Quote:

- "The reasonable man adapts himself to the world around him. The unreasonable man persists in his attempts to adapt the world to himself. Therefore, all progress depends on the unreasonable man."

George Bernard Shaw

Quantum Game theory with Clifford Geometric Algebra (GA)

The natural algebra of 3D space

Vector cross product a x b

Quaternions q

Complex numbers i

Spinors

Rotation matrices R

Why study GA?

- Correct algebra of physical space-3D? *
- Maxwell's equations reduce to a single equation
- 4D spacetime embeds in 3D (Special Theory relativity)
- The Dirac equation in 4D spacetime reduces to a real equation in 3D-removes need for $i=\sqrt{-1}$ in QM.

j

Quaternions (1843)

- 1. The generalization of complex numbers to three dimensions
- $i^2 = j^2 = k^2 = -1$, $i\,j = k$,

k

Non-commutative $i\,j = -j\,i$, try rotating a book

i

Quarternion rotations of vectors

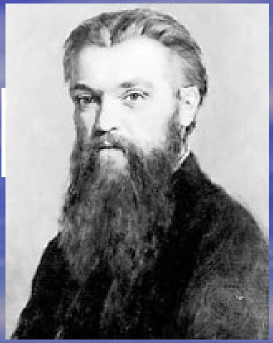
- Bilinear transformation for rotations

  $v' = R\, v\, R^{\dagger}$

  where R is a quaternion

  v is a Cartesian vector
- 1. Initial resistance to these ideas, but actually exactly the properties we need for rotations in 3D

Clifford's Geometric Algebra

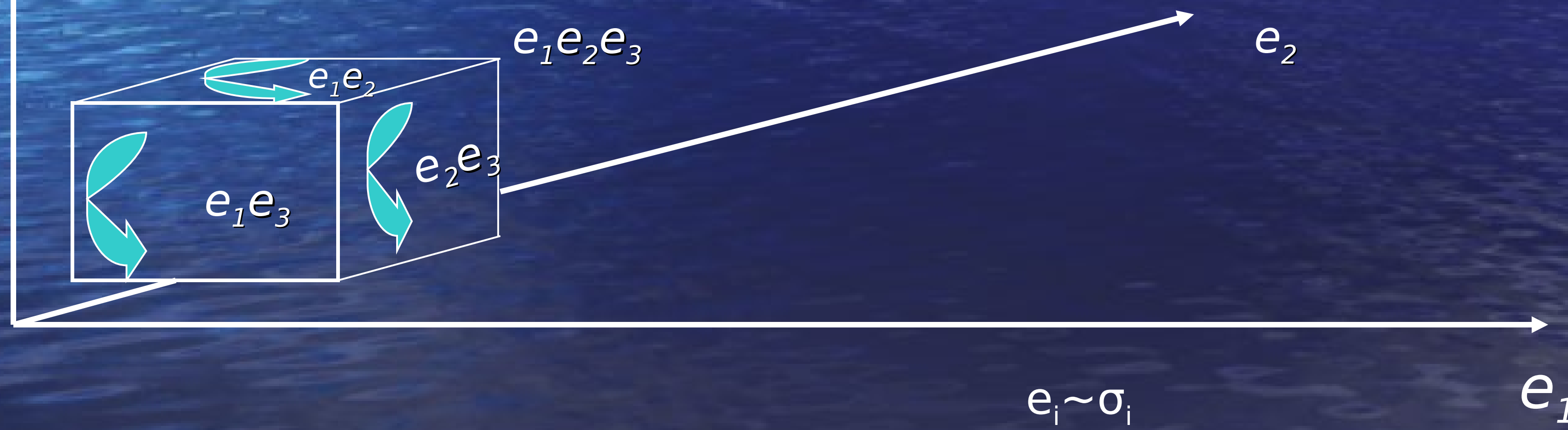


1890

- Define algebraic elements e_1, e_2, e_3
- With $e_1^2=e_2^2=e_3^2=1$, and anticommuting

