On Discrimination of Quantum Operations

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Outline:

- Kraus Representation of General Quantum Operations
- Concept of Discrimination of Quantum operators
- Some discrimination schemes

Quantum Operations

- A quantum operation ξ:T(H) $\rightarrow$ T(H') is a linear trace- nonincreasing, completely positive map represented as

$$\xi(\rho) = \sum_k E_k \rho E_k^{+}$$

where the set $\{E_k\}$ satisfies

$$\sum_k E_k^{+} E_k \leq I_H$$

Quantum Channel

- Probability of occurrence of the Quantum Operator

$$\xi : \rho \text{ ð } \rho' = \frac{\xi(\rho)}{Tr[\xi(\rho)]}$$

- is given by

$$Tr[\xi(\rho)] = Tr\left[\left(\sum_n E_n^+ E_n\right)\rho\right]$$

- A quantum channel is a trace preserving map ξ, whose probability of occurrence is unit .

- There exists an unitary operator U (not unique), acting on some larger space formed by system and environment, corresponding to every quantum operation.

$$\xi(\rho) = Tr_{Env}\left[U(\rho \otimes |e_0\rangle\langle e_0|)U^+\right]$$

- The initial state of the environment can chosen without any loss of generality to be a pure state.

Realization

- A purification scheme for achieving the unitary operator corresponding to a quantum operation ξ:T(H) → T(H’) is given in the Heisenberg picture.
- A quantum operation ξ:T(H) → T(H’) has Kraus form

$$\xi(\rho)=\sum_k E_k \rho E_k^{+} \quad \text{with} \quad Tr[E_i^+ E_j]=Tr[E_i^+ E_i]\delta_{ij}$$

Then all unitary dilation for this operation satisfy the majorization relation

$$\left(Tr\left[A^+ A\right]\right) \quad \left(Tr[E_i^+ E_i]\right) \text{ where } \mathrm{A}=\langle \sigma_{\mathrm{i}} | U | \phi \rangle$$

$\left\{\left|\sigma_{\mathrm{i}}\right\rangle\right\}$ forms an orthonormal basis for Range(Σ), Σ is a nonvanishing projector on a subspace of ancillary system.

Discrimination of Quantum Operations

- Discrimination of quantum operations performed through discriminating the action of them on some quantum state.
- Discrimination of quantum operations is not equivalent with discrimination of quantum states.

Non-orthogonal quantum states are not perfectly distinguishable with finite no of copies of the states, while two unitary operators can be perfectly distinguished with finite no of copies.

- *A. Acin, PRL* **87**, 177901, 2001.

Scheme

- Two unitary operators U_1 , U_2 are perfectly distinguishable with a finite no of copies, say N, if there exists a state $|\Psi\rangle$ such that the states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are orthogonal, where

$$|\Psi_i\rangle = \left(U_i^{\otimes N} \otimes I\right)|\Psi\rangle$$

- It is an Entanglement-assisted scheme, but does entanglement is always necessary?

- No, the discrimination may be processed by a separable input state also.
- R. Y. Duan, Y. Feng and M.S. Ying, *Phys. Rev. Lett.* **98**, 100503, 2007**.**

Discrimination of general quantum operations

Two schemes have been proposed.

- Minimum error discrimination
- The process terminate with a define result that may be incorrect and probability of obtaining an erroneous result is minimized

*M.F. Sacchi, Phys. Rev. A **71**, 062340 (2005)*

- Unambiguous discrimination
- The process fails for a non-zero probability and otherwise the result obtained is correct

*G. Wang and M. Ying, Phys. Rev. A, **73**, 042301 (2006).*

Minimum error discrimination

- The minimum error discrimination of two quantum operations ξ_1 , ξ_2 is the process of finding a suitable state ρ in Hilbert space H such that the error probability of discriminating the output states $\xi_1(\rho)$, $\xi_2(\rho)$ is minimum.

Minimum error discrimination with use of entanglement

- By choosing the input to be a bipartite state ρ of $H \otimes K$, the minimum error discrimination of two quantum operations ξ_1 , ξ_2 is done by discriminating the output states $(\xi_1 \otimes I_K)\rho$, $(\xi_2 \otimes I_K)\, \rho$.
- Does entanglement scheme always improve the probability of discrimination?

- For any two Pauli channels

$$\xi(\rho) = \sum_{i=0}^{3} q_i \, \sigma_i \rho \sigma_i \ ; \quad \sum_{i=0}^{3} q_i = 1$$

  the minimum error probability of unambiguous discrimination can be achieved by non-entanglement strategy also.

- For discriminating the Depolarizing channel and identity map, the use of entanglement necessarily improves the minimum error probability.

