Dynamics of Quantum Correlations in Open Quantum Systems:

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Overview of Talk

- We talk about some measures of quantum correlations, such as, entanglement and discord.
- A brief introduction to Open Quantum Systems is made with relevance to applications to Quantum Information.
- We discuss recent work on studies of these correlations in two-qubit open quantum systems: in which entanglement is generated dynamically by the systems interaction with the environment which we take to be both pure dephasing as well as dissipative.

Quantum Correlations

Entanglement: A Brief Preview and Motivation

- What is entanglement and what is its use?
- Separability versus entanglement: that which is not separable is entangled.
- A pure state is separable if it can be expressed as a tensor product of subsystem states: $|\psi\rangle = |a\rangle \otimes |b\rangle$.
- Examples for pure states:
  (a). separable states: $|00\rangle$, $|11\rangle$
  (b). entangled states: $|\Phi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$; $|\Psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$: Bell states.

Entanglement: A Brief continued...

- A mixed state is separable if it can be represented as a mixture of product states: $\rho = \sum_i p_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|$. Correlations between different subsystems due to incomplete knowledge of quantum states completely characterized by classical probabilities p_i.
- Examples for mixed states:
(a). separable state: $\rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$
(b). entangled state: $\rho_W = (1-p)\frac{1}{4}I + p|\Phi_+\rangle\langle\Phi_+|$, where $1/3 < p \leq 1$: Werner state.

Entanglement: A Brief continued...

- Entanglement can be used to perform tasks not possible classically. E.g.: Using entanglement it is possible to teleport a qubit in state $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle$ using a shared entangled state $|\Phi_+\rangle$.
- Thus entanglement is a resource in quantum communication and information.

Quantifying Entanglement: abstract approach

- A state function can be used to quantify entanglement, if it satisfies some natural properties: entanglement of two systems cannot be increased without, direct or indirect, quantum interaction between them: (Bennett et al. (1997)).
- If the systems are spatially separated, then entanglement cannot increase if only classical communication is allowed between them.
- This is expressed as monotonicity under local operations and classical communication (LOCC) (G. Vidal: (2000)).

Quantifying Entanglement: operational approach

- Entanglement is related to operational tasks... system more entangled if it allows for better performance of certain tasks (impossible without entanglement).
- Example: teleportation... by use of a single pair of two qubits in state $|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and classical communication, a qubit can be transmitted. Impossible by using only classical communication.

Concurrence

- For a pair of qubits there exists a general formula for the entanglement of formation: E_f: based on the quantity "CONCURRENCE". (W. K. Wootters: (1998))
- Consider pure state $|\Phi\rangle$ of a pair of qubits. Concurrence

$$C(\Phi) = |\langle\Phi|\tilde{\Phi}\rangle|$$

, where $|\tilde{\Phi}\rangle = (\sigma_y \otimes \sigma_y)|\Phi^*\rangle$, σ_y is the Pauli operator, $|\Phi^*\rangle$ is the complex conjugate of $|\Phi\rangle$.

- Spin flip operation, via σ_y, when applied to a pure product state, takes the state of each qubit to the orthogonal state, i.e., state diametrically opposite on the Bloch sphere resulting in zero concurrence. A completely entangled state is left invariant by a spin flip, resulting in C taking the maximum value 1.
- Relation between entanglement and concurrence of a pure state is:
$E(\Phi) = \mathcal{E}(C(\Phi))$, where $\mathcal{E}(C) = h\left(\frac{1+\sqrt{1-C^2}}{2}\right)$,
$h(x) = -x\log_2(x) - (1-x)\log_2(1-x)$.

Concurrence continued...

- $\mathcal{E}(C)$ is monotonically increasing for $0 \leq C \leq 1$ implying that concurrence can be regarded as a measure of entanglement in its own right.
- Concurrence of a mixed state of two qubits is defined as the average concurrence of an ensemble of pure states representing ρ, minimised over all decompositions of ρ:
  $C(\rho) = inf \sum_j p_j C(\Phi_j)$.
- $\mathcal{E}(C)$, in addition to being monotonically increasing, is also convex implying

$$\mathcal{E}(C(\rho)) = inf\ \mathcal{E}\left(\sum_j p_j C(\Phi_j)\right) \leq inf \sum_j p_j \mathcal{E}(C(\Phi_j)) = E_f(\rho).$$

- Thus $\mathcal{E}(C(\rho))$ is a lower bound on $E_f(\rho)$.

Concurrence continued...

- There always exists a decomposition of ρ that achieves the minimization required, for $C(\rho)$, with a set of pure states having the same concurrence. This makes $\mathcal{E}(C(\rho)) = E_f(\rho)$.
- An explicit formula for concurrence: (W. K. Wootters: (1998))
  $C(\rho) = max\Big\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\Big\}$,
  where λ_i are the square roots of the eigenvalues of $\rho\tilde{\rho}$ in descending order and $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$.

Quantum Correlations

Discord

- Correlation between two random variables X and Y is: `Mutual Information' $J(X:Y) = H(X) - H(X|Y)$.
- Here $H(X|Y)$ is the conditional entropy of X given that Y has already occured and $H(X)$ is the Shannon entropy of the random variable X.
- $H(X|Y) = H(X,Y) - H(Y)$: an alternative expression for mutual information $I(X:Y) = H(X) + H(Y) - H(X,Y)$.
- Classically: no ambiguity between these two expressions of mutual information and they are same.

Discord continued...

- Situation different in quantum regime (H. Ollivier and W. H. Zurek:(2001); L. Henderson and V. Vedral:(2001); S. Luo: (2008)).
- Consider a bipartite state ρ_{XY}: where ρ_X and ρ_Y are the states of the individual subsystems.
- Shannon entropies $H(X), H(Y)$ are replaced by von-Neumann entropies (e.g: $H(X) = S(\rho_X) = -Tr_X \rho_X Log(\rho_X)$).
- Conditional entropy $S(X|Y)$ requires a specification of the state of X given the state of Y.
- Such a statement in quantum theory is ambiguous until the to-be-measured set of states of Y are selected.
- Focus on perfect measurements of Y defined by a set of one dimensional projectors $\{\pi_j^Y\}$. The subscript j is used for indexing different outcomes of this measurement.

Discord continued...

- The state of X, after the measurement is given by

(1) $$\rho_{X|\pi_j^Y} = \frac{\pi_j^Y \rho_{XY} \pi_j^Y}{Tr(\pi_j^Y \rho_{XY})},$$

with probability $p_j = Tr(\pi_j^Y \rho_{XY})$.

- $S(\rho_{X|\pi_j^Y})$ is the von-Neumann entropy of the system in the state ρ_X, given that projective measurement is carried out on system Y.
- The entropies $S(\rho_{X|\pi_j^Y})$ weighted by the probabilities p_j, yield the conditional entropy of X, given the complete set of measurements $\{\pi_j^Y\}$ on Y: $S(X|\{\pi_j^Y\}) = \sum_j p_j H(\rho_{X|\pi_j^Y})$.
- The quantum analogue of $J(X:Y)$ is thus

(2) $$J(X:Y) = S(X) - S(X|\{\pi_j^Y\}),$$

where a supremum is taken over all $\{\pi_j^Y\}$.

Discord continued...

- $I(X:Y)$ is similar to its classical counterpart

(3) $$I(X:Y) = S(X) + S(Y) - S(X,Y).$$

- It is clearly evident that these two expressions are not identical in quantum theory. Quantum discord is the difference between these two generalizations of classical mutual information,

(4) $$D(X:Y) = I(X:Y) - J(X:Y).$$

- Quantum discord aims to quantify the amount of quantum correlation that remains in the system and also points out that classicality and separability are not synonymous. In other words, it actually reveals the quantum advantage over the classical correlation.

Quantum Correlations

Bell's Inequality

- Bell's inequality: one of the first tools used to detect entanglement. (J. Bell: (1965;1971))
- Consider a bipartite system of two qubits where Alice and Bob share a particle, each supplied and initially prepared by another party, say, Charlie. Each of them are allowed to perform measurements on their respective particle. Once Alice recieves her particle she performs a measurement on it.
- Alice is provided with two sets of measurement operators and she could choose to do one of the two measurements: say P_{M1} and P_{M2}, respectively.
- Since Alice does not know in advance which measurement to apply, she adopts a random method to make her decision. Assume that each of these measurements can have two possible values $\{+1, -1\}$. Let $M1$ and $M2$ be the values revealed by the two measurements P_{M1} and P_{M2}. Similarly, Bobs measurements are labelled by P_{M3} and P_{M4}. Each of these $M1$, $M2$, $M3$ and $M4$ can have the values $\{+1, -1\}$.
- Bob does not decide in advance which measurement he will carry out and waits until he has recieved the particle from Charlie and then chooses randomly.

Bell's Inequality continued...