$$e_i\, e_j = -\, e_j\, e_i$$

1. This algebraic structure unifies Cartesian coordinates, quaternions and complex numbers into a single real framework.

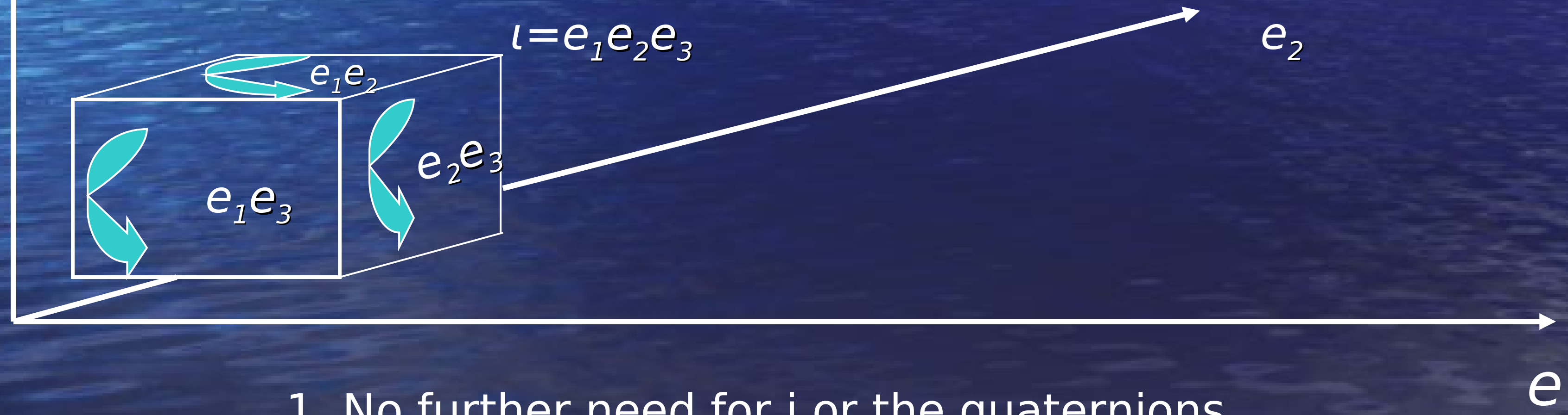


$e_i \sim \sigma_i$

Geometric Algebra-Dual representation

$$e_2 e_3 = \iota e_1, \qquad e_3 e_1 = \iota e_2, \qquad e_1 e_2 = \iota e_3$$

$$\iota = e_1 e_2 e_3$$



1. No further need for i or the quaternions.

The product of two vectors….

1. To multiply 2 vectors we:...simply expand brackets...distributive law of multiplication over addition.

$$
\begin{aligned}
&uv \\
&= (e_1u_1 + e_2u_2 + e_3u_3)(e_1v_1 + e_2v_2 + e_3v_3) \\
&= u_1v_1 + u_2v_2 + u_3v_3 + (u_2v_3 - v_2u_3)e_2e_3 + (u_1v_3 - v_1u_3)e_1e_3 + (u_1v_2 - v_1u_2)e_1e_2 \\
&= u_iv_i + \iota[(u_2v_3 - v_2u_3)e_1 + (u_1v_3 - v_1u_3)e_2 + (u_1v_2 - v_1u_2)e_3] \\
&= u \cdot v + \iota u \times v
\end{aligned}
$$

$$\iota = e_1e_2e_3$$

A complex-type number combining the dot and cross products!

2. Hence we now have an intuitive definition of multiplication and division of vec subsuming the dot and cross products which also now has an inverse.

Spinor mapping

1.How can we map from complex spinors to 3D GA?

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} a_0 + \mathrm{i}a_3 \\ -a_2 + \mathrm{i}a_1 \end{bmatrix} \leftrightarrow \psi = a_0 + a_1\iota\sigma_1 + a_2\iota\sigma_2 + a_3\iota\sigma_3$$

We see that spinors are rotation operators.

$$\iota = e_1 e_2 e_3$$

$$|0\rangle \longleftrightarrow 1, \quad |1\rangle \longleftrightarrow -\iota\sigma_2$$

2. The geometric product is equivalent to the tensor product

$$\begin{aligned}
|0\rangle|0\rangle|0\rangle &\longleftrightarrow 1 \\
|0\rangle|0\rangle|1\rangle &\longleftrightarrow -\iota\sigma_2^3 \\
|0\rangle|1\rangle|0\rangle &\longleftrightarrow -\iota\sigma_2^2 \\
|0\rangle|1\rangle|1\rangle &\longleftrightarrow \iota\sigma_2^2\iota\sigma_2^3, \\
|1\rangle|0\rangle|0\rangle &\longleftrightarrow -\iota\sigma_2^1 \\
|1\rangle|0\rangle|1\rangle &\longleftrightarrow \iota\sigma_2^1\iota\sigma_2^3 \\
|1\rangle|1\rangle|0\rangle &\longleftrightarrow \iota\sigma_2^1\iota\sigma_2^2 \\
|1\rangle|1\rangle|1\rangle &\longleftrightarrow -\iota\sigma_2^1\iota\sigma_2^2\iota\sigma_2^3.
\end{aligned}$$

