Quantum Communication and Quantum Information Splitting (QIS)

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Plan of the talk

- Quantum Entanglement.
- Different states.
- Quantum Communication.
- Non-Destructive Discrimination (NDD) and its examples and uses.
- Quantum Teleportation: Different modes.
- Dense Coding and Quantum Conversation.
- Non-Standard Probabilistic Teleportation
  - Through $|W\rangle$ and Quadripartite ($|P_1\rangle$ and $|P_1\rangle$) states.
- Quantum Information Splitting (QIS):
  - Procedure.
  - Theorems and lemmas.
  - Efficiency.

Quantum Entanglement and Teleportation

- Non-intuitive quantum correlations can exist between two or more particles.
- It enables different types of quantum communication protocols like teleportation, superdense coding, secret sharing, quantum cryptography, one-way quantum computation etc..
- Teleportation of arbitrary single qubit, through an entangled channel of EPR pair [Bennett et al.].
- Experimentally it has been achieved using different quantum systems.
- References : Bennett et al., Phys. Rev. Lett. 70, 1895 (1993); Bouwmeester et al., Nature (London) 390, 575 (1997).

3-qubit States

- Classification is done on the basis of stochastic local operations and classical communications (SLOCC).
  - GHZ state: $|GHZ\rangle_{123} = \frac{1}{\sqrt{2}}(|000\rangle_{123} + |111\rangle_{123}) \longrightarrow$ suitable for perfect teleportation.
  - W state: $|W\rangle_{123} = \frac{1}{\sqrt{3}}(|100\rangle_{123} + |010\rangle_{123} + |001\rangle_{123}) \longrightarrow$ not suitable for perfect teleportation.
- A new W-class of state: $|W_n\rangle_{123} = \frac{1}{\sqrt{2+2n}}(|100\rangle_{123} + \sqrt{n}e^{i\gamma}|010\rangle_{123} + \sqrt{n+1}e^{i\delta}\,|001\rangle_{123})$, is suitable for teleportation [n is real; γ, δ are phases].
- References : P. Agarwal, and A. K. Pati, Phys. Rev. A 74, 062320 (2006); S. Bandyopadhyay, Phys. Rev. A 62, 012308 (2000).

Magnon State

- Most general two magnon four qubit state: $\sum_{i=0}^{1} W_{ii\widetilde{i}}|ii\widetilde{i}\widetilde{i}\rangle + W_{i\widetilde{i}i}|i\widetilde{i}i\widetilde{i}\rangle + W_{\widetilde{i}ii}|i\widetilde{i}\widetilde{i}i\rangle$, ($i$ is a compliment of $\widetilde{i}$).

- Can teleport an arbitrary two qubit state deterministically if: $W_{110}^{*}W_{011}+W_{110}^{*}W_{001} = 0, |W_{101}|^2 = |W_{110}|^2+|W_{100}|^2 = |W_{001}|^2 = |W_{011}|^2+|W_{010}|^2$.

- $\longrightarrow$ Maximum entanglement.

- References: S. Prasath, S. Muralidharan, P. K. Panigrahi, and C. Mitra, 'eprint:quant-ph/0905.1233v2.

Five Qubit State

- [Brown et al., J. Phys. A. Math. Gen. 38(5), 1119 (2005)] Maximally entangled 5-qubit state by extensive numerical optimization:

  $|\psi_5\rangle = \frac{1}{2}[|001\rangle|\phi_-\rangle + |010\rangle|\psi_-\rangle + |100\rangle|\phi_+\rangle + |111\rangle|\psi_+\rangle]$,

  where $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\phi_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$.

- The von Neumann entropy between (1234—5) is equal to 1 and between (123—45) is 2 $\longrightarrow$ Teleportation.

- Reference: Muralidharan, S., Panigrahi, P.K., Phys. Rev. A 77, 032321 (2006).

Composite GHZ-Bell State

- Some of the Bell states are decoherence free under certain environment.
- Being the superposition of two terms, the GHZ state can be less prone to decoherence.
- Five-qubit composite GHZ-Bell state: $|\zeta\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{2}(|00000\rangle + |00011\rangle + |11100\rangle + |11111\rangle)$.
- This state has the maximum possible entanglement between two subsystem.
- Reference: Rao, D.D.B., Panigrahi, P.K., Mitra, C., Phys. Rev. A 78, 022336 (2008).

Composite GHZ-Bell State

- Five-qubit state can teleport one and two-qubit state deterministically.