- The Clauser-Horne-Shimony-Holt inequality (J. F. Clauser and A. Shimony: (1978)), derived on the premises of a local realistic theory is:

(5) $E[(M1)(M3] + E[(M2)(M3] + E[(M2)(M4] - E[(M1)(M4)] \leq 2,$

where E stands for the mean value.

- Interestingly, it can be seen that in standard quantum theory, it is always possible to design experiments for which this inequality gets violated (A. Aspect, P. Grangier and G. Roger: (1981)). This shows that quantum physics can violate local realism.
- It may also provoke the implication that, if measurements on a quantum state violate a Bell's inequality, the state is entangled. However, the converse of this statement need not be true.
- One can express the most general form of Bell-CHSH inequality for the two-qubit mixed state $\rho = \frac{1}{4}[I \otimes I + (r.\sigma) \otimes I + I \otimes (s.\sigma) + \sum_{n,m=1}^{3} t_{mn}(\sigma_m \otimes \sigma_n)]$ as $M(\rho) < 1$, where $M(\rho) = max(u_i + u_j)$, where u_i, u_j are the eigenvalues of the matrix $T^{\dagger}T$ (where the elements of the correlation matrix T is given by, $t_{mn} = Tr[\rho(\sigma_m \otimes \sigma_n)]$) (Horodecki family: (1995)).

Quantum Correlations

Teleportation Fidelity

- In addition to all these measures of quantum correlation one could also attempt to quantify them in terms of an application, for e.g., fidelity of teleportation (C. H. Bennett et al.: (1993)).
- The basic idea is to use a pair of particles in a singlet state shared by sender (Alice) and receiver (Bob). Pairs in a mixed state could be still useful for (imperfect) teleportation (S. Popescu: (1994)).
- The general mixed state of a two-qubit system :

$$
\begin{aligned}
\rho &= \frac{1}{4}[I \otimes I + (r.\sigma) \otimes I \\
&+ I \otimes (s.\sigma) + \sum_{n,m=1}^{3} t_{mn}(\sigma_m \otimes \sigma_n)]. \qquad (6)
\end{aligned}
$$

- The quantities $t_{nm} = Tr[\rho(\sigma_n \otimes \sigma_m)]$ are the coefficients of a real matrix denoted by T. This representation is most convenient when one talks about the inseparability of mixed states. In fact, all the parameters fall into two different classes: those that describe the local behaviour of the state, i.e., (r and s), and those responsible for correlations (T matrix).

Teleportation Fidelity continued...

- In the standard teleportation scheme a mixed state ρ acts as a quantum channel.
- One of the particles is with Bob while the other one and a third particle in an unknown state $|\phi\rangle$ are subjected to joint measurement in Alice's Hilbert space. These measurement operators are given by a family of projectors

$$P_k = |\psi_k\rangle\langle\psi_k|, k = 0, 1, 2, 3, \tag{7}$$

  where ψ_k constitute the so-called Bell basis.
- Using two bits Alice sends Bob the result of outcome k on basis of which he applies some unitary transformation U_k, obtaining in this way his particle in a state k.
- Fidelity of transmission of the unknown state is given by (S. Popescu: (1994); Horodecki family: (1996)),

$$F = \int_S dR(\phi) \sum_k p_k Tr(\rho_k P_\phi), \tag{8}$$

  where the integral is taken over all states (indexed by the angle ϕ) belonging to the Bloch sphere with uniform distribution R and $p_k = Tr[(P_k \otimes I)(P_\phi \otimes \rho)]$ denotes the probability of the k-th outcome.

Teleportation Fidelity continued...

- The task is to find those unitary transformations U_k that produce the highest fidelity (a choice of a quadruple of such U_k is what would be called a strategy).
- Maximizing F over all strategies gives (Horodecki : (1996))

$$
\begin{aligned}
F_{max} &= \frac{1}{2}(1 + \frac{1}{3}N(\rho)) \\
&= \frac{1}{2}(1 + \frac{1}{3}[\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3}]). \qquad (9)
\end{aligned}
$$

  Here u_i and u_j are the eigenvalues of $U = T^\dagger(\rho)T(\rho)$, where $T(\rho) = [T_{ij}], T_{ij} = Tr[\rho(\sigma_i \otimes \sigma_j)]$ and $T^\dagger$ implies the Hermitian conjugate of T. The classicial fidelity of teleportation in the absence of entanglement is obtained as $\frac{2}{3}$. Thus whenever $F_{max} > \frac{2}{3}(N(\rho) > 1)$, teleportation is possible.
- At this point it is interesting to note that there is a non-trivial interplay between Bell's inequality and teleportation fidelity. This is because both $M(\rho), N(\rho)$ are dependent on the correlation matrix T. The relationship between these two quantities is the inequality $N(\rho) > M(\rho)$. Hence, it is clear that states which do violate Bell's inequality are always useful for teleportation. However, this does not rule out the possibility of existence of entangled states that do not violate Bell's inequality, but can still be useful for teleportation.

Open Quantum Systems

- The theory of open quantum systems addresses the problems of damping and dephasing in quantum systems by the assertion that all real systems of interest are 'open' systems, surrounded by their environments (U. Weiss: (1999); H. -P. Breuer and F. Petruccione: (2002)).
- Quantum optics provided one of the first testing grounds for the application of the formalism of open quantum systems (W. H. Louisell: (1973)). Application to other areas was intensified by the works of (Caldeira and Leggett: (1983)) and (Zurek: (1993)), among others.
- The recent upsurge of interest in the problem of open quantum systems is because of the spectacular progress in manipulation of quantum states of matter, encoding, transmission and processing of quantum information, for all of which understanding and control of the environmental impact are essential (Turchette *et al.*: (2000); Myatt *et al.*: (2000)). This increases the relevance of open system ideas to quantum computation and quantum information.

Open Quantum Systems: continued...

- Hamiltonian of the total (closed system):

$$H = H_S + H_R + H_{SR}.$$

- S- system, R- reservoir (bath), $S - R$-interaction between them.
- System-reservoir complex evolves unitarily by:

$$\rho(t) = e^{-\frac{i}{\hbar}Ht}\rho(0)e^{\frac{i}{\hbar}Ht}.$$

- We are interested in the reduced dynamics of the system S, taking into account the influence of its environment. This is done by taking a trace over the reservoir degrees of freedom, making the reduced dynamics *non-unitary*:

$$\rho^s(t) = \mathrm{Tr}_\mathrm{R}(\rho(\mathrm{t})) = \mathrm{Tr}_\mathrm{R}\left[\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathrm{Ht}}\rho(0)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathrm{Ht}}\right].$$

Open Quantum Systems: continued...

- Open quantum systems can be broadly classified into two categories:
  (A). Quantum non-demolition (QND), where $[H_S, H_{SR}] = 0$ resulting in decoherence without any dissipation (Braginsky *et al.*: (1975), (1980); Caves *et al.*: (1980); G. Gangopadhyay, S. M. Kumar and S. Duttagupta: (2001); SB and R. Ghosh: (2007)) and
  (B). Quantum dissipative systems, where $[H_S, H_{SR}] \neq 0$ resulting in decoherence with dissipation (Caldeira and Leggett: (1983); H. Grabert, P. Schramm and G-L. Ingold: (1988); SB and R. Ghosh: (2003), (2007)).
- In the parlance of quantum information theory, the noise generated by a QND open system would be a "phase damping channel", while that generated by a dissipative (Lindblad) evolution would be a "(generalized) amplitude damping channel".

Time scales associated with the Open System Evolution

- The open system evolution is characterized by a number of time-scales, the salient ones being:
- Scale associated with the natural frequency of the system.
- Relaxation time scale determined by the S-R coupling strength.
- Reservoir correlation time (memory time) associated with the high-frequency cutoff in the reservoir spectral density and the time scale associated with the reservoir temperature, which measures the relative importance of quantum to thermal effects.

Quantum Operations

- Any evolution consistent with the general rules of quantum mechanics can be described by a linear, completely positive map, called quantum operation ($\mathcal{E}$). (M. A. Nielsen and I. L. Chuang: (2000))
- Complete positivity: Consider any positive map $\mathcal{E}$: $\mathcal{E}$ maps density operators of system Q_1 to density operators of system Q_2, such that $\mathcal{E}(A)$ is positive for any positive operator A.
  If an extra system R of arbitrary dimensionality is introduced, and $(\mathcal{I} \otimes \mathcal{E})(A)$ is positive on any positive operator A on the combined system RQ_1, where $\mathcal{I}$ denotes the identity map on system R: then $\mathcal{E}$ is completely positive.
- A unitary evolution is a special case of a quantum operation: general quantum operations can describe non-unitary evolutions, due to coupling with environment.

Quantum Operations continued...

- Any such quantum operation can be composed from elementary operations:

  • unitary transformations: $\mathcal{E}_1(\rho) = U\rho U^\dagger$

  • addition of an auxiliary system: $\mathcal{E}_2(\rho) = \rho \otimes \sigma$: here ρ is the original system and σ is the auxiliary one

  • partial traces: $\mathcal{E}_3(\rho) = \mathrm{Tr}_\mathrm{B}(\rho)$

  • projective measurements: $\mathcal{E}_4(\rho) = P_k\rho P_k/\mathrm{Tr}(\mathrm{P}_\mathrm{k}\rho)$, with $P_k^2 = P_k$.