Mini-max discrimination

- A process of optimal discrimination of a given set of quantum operations is by maximizing the smallest of the probabilities of correct identification of the channel.
- *M.F. Sacchi, Phys. Rev. A **71**, 062340 (2005)*
- *G. M. D'Ariano, M.F. Sacchi and J. Kahn, Phys. Rev. A **72**, 052302 (2005)*

Unambiguous discrimination

Support of a quantum operation $\xi(\rho) = \sum_k E_k \rho E_k^+$ is given by span$\{E_k\}$ of bounded operators in its Kraus form.

As each set $\{E_k\}$ chosen, is unitarily connected with another, so support is independent of a specific choice of $\{E_k\}$.

The condition for unambiguous discrimination of a finite number of quantum operations $\{\xi_1, \xi_2, \ldots, \xi_n\}$ is

$$\mathrm{supp}(\xi_i) \not\subseteq \sum_{k=1}^{n} \mathrm{supp}(\xi_k) \qquad \text{for each } i=1,2,\ldots,n$$

Condition for perfect distinguishability

- Dual et.al provide a scheme for perfectly distinguishability of two quantum operations

$$\xi^{i}(\rho) = \sum_{k=1}^{n_i} E^{i}{}_{k} \rho E^{i}{}_{k}{}^{+} \ , \quad \mathrm{i} = 1{,}2$$

- In that scheme the operators are perfectly distinguishable, iff
- The operators are disjoint
- $I_d \notin span\left\{ E_i^{1^+} E_j^{2} \right\}$

- The scheme thus cannot perfectly discriminate a minimum number of two arbitrary quantum operation acting on same system.

Class of Pauli Channels

- A class of operators acting on single-qubit system defined as

$$\xi(\rho) = \sum_{i=0}^{3} q_i \, \sigma_i \rho \sigma_i \quad ; \quad \sum_{i=0}^{3} q_i = 1$$

- Then there exists unitary operator

$$U = \sum_{i=0}^{3} \sqrt{q_i} \, \sigma_i \otimes |e_i\rangle\langle e_0|$$

such that

$$\xi(\rho) = Tr_E\left[U\left(\rho_S \otimes |e_0\rangle_E\langle e_0|\right)U^+\right]$$

We may choose environment as 2 qubit system with basis $\{|e_0\rangle = |00\rangle, |e_1\rangle = |01\rangle, |e_2\rangle = |10\rangle, |e_3\rangle = |11\rangle\}$

Discrimination

- Though any two operators of this class can not be perfectly discriminated with finite no of copies, the unitary operators corresponding to this two operators, acting on a larger system are one-copy perfectly distinguishable.

- Consider two Pauli operators

$$\xi_k(\rho) = \sum_{i=0}^{3} q_i^{(k)} \sigma_i \rho \sigma_i$$

with

$$U_k = \sum_{i=0}^{3} \sqrt{q_i^{(k)}} \sigma_i \otimes |e_i\rangle\langle e_0|$$

- Then U1 , U2 can be perfectly distinguishable by any state of the form

$$|\Psi\rangle = \sum_{\substack{i=0,1 \\ j=1,2,3}} p_{ij} |i\rangle |e_j\rangle \quad \text{as} \quad \langle \Psi | U_1^{+} U_2 | \Psi \rangle = 0$$

- We may choose a product input state $|\Psi\rangle = |0\rangle|01\rangle$

Generalized Pauli Operations

- Class of operators

$$\xi(\rho) = \sum_{n=0}^{d^2-1} q_n \, \mathrm{U}_n \rho U_n^{+} \quad ; \quad \sum_{n=0}^{d^2-1} q_n = 1$$

- The 2nd Kraus representation of this operator is

$$\mathrm{V} = \sum_{n=0}^{d^2-1} \sqrt{q_n} \, \mathrm{U}_\mathrm{n} \otimes |e_n\rangle\langle e_0|$$

where environment can be a d² dimensional system with basis $\{|00\rangle, |01\rangle, \cdots |d-1, d-1\rangle\}$

Discrimination

- The unitary operators $\mathrm{V_k} = \sum_{n=0}^{d^2-1} \sqrt{q_n^{(k)}}\, \mathrm{U_n} \otimes |e_n\rangle\langle e_0|$

corresponding to two General Pauli operators

$$\xi_k(\rho) = \sum_{n=0}^{d^2-1} q_n^{(\mathrm{k})}\, \mathrm{U}_n \rho U_n^{+} \quad ; \quad \mathrm{k} = 1,2$$

can be perfectly discriminated with a single copy by a product state $|\Psi\rangle = |\phi\rangle \otimes |01\rangle$

For discriminating two general quantum operations we consider both of them to be acting on same system. Adding a pure product state as ancilla (environment), if we consider the evolution to be an unitary operator, then we may proceed to discriminate the given operation by discriminating the orthogonal output states obtain by acting the unitary operators on a product input state, the ancillary part of the state is chosen to be orthogonal to the initial state of ancilla system(in preparing the unitary operator).

Thank you