- The Generalized '$2N+1$' qubit state can be represented by the product of one GHZ and $(N-1)$ number of Bell states :
  $|\zeta_0\rangle = |\xi^+\rangle_{AB}|\psi^+\rangle_{AB}........|\psi^+\rangle_{AB}$,
  where $|\xi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|0_A 0_A 0_B\rangle + |1_A 1_A 1_B\rangle)$ and $|\psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)$.

- For deterministic N-qubit teleportation, the basis states of Alice's system can be decomposed into GHZ-Bell pair.

- The capacity of superdense coding of this channel is (2N +1), satisfying 'Holevo bound'.

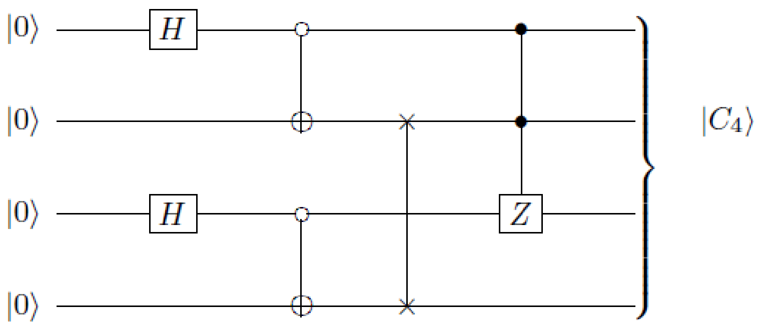
- References : D. Saha and P. K. Panigrahi, Quantum Inf. Process, DOI 10.1007/s11128-011-0270-x.

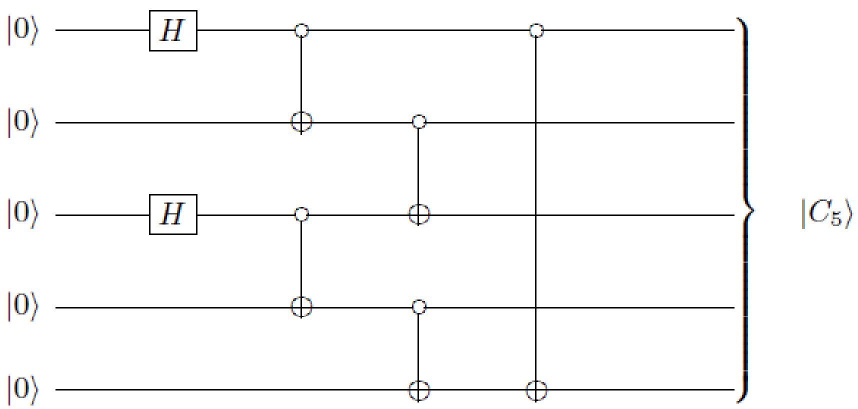
Quantum Communication

- All the earlier channels are useful for dense coding, as well as for QIS.
- It is found that the number of protocols for QIS are less for certain quantum channels then expected.
- Different quantum communication protocols [teleportation, state sharing and dense coding etc.] require entangled [e.g. Bell]bases.
- It can be used for quantum communication between two and more parties. For this purpose, we may require Non Destructive Discrimination (NDD).
- NDD of Bell states are analyzed [M. Gupta and P. K. Panigrahi, arXiv:quant-ph/0504183v1] and experimentally realized [J. R. Samal et al., J. Phys. B: At. Mol. Opt. Phys. 43 095508 (2010)].

Generation of cluster states:Q-Circuit

- Cluster states containing 4 and 5 particles can be generated through following circuitry:

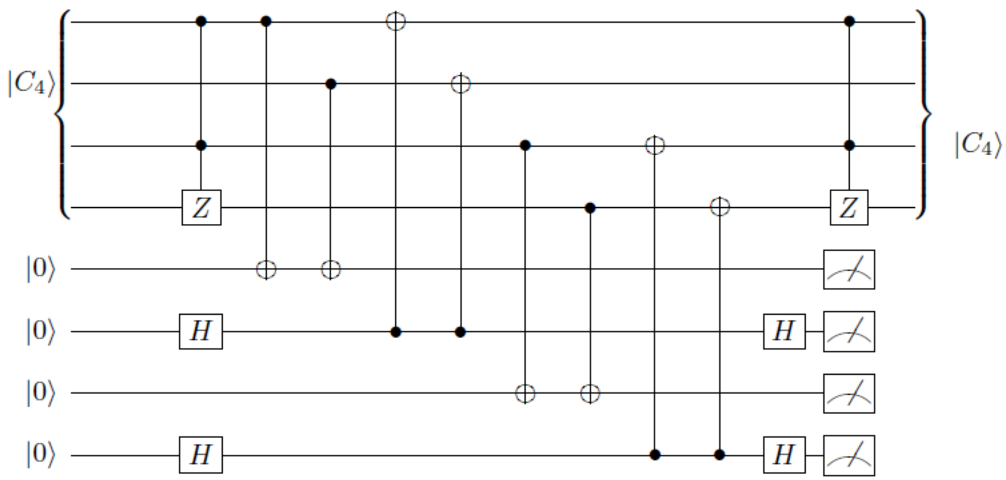




Non-Destructive Discrimination (NDD)

- Sixteen orthogonal cluster states are distinguished without disturbing: Measurements on four ancilla bits.