Quantum Operations continued...

Connection to quantum noise processes

- Interpret results from open quantum systems in terms of familiar noisy channels. How these environmental effects can affect quantum computing. In operator-sum representation, action of superoperator $\mathcal{E}$ due to environmental interaction

  $$\rho \longrightarrow \mathcal{E}(\rho) = \sum_k \langle e_k | U (\rho \otimes |f_0\rangle\langle f_0|) U^\dagger | e_k \rangle = \sum_k E_k \rho E_k^\dagger,$$

  unitary U acts jointly on system-environment $|f_0\rangle$: environment's initial state; $\{|e_k\rangle\}$ a basis for the environment.
- environment-system assumed to start in a separable state.
- $E_k \equiv \langle e_k | U | f_0 \rangle$ are the Kraus operators; partition of unity: $\sum_k E_k^\dagger E_k = \mathcal{I}$. Any transformation representatable as operator-sum is a completely positive (CP) map.

Model: (A). Dynamics of the Reduced Density Matrix for two-qubit QND system

(SB, V. Ravishankar and R. Srikanth: (2009))

- Hamiltonian, describing the QND interaction of two qubits with the bath:

$$\begin{aligned} H &= H_S + H_R + H_{SR} \\ &= \sum_{n=1}^{L=2} \hbar \varepsilon_n J_z^n + \sum_k \hbar \omega_k b_k^\dagger b_k + \sum_{n,k} \hbar J_z^n (g_k^n b_k^\dagger + g_k^{n*} b_k). \end{aligned}$$

- H_S, H_R and H_{SR} stand for the Hamiltonians of the system, reservoir and system-reservoir interaction, respectively. $b_k^\dagger$, b_k denote the creation and annihilation operators for the reservoir oscillator of frequency ω_k, g_k^n stands for the coupling constant (assumed to be position dependent) for the interaction of the oscillator field with the qubit system and are taken to be

$$g_k^n = g_k e^{-ik.r_n},$$

where r_n is the qubit position. Since $[H_S, H_{SR}] = 0$, the Hamiltonian (1) is of QND type.

Dynamics of the Reduced Density Matrix continued...

- The position dependence of the coupling of the qubits to the bath helps to bring out the effect of entanglement between qubits through the qubit separation: $r_{mn} \equiv r_m - r_n$. This allows for a discussion of the dynamics in two regimes:

  (a). Localized (independent) Decoherence:

  where $k.r_{mn} \sim \frac{r_{mn}}{\lambda} \geq 1$

  and

  (b). Collective Decoherence:

  where $k.r_{mn} \sim \frac{r_{mn}}{\lambda} \rightarrow 0.$

Dynamics of the Reduced Density Matrix continued...

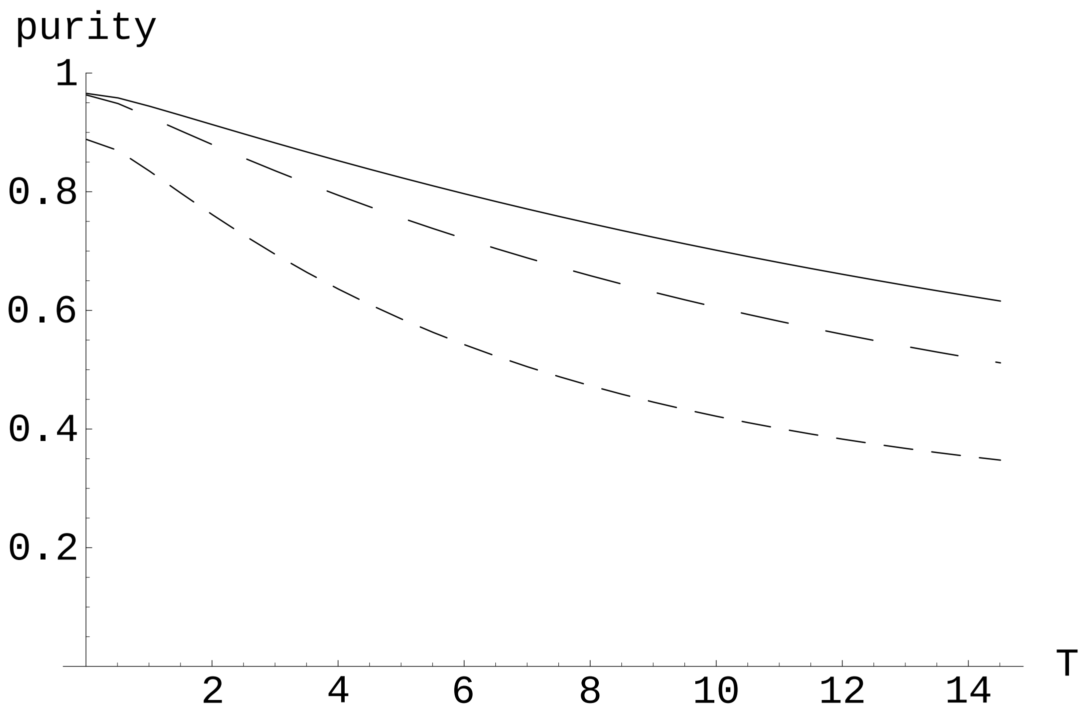


Fig.1 : Purity, $\mathrm{Tr}(\rho^2(t))$, as a function of temperature T (in units where $\hbar \equiv k_B = 1$) for the localized decoherence model. The bold, large-dashed and small-dashed curves correspond to evolution time $t = 2.0$, 3.0 and 2.0, respectively. The bath squeezing parameter α is equal to 0.2, 0.2 and 1.0, respectively, for the three curves. Also the bath parameters γ_0, ω_c are equal to 0.01 and 100.0, respectively. It can be shown from these results that with the increase in temperature, as also evolution time t and bath squeezing α, the system becomes more mixed and hence looses its purity.

Dynamics of the Reduced Density Matrix continued...

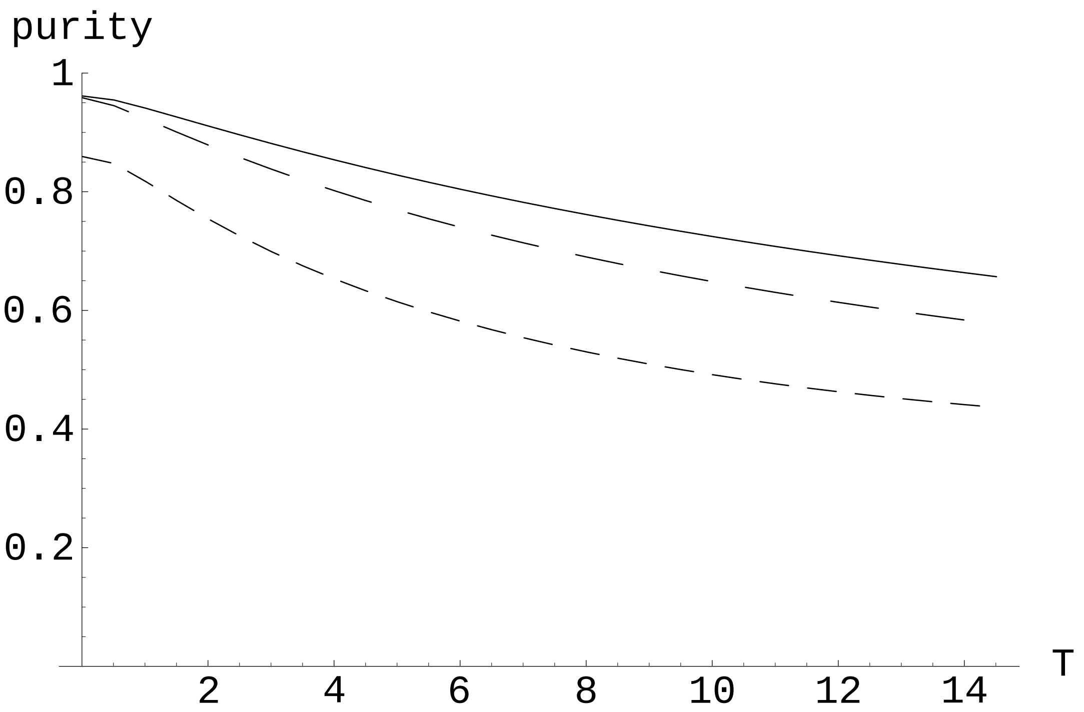


Fig.2 : Purity, $\mathrm{Tr}(\rho^2(t))$, as a function of temperature T (in units where $\hbar \equiv k_B = 1$) for the collective decoherence model. The bold, large-dashed and small-dashed curves correspond to evolution time $t = 2.0$, 3.0 and 2.0, respectively. The bath squeezing parameter α is equal to 0.2, 0.2 and 1.0, respectively, for the three curves. Also the bath parameters γ_0, ω_c are equal to 0.01 and 100.0, respectively. It can be shown from these results that with the increase in temperature, as also evolution time t and bath squeezing α, the system becomes more mixed and hence looses its purity.