Quantum game theory

- 1. Extension of classical game theory to the quantum regime

Motivation example:

A GAME THEORETIC APPROACH TO STUDY THE QUANTUM KEY DISTRIBUTION BB84 PROTOCOL, IJQI 2011, HOUSHMAND

EPR setting for Quantum games

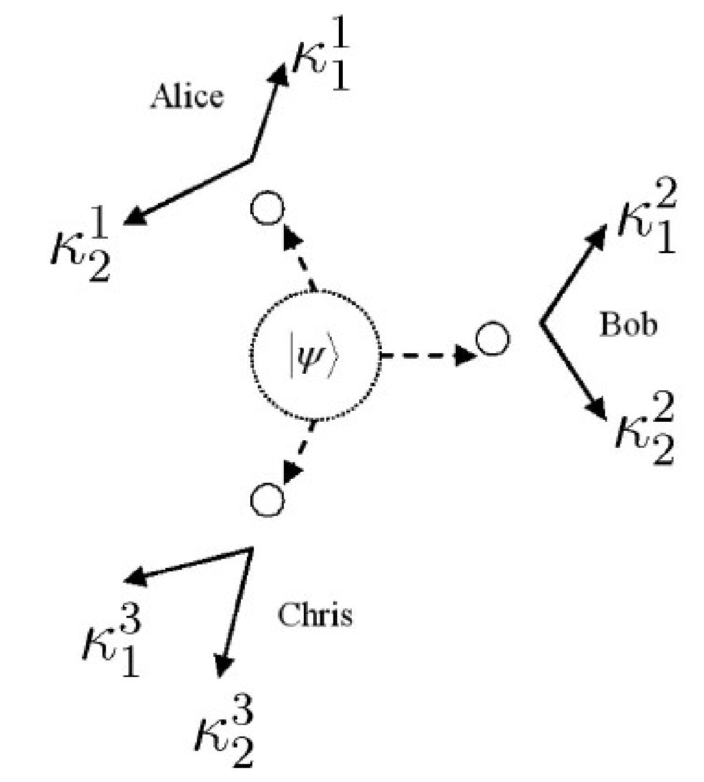


Figure 1. The EPR setup for three-player quantum game. A three-qubit entangled quantum state is distributed to the three players, who each choose between two possible measurement directions.

1. Naturally extends classical games as player choices remain classical

GHZ state

$$|\mathrm{GHZ}\rangle = \cos\frac{\gamma}{2}|000\rangle + \sin\frac{\gamma}{2}|111\rangle \longrightarrow \psi = ABC(\cos\frac{\gamma}{2} - \sin\frac{\gamma}{2}\iota\sigma_2^1\iota\sigma_2^2\iota\sigma_2^3)$$

$Ø=KLM$

$$K = e^{-\iota\kappa\sigma_2^1/2},\ L = e^{-\iota\kappa\sigma_2^2/2}, M = e^{-\iota\kappa\sigma_2^3/2}$$

Probability distribution

1. General algebraic expression for probability of outcomes

$$P(\psi,\phi) = 2^{N-2}[\langle \psi E \psi^\dagger \phi E \phi^\dagger \rangle_0 - \langle \psi J \psi^\dagger \phi J \phi^\dagger \rangle_0]$$

Where

$$E = \prod_{i=2}^{N} \frac{1}{2}(1 - \iota\sigma_3^1 \iota\sigma_3^i) = \frac{1}{4}(1 - \iota\sigma_3^1\iota\sigma_3^2 - \iota\sigma_3^1\iota\sigma_3^3 - \iota\sigma_3^2\iota\sigma_3^3)$$

$$J = E\iota\sigma_3^1 = \frac{1}{4}(\iota\sigma_3^1 + \iota\sigma_3^2 + \iota\sigma_3^3 - \iota\sigma_3^1\iota\sigma_3^2\iota\sigma_3^3).$$