- Four qubit cluster states are made to interact with these ancillas:

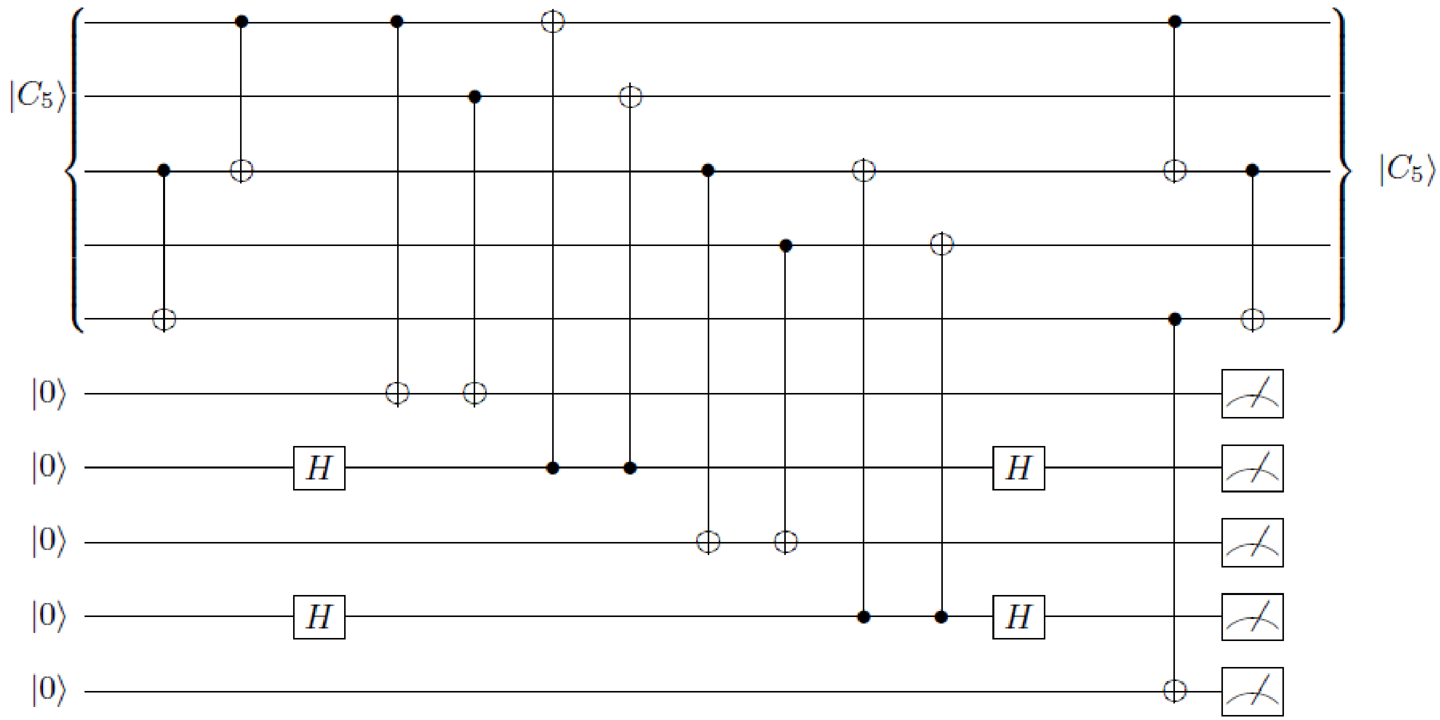


4-qubit NDD

TABLE I: Four-qubit Cluster State discrimination from the ancilla measurements

Ancilla Measurement	Corresponding four-qubit Cluster State
$\|0000\rangle$	$\|0000\rangle + \|0011\rangle + \|1100\rangle - \|1111\rangle$
$\|0001\rangle$	$\|0000\rangle - \|0011\rangle + \|1100\rangle + \|1111\rangle$
$\|0010\rangle$	$\|0001\rangle + \|0010\rangle - \|1101\rangle + \|1110\rangle$
$\|0011\rangle$	$\|0001\rangle - \|0010\rangle - \|1101\rangle - \|1110\rangle$
$\|0100\rangle$	$\|0000\rangle + \|0011\rangle - \|1100\rangle + \|1111\rangle$
$\|0101\rangle$	$\|0000\rangle - \|0011\rangle - \|1100\rangle - \|1111\rangle$
$\|0110\rangle$	$\|0001\rangle + \|0010\rangle + \|1101\rangle - \|1110\rangle$
$\|0111\rangle$	$\|0001\rangle - \|0010\rangle + \|1101\rangle + \|1110\rangle$
$\|1000\rangle$	$\|0100\rangle + \|0111\rangle + \|1000\rangle - \|1011\rangle$
$\|1001\rangle$	$\|0100\rangle - \|0111\rangle + \|1000\rangle + \|1011\rangle$
$\|1010\rangle$	$\|0101\rangle + \|0110\rangle - \|1001\rangle + \|1010\rangle$
$\|1011\rangle$	$\|0101\rangle - \|0110\rangle - \|1001\rangle - \|1010\rangle$
$\|1100\rangle$	$\|0100\rangle + \|0111\rangle - \|1000\rangle + \|1011\rangle$
$\|1101\rangle$	$\|0100\rangle - \|0111\rangle - \|1000\rangle - \|1011\rangle$
$\|1110\rangle$	$\|0101\rangle + \|0110\rangle + \|1001\rangle - \|1010\rangle$
$\|1111\rangle$	$\|0101\rangle - \|0110\rangle + \|1001\rangle + \|1010\rangle$

5-qubit NDD



$|C_5\rangle$
$|C_5\rangle$
$|0\rangle$
$|0\rangle$
$|0\rangle$
$|0\rangle$
$|0\rangle$
H
H
H
H