Model: (B). Dynamics of the Reduced Density Matrix for two-qubit Dissipative system

(SB, V. Ravishankar and R. Srikanth: (2010))

- Hamiltonian, describing the dissipative, position dependent, interaction of two qubits with bath (modelled as a 3-D electromagnetic field (EMF)) via dipole interaction as:

$$
\begin{aligned}
H &= H_S + H_R + H_{SR} \\
&= \sum_{n=1}^{N=2} \hbar\omega_n S_n^z + \sum_{\vec{k}s} \hbar\omega_k (b^\dagger_{\vec{k}s} b_{\vec{k}s} + 1/2) - i\hbar \sum_{\vec{k}s} \sum_{n=1}^{N} [\vec{\mu}_n . \vec{g}_{\vec{k}s}(\vec{r}_n)(S_n^+ + S_n^-) b_{\vec{k}s} - h.c.].
\end{aligned}
$$

$\vec{\mu}_n$: transition dipole moments, dependent on the different atomic positions $\vec{r}_n$

$$S_n^+ = |e_n\rangle\langle g_n|, \; S_n^- = |g_n\rangle\langle e_n| :$$

dipole raising and lowering operators satisfying the usual commutation relations

$$S_n^z = \frac{1}{2}(|e_n\rangle\langle e_n| - |g_n\rangle\langle g_n|) :$$

energy operator of the nth atom

Dynamics of the Reduced Density Matrix continued...

$b^\dagger_{\vec{k}s}$, $b_{\vec{k}s}$: creation and annihilation operators of the field mode (bath) $\vec{k}s$ with the wave vector $\vec{k}$, frequency ω_k and polarization index $s = 1, 2$

- System-Reservoir (S-R) coupling constant:

$$\vec{g}_{\vec{k}s}(\vec{r}_n) = (\frac{\omega_k}{2\varepsilon\hbar V})^{1/2} \vec{e}_{\vec{k}s} e^{i\vec{k}.r_n}.$$

  V: the normalization volume and $\vec{e}_{\vec{k}s}$: unit polarization vector of the field.

- S-R coupling constant: dependent on the atomic position r_n.
  This leads to a number of interesting dynamical aspects.

Dynamics of the Reduced Density Matrix continued...

- Assuming separable initial conditions, and taking a trace over the bath the reduced density matrix of the qubit system in the interaction picture and in the usual Born-Markov, rotating wave approximation (RWA) is obtained as

$$
\begin{aligned}
\frac{d\rho}{dt} &= -\frac{i}{\hbar}[H_{\tilde{S}}, \rho] - \frac{1}{2}\sum_{i,j=1}^{2} \Gamma_{ij}[1+\tilde{N}](\rho S_i^+ S_j^- + S_i^+ S_j^- \rho - 2 S_j^- \rho S_i^+) \\
&- \frac{1}{2}\sum_{i,j=1}^{2} \Gamma_{ij}\tilde{N}(\rho S_i^- S_j^+ + S_i^- S_j^+ \rho - 2 S_j^+ \rho S_i^-) \\
&+ \frac{1}{2}\sum_{i,j=1}^{2} \Gamma_{ij}\tilde{M}(\rho S_i^+ S_j^+ + S_i^+ S_j^+ \rho - 2 S_j^+ \rho S_i^+) \\
&+ \frac{1}{2}\sum_{i,j=1}^{2} \Gamma_{ij}\tilde{M}^*(\rho S_i^- S_j^- + S_i^- S_j^- \rho - 2 S_j^- \rho S_i^-).
\end{aligned}
$$

Dynamics of the Reduced Density Matrix continued...

$$\tilde{N} = N_{\rm th}(\cosh^2(r) + \sinh^2(r)) + \sinh^2(r),$$

$$\tilde{M} = -\frac{1}{2}\sinh(2r)e^{i\Phi}(2N_{\rm th}+1) \equiv Re^{i\Phi(\omega_0)},$$

with

$$\omega_0 = \frac{\omega_1 + \omega_2}{2},$$

and

$$N_{\rm th} = \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}.$$

- Here $N_{\rm th}$ is the Planck distribution giving the number of thermal photons at the frequency ω and r, Φ are squeezing parameters. The analogous case of a thermal bath without squeezing can be obtained from the above expressions by setting these squeezing parameters to zero, while setting the temperature (T) to zero one recovers the case of the vacuum bath.

Dynamics of the Reduced Density Matrix continued...

$$H_{\tilde{S}} = \hbar \sum_{n=1}^{2} \omega_n S_n^z + \hbar \sum_{\substack{i,j \\ (i \neq j)}}^{2} \Omega_{ij} S_i^+ S_j^- ,$$

where

$$\begin{aligned}
\Omega_{ij} &= \frac{3}{4}\sqrt{\Gamma_i \Gamma_j}\Bigg[-[1-(\hat{\mu}.\hat{r}_{ij})^2]\frac{\cos(k_0 r_{ij})}{k_0 r_{ij}} + [1-3(\hat{\mu}.\hat{r}_{ij})^2] \\
&\times [\frac{\sin(k_0 r_{ij})}{(k_0 r_{ij})^2} + \frac{\cos(k_0 r_{ij})}{(k_0 r_{ij})^3}]\Bigg] .
\end{aligned}$$

$\hat{\mu} = \hat{\mu}_1 = \hat{\mu}_2$ and $\hat{r}_{ij}$ are unit vectors along the atomic transition dipole moments and $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, respectively.
$k_0 = \omega_0/c$, $r_{ij} = |\vec{r}_{ij}|$.

Dynamics of the Reduced Density Matrix continued...

- Wavevector $k_0 = 2\pi/\lambda_0$, λ_0 being the resonant wavelength, occuring in the term $k_0 r_{ij}$ sets up a length scale into the problem depending upon the ratio r_{ij}/λ_0. This is thus the ratio between the interatomic distance and the resonant wavelength, allowing for a discussion of the dynamics in two regimes:
  (a). localized decoherence: where $k_0.r_{ij} \sim \frac{r_{ij}}{\lambda_0} \geq 1$
  and
  (b). collective decoherence: where $k_0.r_{ij} \sim \frac{r_{ij}}{\lambda_0} \rightarrow 0$.
- Collective decoherence would arise when the qubits are close enough for them to feel the bath collectively or when the bath has a long correlation length (set by the resonant wavelength λ_0) in comparison to the interqubit separation r_{ij}.

Dynamics of the Reduced Density Matrix continued...

- Ω_{ij}: a collective coherent effect due to the multi-qubit interaction and is mediated via the bath through the terms

$$\Gamma_i = \frac{\omega_i^3 \mu_i^2}{3\pi\varepsilon\hbar c^3}.$$

- The term Γ_i is present even in the case of single-qubit dissipative system bath interaction and is the spontaneous emission rate, while

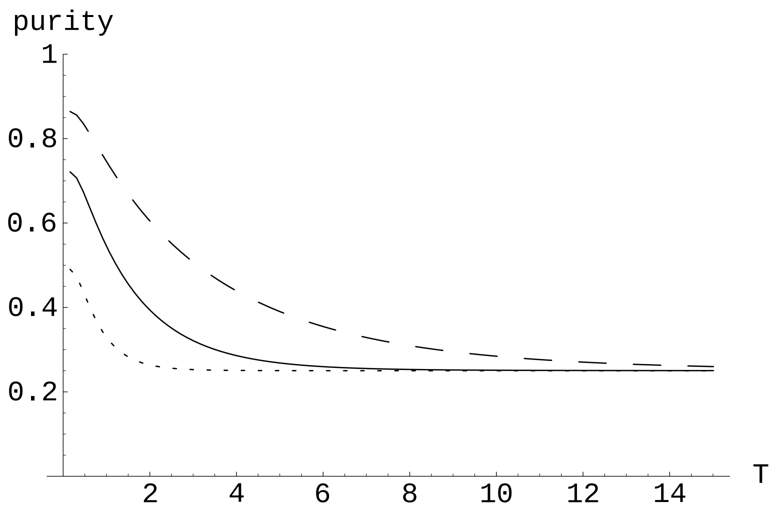
$$\Gamma_{ij} = \Gamma_{ji} = \sqrt{\Gamma_i \Gamma_j} F(k_0 r_{ij}),$$

where $i \neq j$ with

$$\begin{aligned} F(k_0 r_{ij}) &= \frac{3}{2}\left[[1 - (\hat{\mu}.\hat{r}_{ij})^2] \frac{\sin(k_0 r_{ij})}{k_0 r_{ij}} + [1 - 3(\hat{\mu}.\hat{r}_{ij})^2] \right. \\ &\times \left. [\frac{\cos(k_0 r_{ij})}{(k_0 r_{ij})^2} - \frac{\sin(k_0 r_{ij})}{(k_0 r_{ij})^3}] \right]. \end{aligned}$$

- Γ_{ij}: collective incoherent effect due to the dissipative multi-qubit interaction with the bath. For the case of identical qubits, as considered here, $\Omega_{12} = \Omega_{21}$, $\Gamma_{12} = \Gamma_{21}$ and $\Gamma_1 = \Gamma_2 = \Gamma$.