Doran C, Lasenby A (2003) Geometric algebra for physicists

$\langle \rho Q \rangle_0 \sim \mathrm{Tr}[\rho Q]$

Phase structure for three-qubit EPR

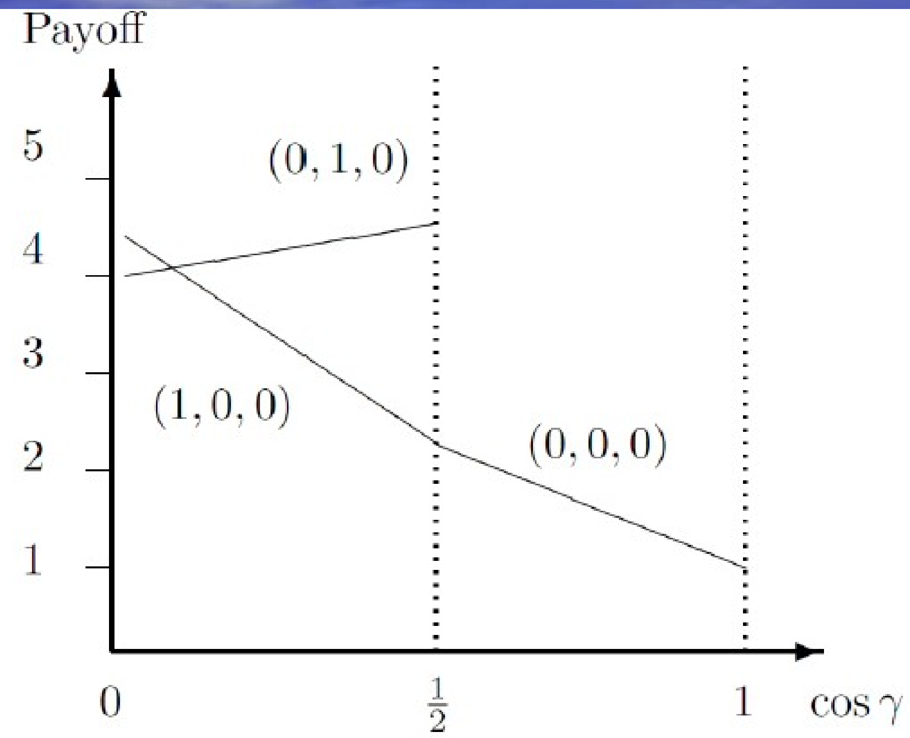


Figure 2. Phase structure for Alice in quantum PD game using EPR setting. For the PD

Table 1. An example of three-player Prisoners' Dilemma.

State	$\|000\rangle$	$\|001\rangle$	$\|010\rangle$	$\|100\rangle$	$\|011\rangle$	$\|101\rangle$	$\|110\rangle$	$\|111\rangle$
Payoff	(6, 6, 6)	(3, 3, 9)	(3, 9, 3)	(9, 3, 3)	(0, 5, 5)	(5, 0, 5)	(5, 5, 0)	(1, 1, 1)

References:

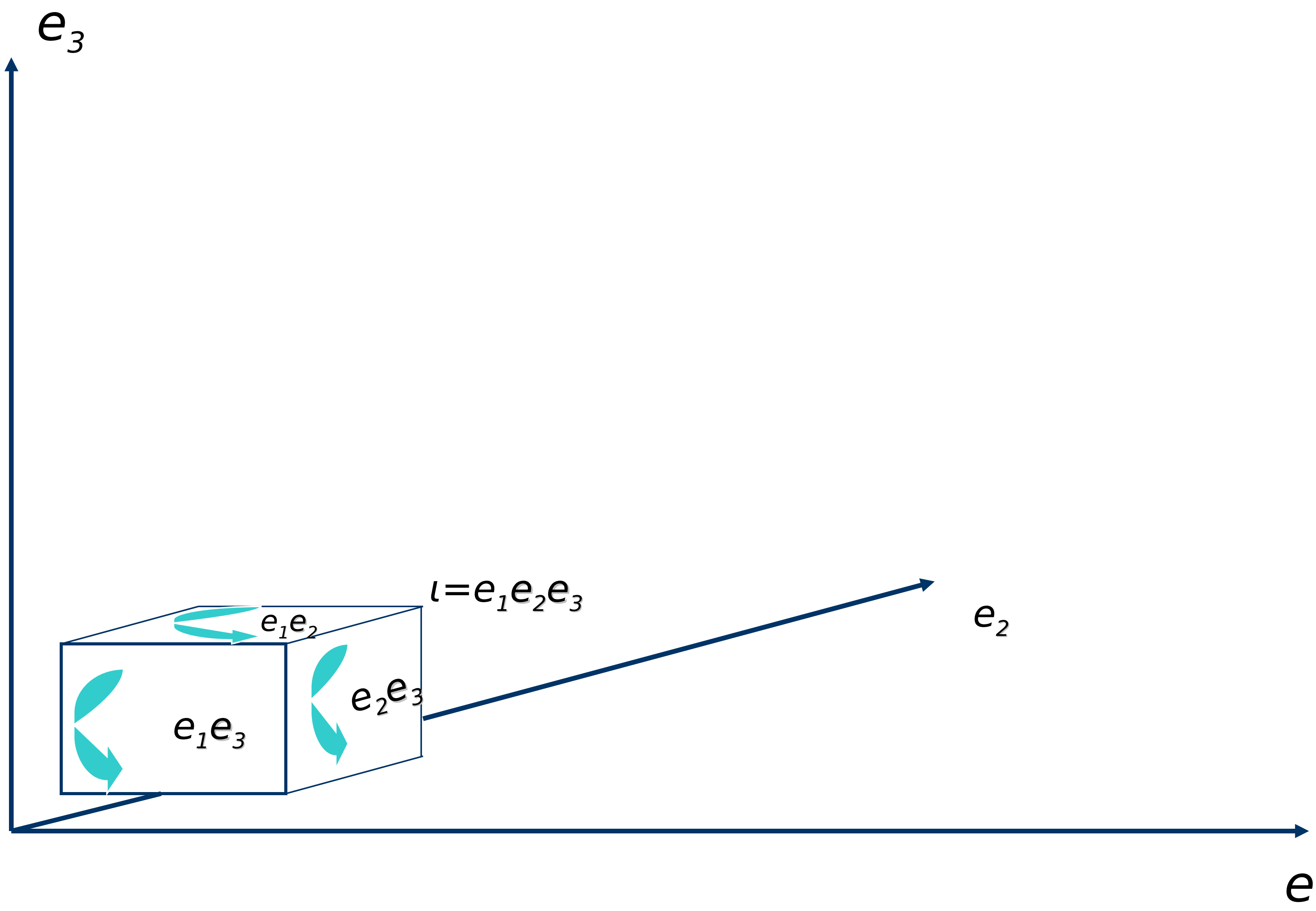
- Analysis of two-player quantum games in an EPR setting using Clifford's geometric algebra
- Analyzing three-player quantum games in an EPR type setup
- N player games...

Copies on arxiv, Chappell

Google: Cambridge university geometric algebra

Conclusion

- Natural description of 3D physical space
- Integration of complex numbers and quaternions into 3D Cartesian space
- Allows removal of unit imaginary from QM
- Analysis of EPR experiments within a real formalism
- Use EPR setting for quantum games which authentically extends the underlying classical game.



e3
ι=e1e2e3
e1e2
e2
e2e3
e1e3
e1

The geometric product magnitudes

$$
\begin{aligned}
ab &= a.b + a \wedge b \\
|a.b| &= |a||b| \cos\theta \\
|a \wedge b| &= |a||b| \sin\theta
\end{aligned}
$$

In three dimensions we have:

$$a \wedge b = \imath a \times b$$

Negative Numbers

- Interpreted financially as debts by Leonardo di Pisa,(A.D. 1170-1250)
- Recognised by Cardano in 1545 as valid solutions to cubics and quartics, along with the recognition of imaginary numbers as meaningful.
- Vieta, uses vowels for unknowns and use powers. Liebniz 1687 develops rules for symbolic manipulation.