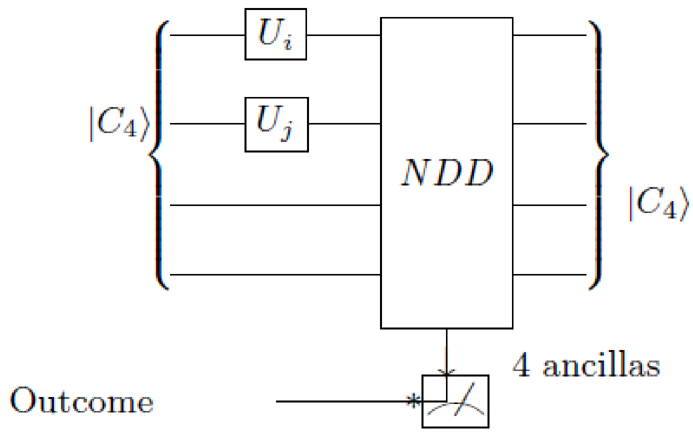
5-qubit NDD

TABLE II: Five qubit cluster state discrimination from the ancilla measurements

Ancilla Measurement	Corresponding five qubit Cluster State
$\|00000\rangle$	$\|00000\rangle + \|00111\rangle + \|11101\rangle + \|11010\rangle$
$\|00001\rangle$	$\|00000\rangle - \|00111\rangle + \|11101\rangle - \|11010\rangle$
$\|00010\rangle$	$\|00010\rangle + \|00101\rangle + \|11111\rangle - \|11000\rangle$
$\|00011\rangle$	$\|00010\rangle - \|00101\rangle + \|11111\rangle - \|11000\rangle$
$\|00100\rangle$	$\|00000\rangle + \|00111\rangle - \|11101\rangle - \|11010\rangle$
$\|00101\rangle$	$\|00000\rangle - \|00111\rangle - \|11101\rangle + \|11010\rangle$
$\|00110\rangle$	$\|00010\rangle + \|00101\rangle - \|11111\rangle - \|11000\rangle$
$\|00111\rangle$	$\|00010\rangle - \|00101\rangle - \|11111\rangle + \|11000\rangle$
$\|01000\rangle$	$\|01000\rangle + \|01111\rangle + \|10101\rangle + \|10010\rangle$
$\|01001\rangle$	$\|01000\rangle - \|01111\rangle + \|10101\rangle - \|10010\rangle$
$\|01010\rangle$	$\|01010\rangle + \|01101\rangle + \|10111\rangle + \|10000\rangle$
$\|01011\rangle$	$\|01010\rangle - \|01101\rangle + \|10111\rangle - \|10000\rangle$
$\|01100\rangle$	$\|01000\rangle + \|01111\rangle - \|10101\rangle - \|10010\rangle$
$\|01101\rangle$	$\|01000\rangle - \|01111\rangle - \|10101\rangle + \|10010\rangle$
$\|01110\rangle$	$\|01101\rangle + \|01010\rangle - \|10000\rangle - \|10111\rangle$
$\|01111\rangle$	$\|01010\rangle - \|01101\rangle - \|10111\rangle + \|10000\rangle$
$\|10000\rangle$	$\|00001\rangle + \|00110\rangle + \|11100\rangle + \|11011\rangle$
$\|10001\rangle$	$\|00001\rangle - \|00110\rangle + \|11100\rangle - \|11011\rangle$
$\|10010\rangle$	$\|00011\rangle + \|00100\rangle + \|11110\rangle + \|11001\rangle$
$\|10011\rangle$	$\|00011\rangle - \|00100\rangle + \|11110\rangle - \|11001\rangle$
$\|10100\rangle$	$\|00001\rangle + \|00110\rangle - \|11100\rangle - \|11011\rangle$
$\|10101\rangle$	$\|00001\rangle - \|00110\rangle - \|11100\rangle + \|11011\rangle$
$\|10110\rangle$	$\|00011\rangle + \|00100\rangle - \|11110\rangle - \|11001\rangle$
$\|10111\rangle$	$\|00011\rangle - \|00100\rangle - \|11110\rangle + \|11001\rangle$
$\|11000\rangle$	$\|01001\rangle + \|01110\rangle + \|10100\rangle + \|10011\rangle$
$\|11001\rangle$	$\|01001\rangle - \|01110\rangle + \|10100\rangle - \|10011\rangle$