Dynamics of the Reduced Density Matrix continued...



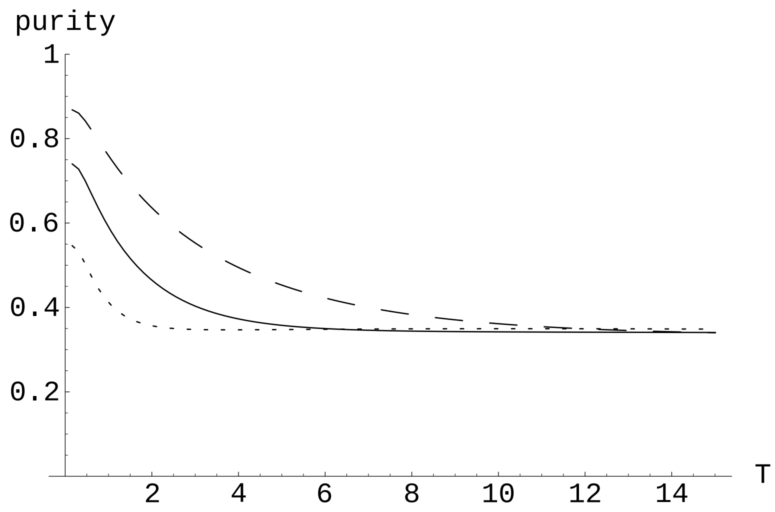


Fig.3 & 4 : Purity, $\mathrm{Tr}(\rho^2(t))$, as a function of temperature T for (3) the localized decoherence model, where r_{ij}/λ_0 is ≥ 1 and (4) the collective decoherence model, where r_{ij}/λ_0 is ≈ 0. Here with r_{12} is the inter-qubit distance. The large-dashed, bold and dotted curves correspond to evolution time $t = 1.0$ and bath squeezing parameter $r = -0.5$, 1.0 and 1.5, respectively. Also ω_0 and the bath parameter Γ, are set equal to 1.0 and 0.05, respectively. Both for the independent as well as the collective decoherence model, with increase in temperature, as also evolution time t and bath squeezing r, the system becomes more mixed and hence looses its purity.

Dynamics of Entanglement

Quantum Non-Demolition (QND) Evolution

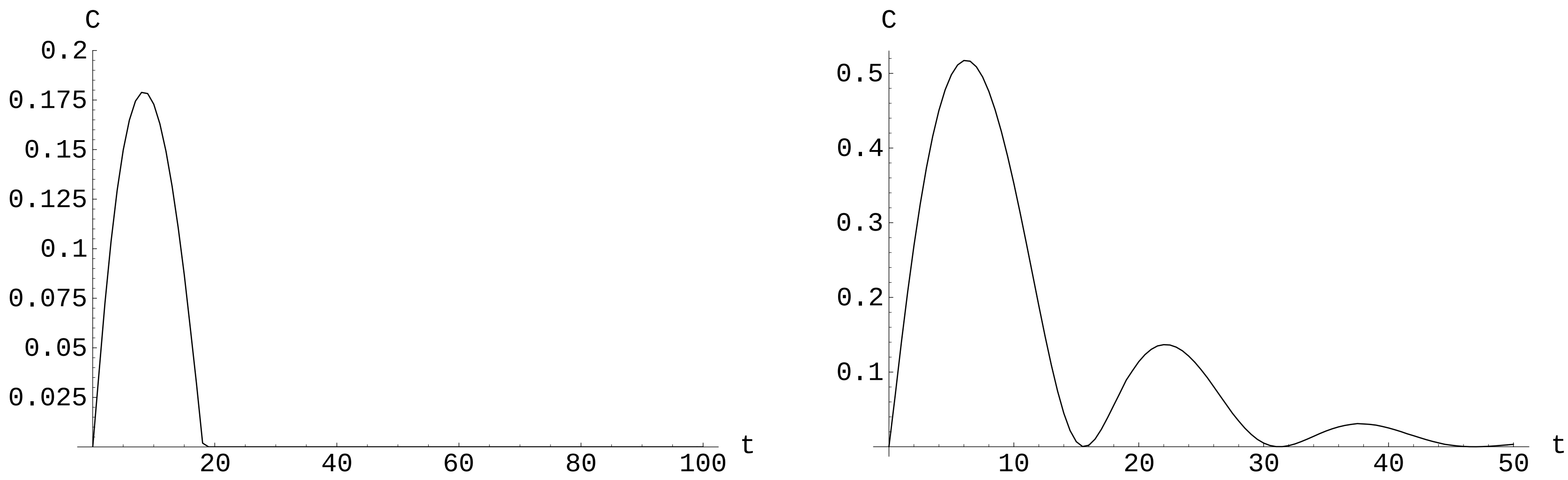
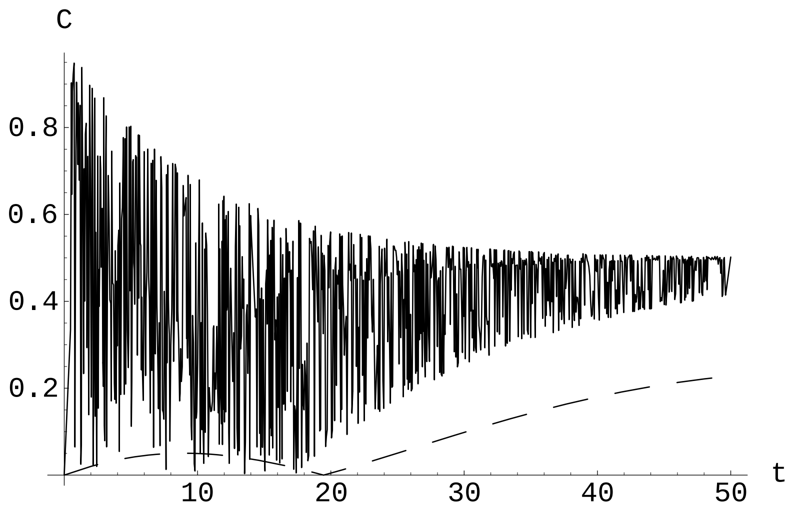


Fig.5 & 6 : Concurrence $\mathcal{C}$ as a function of time of evolution t at $T = 5.0$ and bath squeezing parameter α equal to 0.2. Figure (5) refers to the localized decoherence model and (6) the collective decoherence model.

Dynamics of Entanglement: QND continued...

- In figure 5, concurrence is plotted with respect to time for the case of the localized decoherence model, while figure 6 depicts the temporal behavior of concurrence for the collective decoherence model. It is clearly seen from the figures that the two qubit system is initially unentangled, but with time there is a build up of entanglement between them as a result of their interaction with the bath. Also the entanglement builds up more quickly in the collective decoherence model when compared to the localized model. This is expected as the effective interaction between the two qubits is stronger in the collective case. Another interesting feature that can be inferred from figure 5 is the phenomena of entanglement birth and death (T. Yu and J. H. Eberly: (2009)) in the localized decoherence model.

Dissipative Evolution



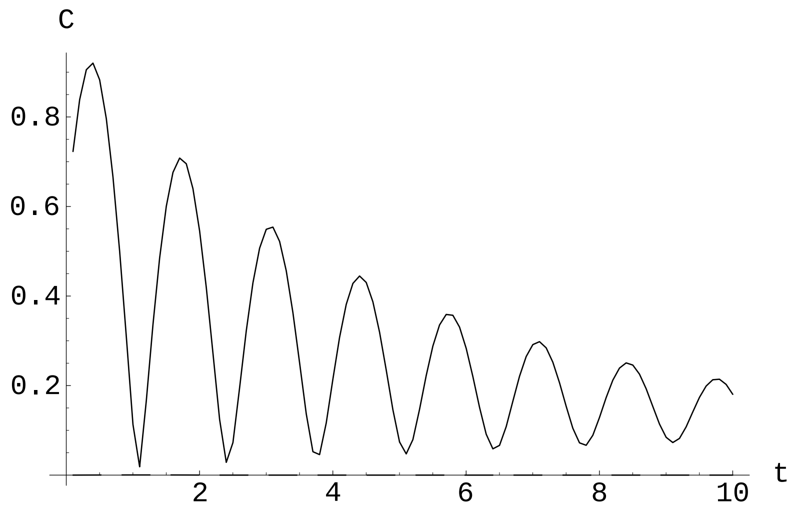
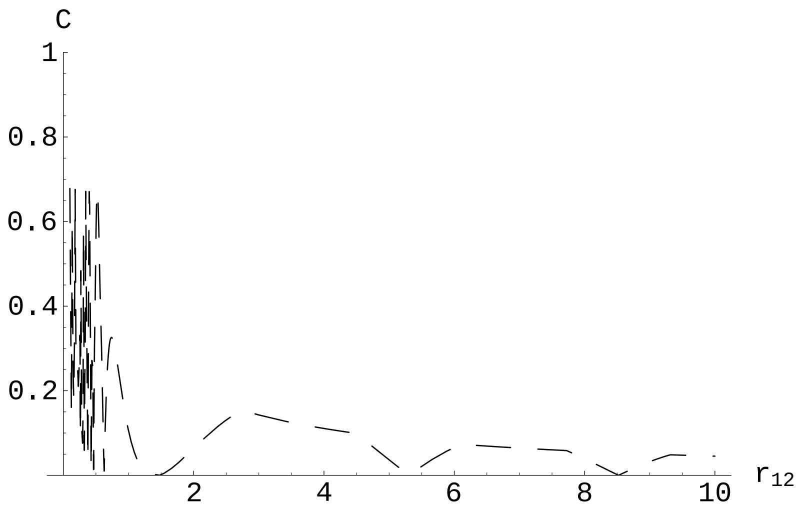