Diophantus 200AD

Modern

$$x^3 - 2x^2 + 10x - 1 = 5$$

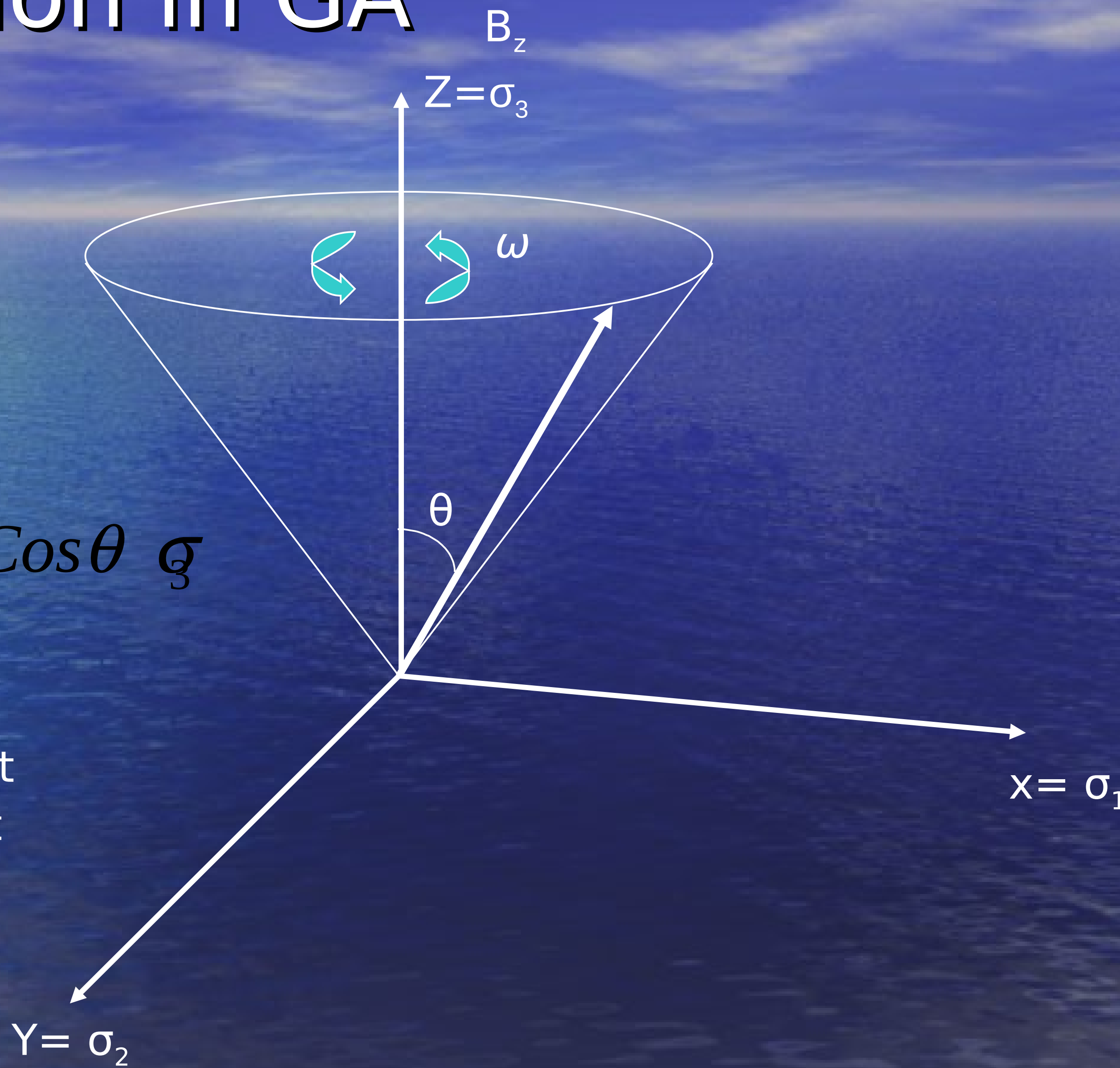
Precession in GA

Spin-1/2

$$R = e^{\omega t I z}$$

$$S = R v_0 \tilde{R}$$

$$v_0 = Sin\theta \;\; \sigma_1 + Cos\theta \;\; \sigma_3$$



$<S_x>=$ Sin θ Cos ω t

$<S_y>=$ Sin θ Sin ω t

$<S_z>=$ Cos θ

$\omega = \gamma B_z$

Greek concept of the product Euclid Book VII(B.C. 325-265)

"1. A **unit** is that by virtue of which each of the things that exist is called one."

"2. A **number** is a multitude composed of units."

"16. When two numbers having multiplied one another make some number, the number so produced is called plane, and its sides are the numbers which have multiplied one another."

