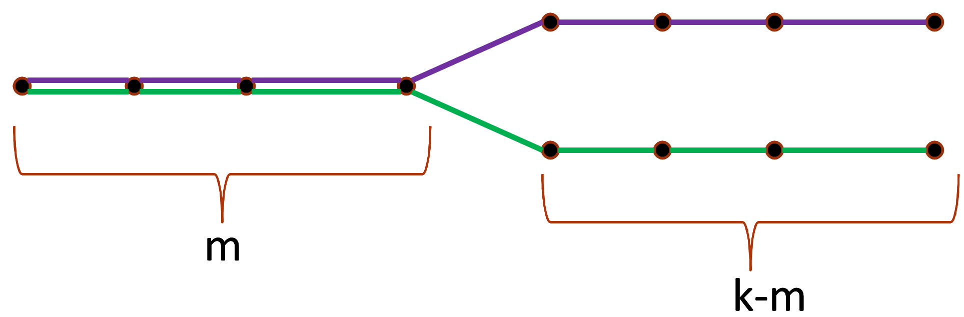
- Consider N qubits trying to violate a set of bipartite B.I.s - Graph G (Black) with observers as vertices, inequalities as edges.



- Method: For arbitrary graph G, construct its line graph L(G) (Red) placing vertices of L on every edge of G & connecting vertices of L(G) whenever the corresponding edges of G share a vertex.
- Every edge of L(G) is an elementary monogamy: $L^2_{AB} + L^2_{AC} \leq 2$.
- $\boldsymbol{\sum_v d_v L^2_v \leq 2\, \varepsilon}$, where d_v :degree of vertex v and ε: number of edges in L(G).
- Inequality is tight for arbitrary graph of bipartite inequalities! ☺

Multipartite EMRs

- Two-qubits: single EMR. Multiple qubits: k-1 EMR's generate monogamies via the line graph.
- All EMRs are tight independent of the number of common observers: $\mathbf{L^2_1 + L^2_2 \leq 2^{k-1}}$.

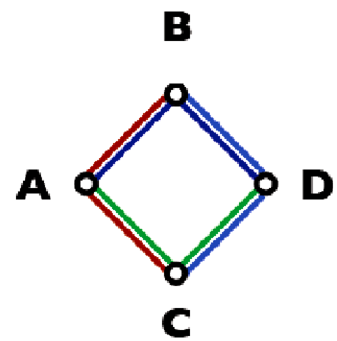


- Monogamies for arbitrary graphs of k-qubit inequalities constructed via line graph.
- Condition for tightness: line graph of the multipartite graph must be bipartite!! ☺

Multipartite Polygamy – Complete hypergraph

- Consider parties A, B, C, D trying to violate a correlation Bell inequality in the graph shown.

$$\mathbf{L^2_{ABC} + L^2_{BCD} + L^2_{CDA} + L^2_{DAB} \leq 4.}$$

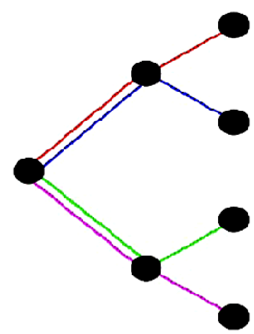


- Mermin monogamy: $\mathbf{M^2_{ABC} + M^2_{BCD} + M^2_{CDA} + M^2_{DAB} \leq 16}$.
- Possibilities: two and three triples violate Mermin inequality non-maximally, for example:
  - The state ½ (|0001> + |0010> + i √2 |1111>) allows ABC and ABD to obtain $\mathbf{M} = \mathbf{2\sqrt{2}}$.
  - The state 1/√6 (|0001> + |0010> + |0100> + i √3|1111>) allows ABC, ABD and ACD to obtain $\mathbf{M} = \mathbf{4/\sqrt{3}}$.
  - All four inequalities cannot be simultaneously violated.
- Tightness for three-qubit inequalities in complete graphs of arbitrary number of qubits! ☺