Fig. 7 & 8 : Concurrence $\mathcal{C}$ as a function of time of evolution t. Figure (7) deals with the case of vacuum bath ($T = r = 0$), while figure (8) considers concurrence in the two-qubit system interacting with a squeezed thermal bath, for a temperature $T = 1$ and and bath squeezing parameter r equal to 0.1. In both the figures the bold curve depicts the collective decoherence model ($kr_{12} = 0.05$), while the dashed curve represents the independent decoherence model ($kr_{12} = 1.1$). In figure (8) for the given settings, the concurrence for the independent decoherence model is negligible and is thus not seen.

Dynamics of Entanglement continued...

Dissipative Evolution



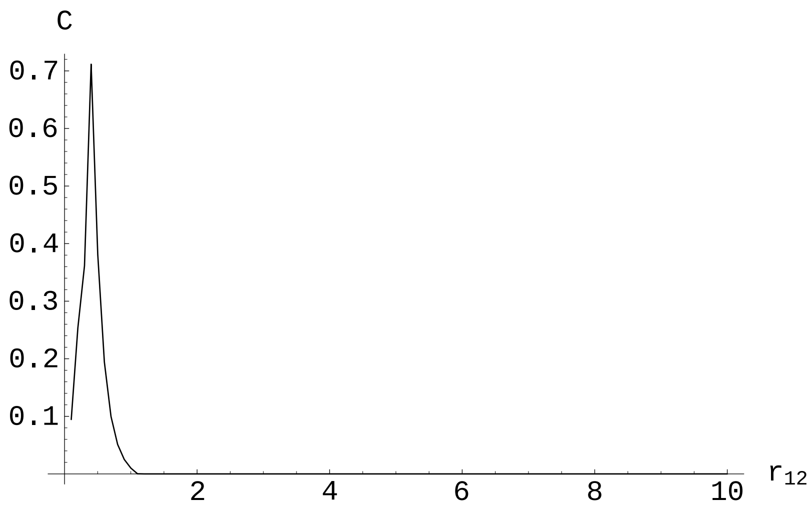
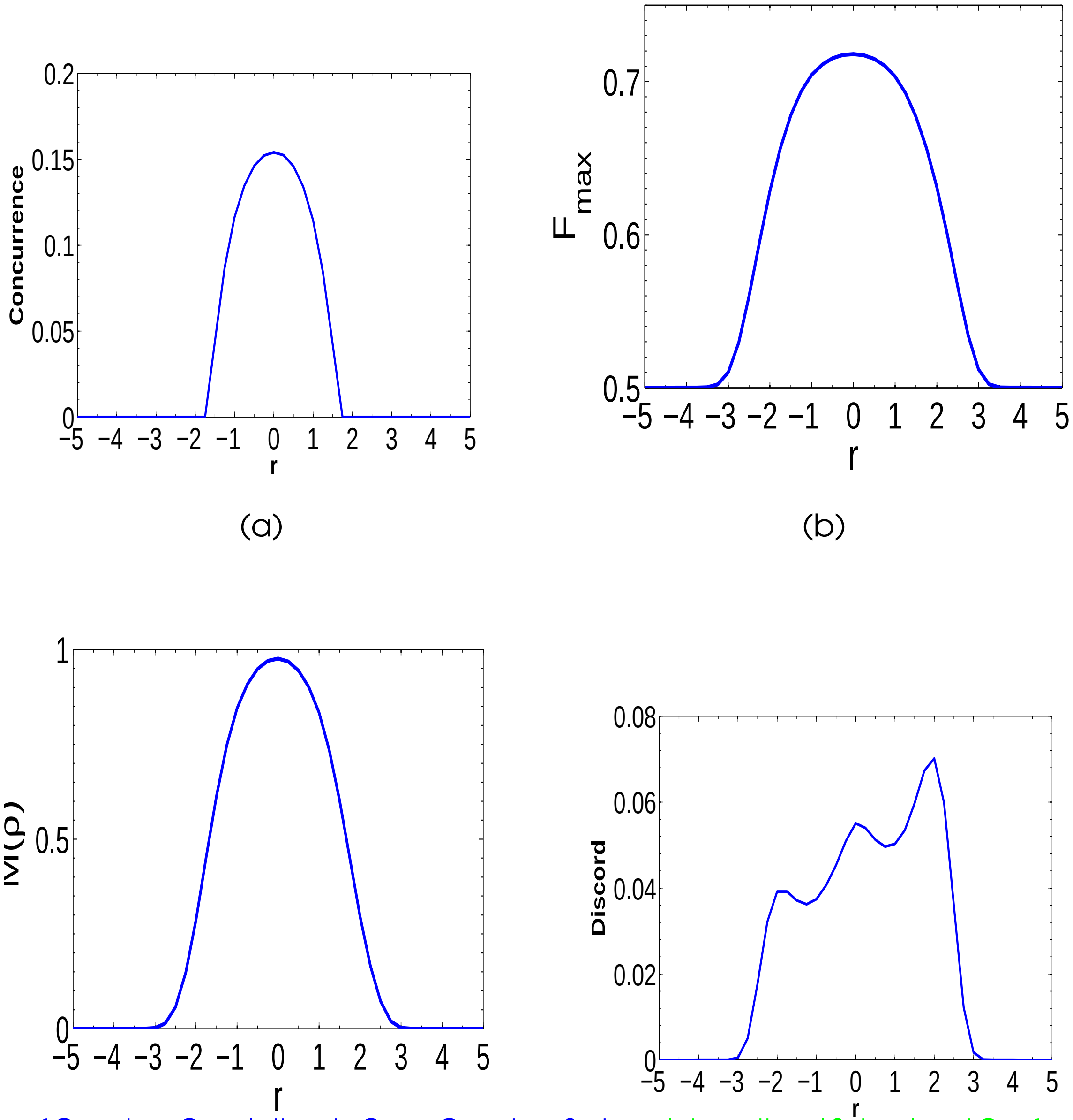


Fig. 9 & 10 : Concurrence $\mathcal{C}$ with respect to inter-qubit distance r_{12}. Figure (9) deals with the case of vacuum bath ($T = r = 0$), while figure (10) considers concurrence in the two-qubit system interacting with a squeezed thermal bath, for $T = 1$, evolution time $t = 1$ and bath squeezing parameter r equal to 0.1. In figure (a) the oscillatory behavior of concurrence is stronger in the collective decoherence regime, in comparison with the independent decoherence regime ($kr_{12} \geq 1$). In figure (10), the effect of finite bath squeezing and T has the effect of diminishing the concurrence to a great extent in comparison to the vacuum bath case. Here the concurrence for the localized decoherence regime is negligible, in agreement with the previous figure.

Dynamics of Quantum Correlations

- We made a comparative study, on states generated by our model, of various features of quantum correlations like teleportation fidelity (F_{max}), violation of Bell's inequality $M(\rho)$ (violation takes place for $M(\rho) \geq 1$), concurrence $C(\rho)$ and discord with respect to various experimental parameters like, bath squeezing parameter r, inter-qubit spacing r_{12}, temperature T and time of evolution t (I. Chakrabarty, SB, N. Siddharth: (2011)).
- A basic motivation of this work is to have realistic open system models that generate entangled states which can be useful for teleportation, but at the same time, not violate Bell's inequality. We provide below some examples of such states. Interestingly, we also find examples of states with positive discord, but zero entanglement, reiterating the fact that entanglement is a subset of quantum correlations.