Conventional Dirac Equation

$$-i\hbar\gamma^\mu\partial_\mu\psi + mc\psi = 0.$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -i \end{pmatrix}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

"Dirac has redisovered Clifford algebra..", Sommerfield

That is for Clifford basis vectors we have: $\{e_i, e_j\} = e_i e_j + e_j e_i = 2e_i \cdot e_j = 2\delta_{ij}$

isomorphic to the Dirac agebra.

Dirac equation in real space

$$\Box F = m F^{*} \iota e_3$$

$$F = a + E + \iota B + \iota b$$

$$\iota = e_1 e_2 e_3$$

ame as the free Maxwell equation, except for the addition
f a mass term and expansion of the field to a full multivector

Free Maxwell equation(*J=0*): $\Box F = J$

Special relativity

Its simpler to begin in 2D, which is sufficient to describe most phenomena. We define a 2D spacetime event as

$$X = x\iota + t\iota$$

So that time is represented as the bivector of the plane and so an extra Euclidean-type dimension is not required. This also implies 3D GA is sufficient describe 4D Minkowski spacetime.

We find: $X^2 = x^2 - t^2$ the correct spacetime distance.

We have the general Lorentz transformation given by:

$$X' = e^{-\phi\hat{v}/2} e^{-\iota\theta/2} X e^{\iota\theta/2} e^{\phi\hat{v}/2}$$

Consisting of a rotation and a boost, which applies uniformly to both coordinate and field multivectors.

$$P' = -\hat{\mathbf{v}}\mathrm{e}^{-\phi\hat{\mathbf{v}}/2} P \mathrm{e}^{\phi\hat{\mathbf{v}}/2}\hat{\mathbf{v}}$$

Compton scattering formula

Time after time

- “Of all obstacles to a thoroughly penetrating account of existence, none looms up more dismayingly than time.” Wheeler 1986
- In GA time is a bivector, ie rotation.
- Clock time and Entropy time

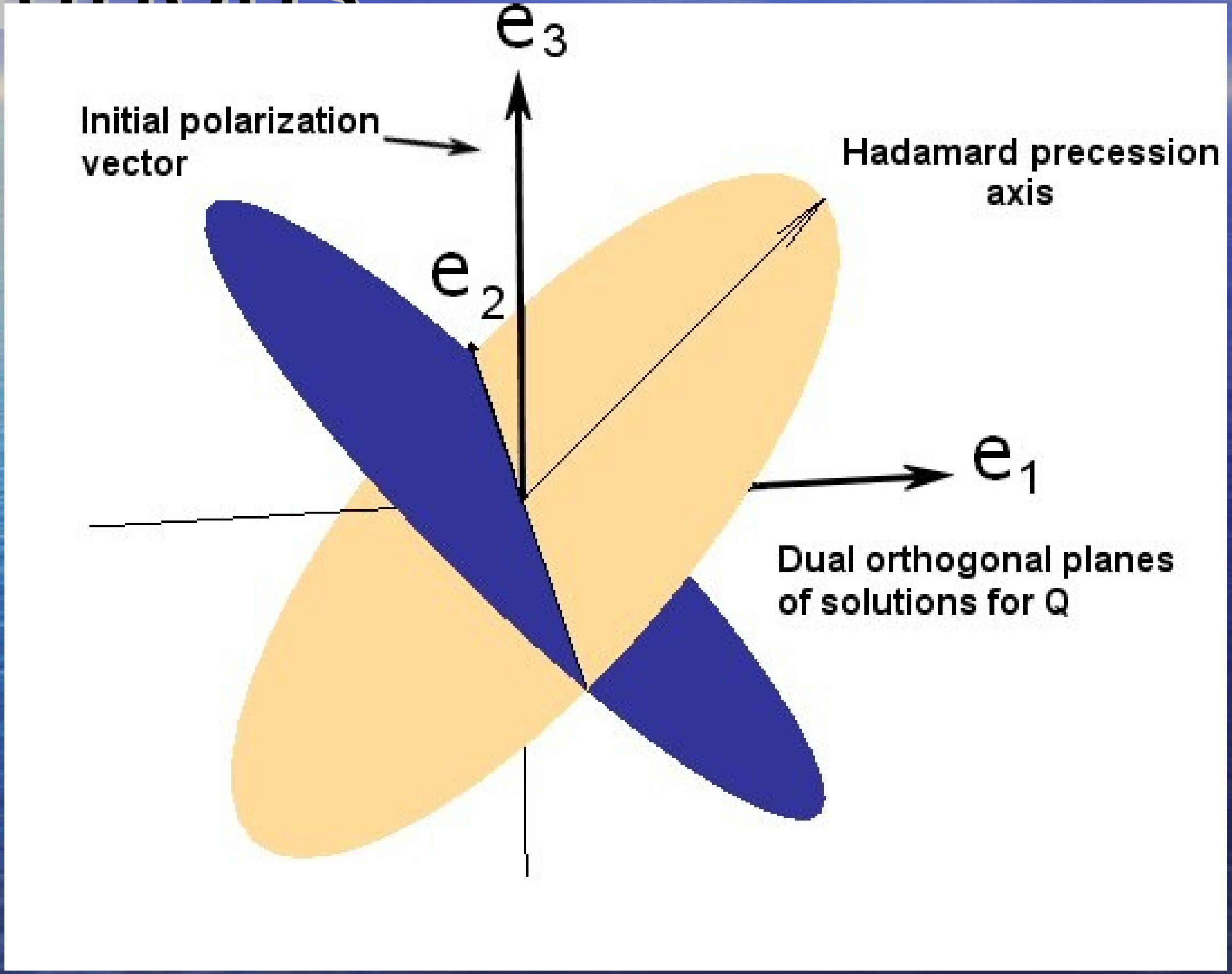
The versatile multivector (a generalized number)