Multipartite Polygamy – Tree hypergraph

- The 2^{k-1} inequalities obey $\mathbf{L^2_1 + \ldots + L^2_{2^{k-1}} \leq 2^{k-1}}$ for arbitrary k.

k = 3



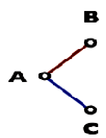
- All patterns of violation except simultaneous violation of all.
- For any $m < 2^{k-1}$ of Bell inequalities, the state shows spherical tightness:

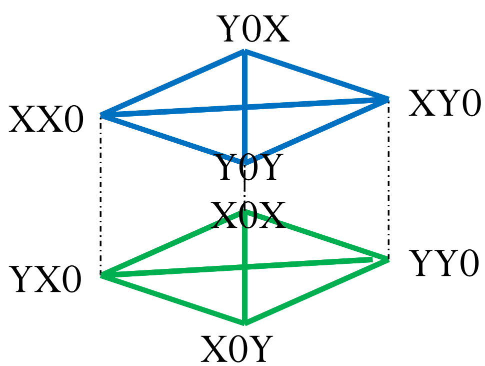
$$|\psi_n\rangle = \frac{1}{\sqrt{2}}|\underbrace{0\ldots0}_{n}\rangle + \frac{1}{\sqrt{2m}}\sum_{j=1}^{m}|0\ldots0\underbrace{1\ldots1}_{P_j}0\ldots0\rangle,$$

- $\mathbf{L_j^2 = 2^{k-1}/m}$ for each Bell inequality $j = 1, \ldots, m$ - remaining Bell parameters vanish.
- Maximal violation in a branch of an “arbitrary graph” ⟶ no violation in any connected branch! ☺

Practical matters

- Construct the operator graph H(G) from G: vertices correspond to the operators, edges connect anti-commuting vertices.
- Clique partitioning: partition the operator graph into sets of vertices that are fully connected.
- All spherically tight monogamy relations for given k correspond to a single operator graph!
- The operator graph for the bipartite case AB vs. AC:

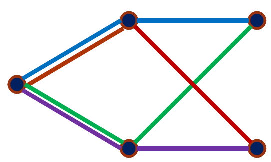




- Compare with checking positivity of quantum states under different values of Bell parameters.

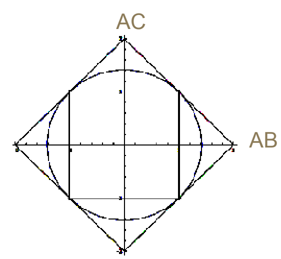
No-signaling monogamy

- Method: Decomposition of the graph G (J B.I.'s of N qudits each) into J graphs each corresponding to a single B.I. – (Pawlowski, Brukner PRL (2009)).
- One measurement setting per qudit in each of the J graphs $\longrightarrow$ joint probability distribution.
- Result: For the graph with each qudit involved in as many BI's as settings, violation of the inequality implies signaling.



$$\sum_{p_1,p_2,\ldots,p_N} B(\vec{A}^{(1)}_{p_1}\vec{A}^{(2)}_{p_2}\ldots\vec{A}^{N}_{p_N}) \leq n^{N-1}R$$

- Linear monogamies from no-signaling bound the spherical monogamies in quantum theory !



Conclusions & Outlook

- Take-home: Correlation complementarity implies tight bounds on violation of single and multiple Bell inequalities in quantum theory.

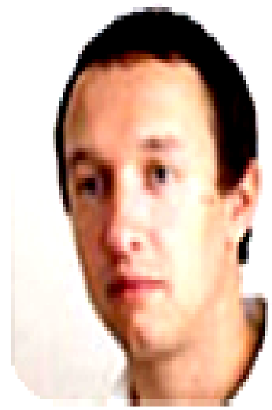
- Quadratic monogamies tighter than linear No-signaling monogamies.

- Possible applications: secure communication in tree networks, properties of condensed matter systems, etc.

- Possible extensions
  - Derivation of Entanglement monogamy.
  - Sub-determinants of density matrix to derive complementarities.
  - More measurement settings
  - Qudit inequalities

Thank You !

P. KURZYŃSKI



T. PATEREK

W. LASKOWSKI

D. KASZLIKOWSKI