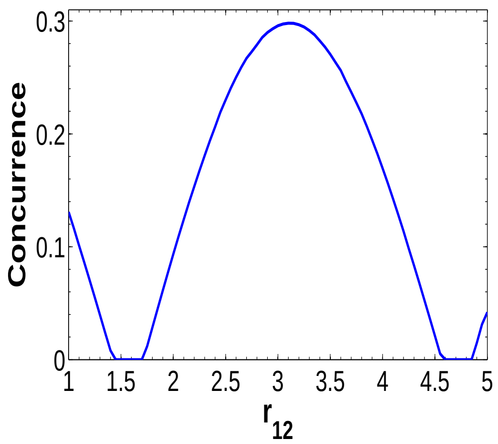
Dynamics of Quantum Correlations: QND



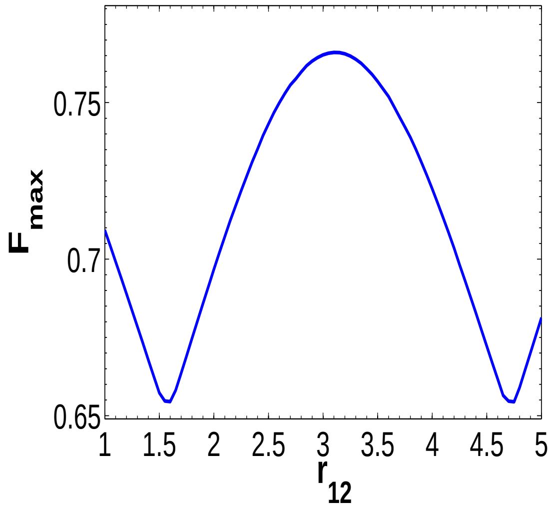
Dynamics of Quantum Correlations: QND continued...

Fig. 11 : The example depicted in Figs. (13 (a)), (b), (c), (d), study two-qubit density matrices, as a function of bath squeezing parameter r, from the independent model. In Fig. (11 (a)), concurrence is plotted with respect to bath squeezing parameter r. It is seen that states are entangled when r lies in the range $[-1.8, 1.8]$. Teleportation fidelity, as in the Fig. (11 (b)), indicates that the states are useful for teleportation for the same range of r, i.e., when they are entangled. However, from Bell's inequality, as shown in Fig. (11 (c)), in the same range, we see that the states do not violate Bell's inequality. Interestingly, from Fig. (11 (d)), we find a non zero quantum discord in the range $[-3, 3]$ and particularly, in the range $[-3, -1.8] \cup [1.8, 3]$, i.e., where entanglement as depicted in Fig. (11 (a)) is zero, discord is non-zero. This brings out the fact that the amount of entanglement present in a system is not equivalent to the the total amount of quantum correlation in it.

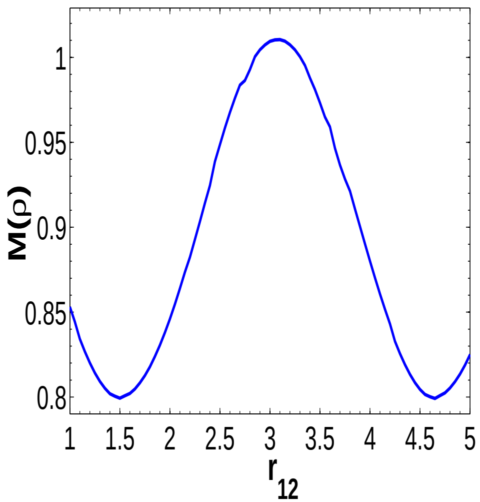
Dynamics of Quantum Correlations: QND continued...



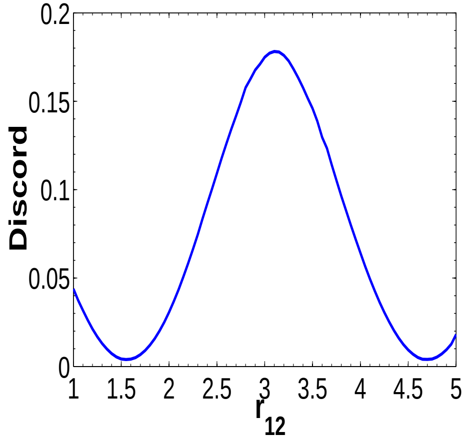
(e)



(f)



(g)

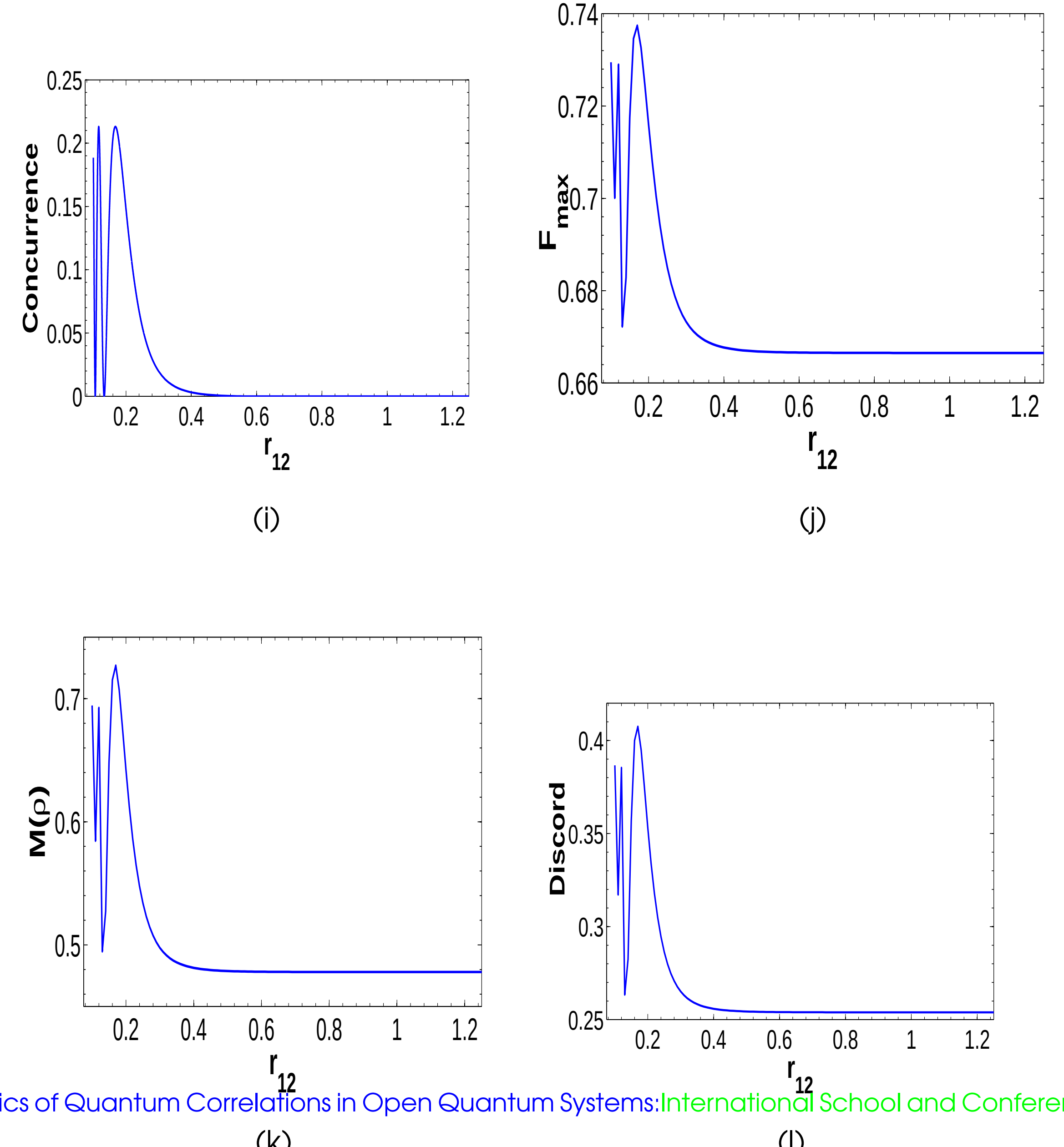


(h)

Dynamics of Quantum Correlations: QND continued...

Fig. 12 : Figs. (12 (a)), (b), (c), (d) depict, in the independent regime of the model, concurrence, teleportation fidelity, Bell's inequality and discord, respectively, with respect to the inter-qubit spacing r_{12}. In Fig. (12 (a)), from the plot of concurrence, it can be seen that the states are entangled with a positive concurrence for all values of r_{12}, except in the range $[1.5, 1.6], [4.6, 4.8]$. Figure (12 (b)) shows that the states useful for teleportation are from the same range of r_{12}, for which they are entangled. From Fig. (12 (c)), it can be seen that there are certain regions where Bell's inequality is violated, but mostly it is satisfied. Like in the previous example, we see non-vanishing value of discord in the range where there is no entanglement, as shown in Fig. (12 (d)).