$$M = a + v_1 e_1 + v_2 e_2 + v_3 e_3 + w_1 e_2 e_3 + w_2 e_3 e_1 + w_3 e_1 e_2 + b e_1 e_2 e_3$$

$$= a + v + \iota w + \iota b$$

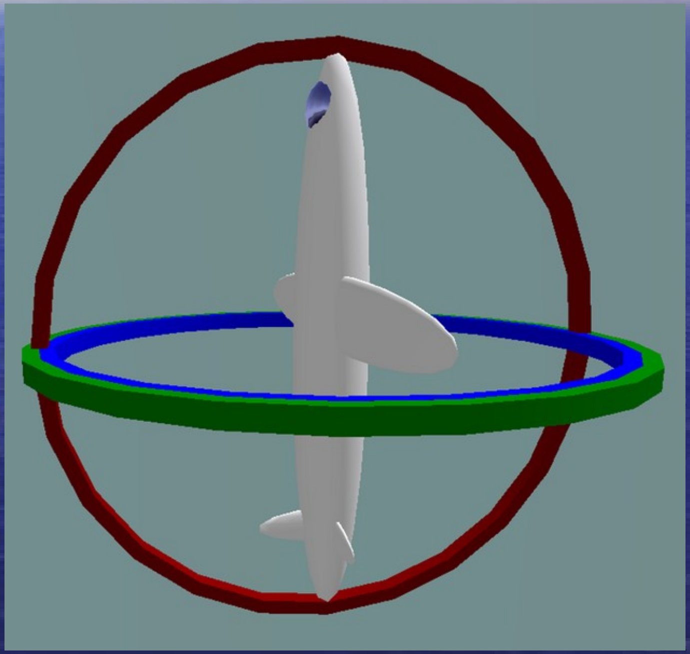
$a+\iota b$	Complex numbers
v	Vectors
ιw	Pseudovectors
ιb	Pseudoscalars
$v+\iota w$	Anti-symmetric EM field tensor $E+iB$
$a+\iota w$	Quaternions or Pauli matrices
$a+v$	Four-vectors

Penny Flip game Qubit Solutions



Use of quaternions

Used in airplane guidance systems to avoid Gimbal lock



How many space dimensions do we have?

- The existence of five regular solids implies three dimensional space(6 in 4D, 3 > 4D)
- Gravity and EM follow inverse square laws to very high precision. Orbits(Gravity and Atomic) not stable with more than 3 D.
- Tests for extra dimensions failed, must be sub-millimetre

Quotes

- “The reasonable man adapts himself to the world around him. The unreasonable man persists in his attempts to adapt the world to himself. Therefore, all progress depends on the unreasonable man.” George Bernard Shaw,
- Murphy’s two laws of discovery:

  “All great discoveries are made by mistake.”

  “If you don't understand it, it's intuitively obvious.”
- “It's easy to have a complicated idea. It's very hard to have a simple idea.” Carver Mead.