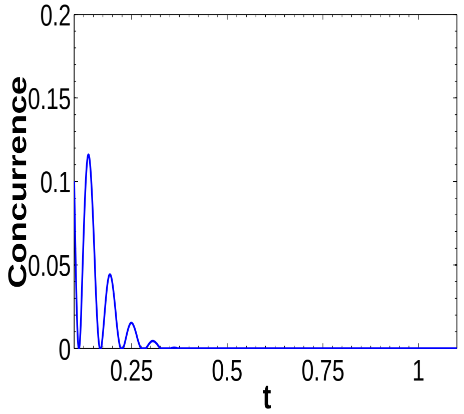
Dynamics of Quantum Correlations: Dissipative

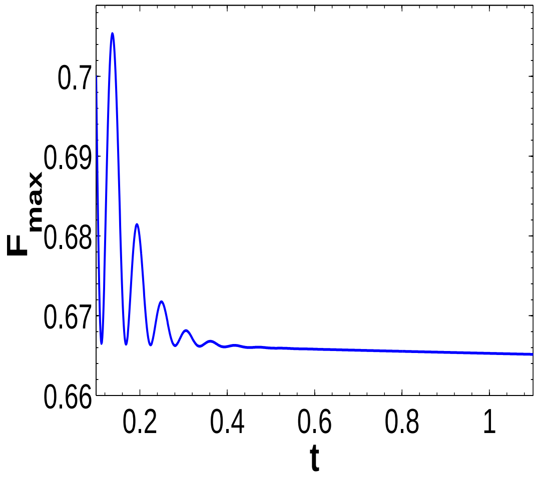


Dynamics of Quantum Correlations: Dissipative continued...

Fig. 13 : Quantum correlations in a two-qubit system undergoing a dissipative evolution. The Figs. (a), (b), (c) and (d) represent the evolution of concurrence, maximum teleportation fidelity F_{max}, test of Bell's inequality $M(\rho)$, discord as a function of inter-qubit distance r_{12}. Here temperature $T = 300$, evolution time t is 0.1 and bath squeezing parameter $r = -1$. From Fig. (15 (a)), we find that the two qubit density matrix is entangled with a positive concurrence except at the point 0.133 (approx) and for $r_{12} \geq 0.4$. Figure (13 (b)) illustrates that $F_{max} > \frac{2}{3}$, for all values of r_{12} except where there is no entanglement. However, from Fig. (13 (c)) we find that $M(\rho) < 1$ for all values of r_{12}, clearly demonstrating that the states can be useful for teleportation despite the fact that they satisfy Bell's inequality. Moreover, from Fig. (13 (d)), a positive discord is seen for the complete range of r_{12}, even in the range where there is no entanglement. As a function of the inter-qubit distance, the various correlation measures exhibit oscillatory behavior, in the collective regime of the model, but flatten out subsequently to attain almost constant values in the independent regime of the model. This oscillatory behavior is due to the strong collective behavior exhibited by the dynamics due to the relatively close proximity of the qubits in the collective regime.

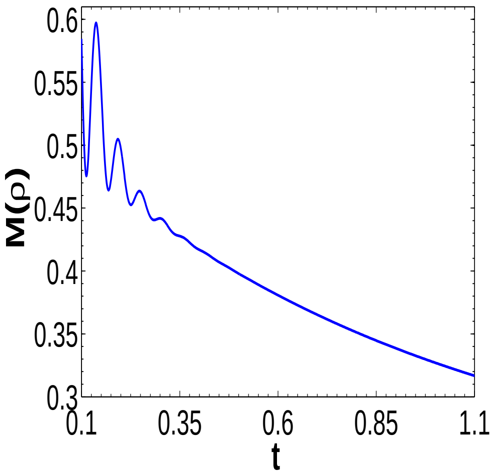
Dynamics of Quantum Correlations: Dissipative continued...

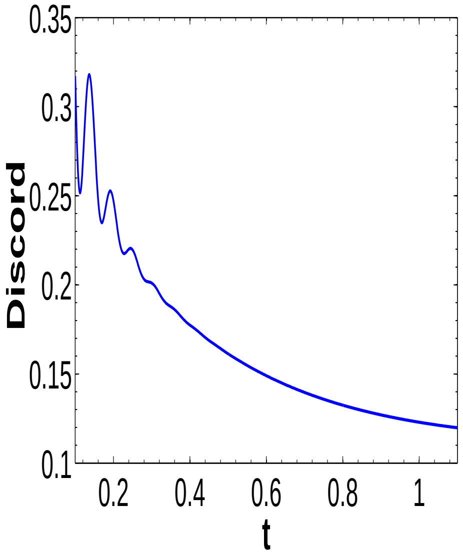




(m)

(n)

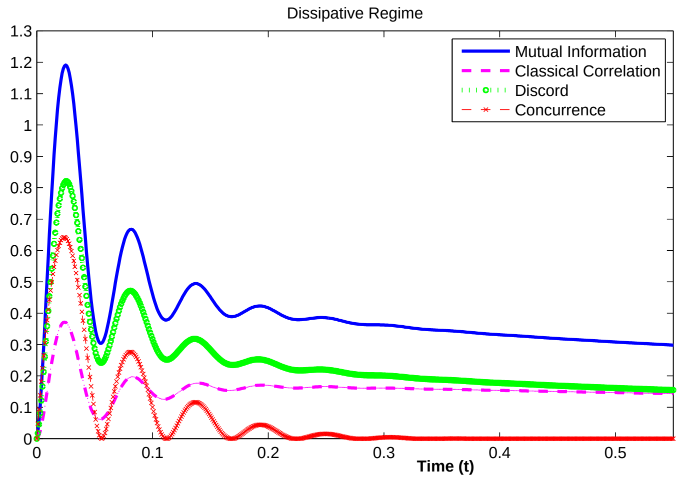




Dynamics of Quantum Correlations: Dissipative continued...

Fig. 14 : Figures (a), (b), (c) and (d) represent the evolution of concurrence, maximum teleportation fidelity F_{max}, test of Bell's inequality $M(\rho)$, discord with respect to the time of evolution t, evolving under a dissipative interaction. Here temperature $T = 300$, inter-qubit distance $r_{12} = 0.11$ and bath squeezing parameter $r = -1$. In Fig. (14 (a)), concurrence is seen to exhibit damped oscillations. Figure (14 (b)), for teleportation fidelity F_{max}, also shows a damped behavior and can be seen, in general, to be greater than $\frac{2}{3}$, except at the points where there is no entanglement (zero concurrence), where it touches $\frac{2}{3}$. From Fig. (14 (c)), we find that the states satisfy Bell's inequality. Discord, as in Fig. (14 (d)), is positive, though its value is decreasing with time.

Dynamics of Quantum Correlations: Dissipative continued...



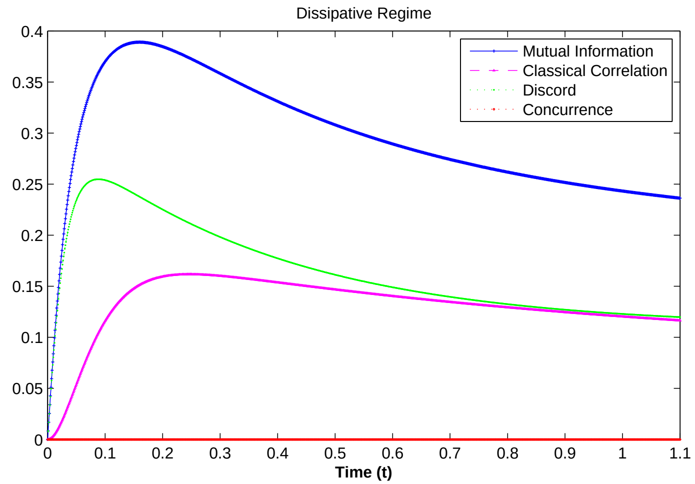


Fig. 15 & 16 : An example showing vanishing entanglement, but non vanishing discord, for a dissipative two-qubit evolution. Figures (15) and (16) represent the evolution of mutual information (blue), quantum discord (quantum correlation) (green), concurrence (red) and classical information (pink) with respect to the time of evolution t, evolving under a dissipative interaction for collective and independent models, respectively. We find that in the absence of entanglement from a certain time $t > \bar{t}$, the classical correlation and the quantum discord becomes identical. Here for (15) temperature $T = 10$, $r = 0$ and inter-qubit distance $r_{12} = 0.11$ and for (16) temperature $T = 10$, $r = 0$ and inter-qubit distance $r_{12} = 1.5$.

Conclusions

- We have discussed the dynamics of the evolution of measures of quantum correlations such as entanglement and discord in open system models consisting of two qubits interacting with their baths under general settings.
- We discussed examples, generated by our two-qubit evolution, where entangled states, generated via a QND or dissipative evolution, in both the collective as well as independent regimes, do not violate Bell's inequality, but can still be useful for teleportation. These examples also illustrate that quantum correlations, when quantified in terms of discord, can be non zero even in the absence of entanglement.