$\|11010\rangle$	$\|01011\rangle + \|01100\rangle + \|10110\rangle + \|10001\rangle$
$\|11011\rangle$	$\|01011\rangle - \|01100\rangle + \|10110\rangle - \|10001\rangle$
$\|11100\rangle$	$\|01001\rangle + \|01110\rangle - \|10100\rangle - \|10011\rangle$
$\|11101\rangle$	$\|01001\rangle - \|01110\rangle - \|10100\rangle + \|10011\rangle$
$\|11110\rangle$	$\|01011\rangle + \|01100\rangle - \|10110\rangle - \|10001\rangle$
$\|11111\rangle$	$\|01011\rangle - \|01100\rangle - \|10110\rangle + \|10001\rangle$

Quantum Dialogue using NDD

- Holevo bound: Four classical bits transmitted by sending two qubits and five classical bits transmitted by sending three qubits respectively.
- Alice encodes four classical bits into two qubits by performing a $(\sigma_i \otimes \sigma_j)$ operation, $i, j \in (0, 1, 2, 3)$.
- On receiving the qubits from Alice, Bob performs NDD to decode:



Quantum Dialogue using NDD

- Bob can send two qubits of $|C_4\rangle$ to Alice and encode four cbits of information by operating on her two qubits and send them to Alice.
- Alice can decode this information through NDD and the dialogue continues without destroying the entanglement.
- Same can be done using $|C_5\rangle$.
- Position of errors [bit flip and phase flip] in initial cluster states can be known: No cluster basis measurement required.

Dense Coding Protocol

- Alice and Bob possess, the first three and the last two qubits of $|\psi_5\rangle$ respectively.

- To send a secret message "$m_1m_2m_3m_4m_5$" to Bob, where $m_i \in \{0, 1\}$, Alice operates unitarily on her first three qubits:

$$((\zeta^1.\zeta^2.\zeta^3.\zeta^4.\zeta^5) \otimes \mathbf{I} \otimes \mathbf{I})|\psi_5\rangle = |\psi_5\rangle_{\mathbf{m_1m_2m_3m_4m_5}}. \quad (1)$$

where,

$$\begin{aligned}
\zeta^1 &= \sigma_x \otimes I \otimes I, \quad m_1 = 1; \qquad I \otimes I \otimes I, \quad m_1 = 0,\\
\zeta^2 &= I \otimes I \otimes \sigma_x, \quad m_2 = 1; \qquad I \otimes I \otimes I, \quad m_2 = 0,\\
\zeta^3 &= \sigma_z \otimes \sigma_z \otimes I, \quad m_3 = 1; \qquad I \otimes I \otimes I, \quad m_3 = 0,\\
\zeta^4 &= \sigma_x \otimes I \otimes \sigma_x, \quad m_4 = 1; \qquad I \otimes I \otimes I, \quad m_4 = 0,\\
\zeta^5 &= I \otimes \sigma_x \otimes \sigma_x, \quad m_5 = 1; \qquad I \otimes I \otimes I, \quad m_5 = 0.
\end{aligned} \quad (2)$$

- Example: For 5-bit classical information (10010),

$$(((\sigma_x \otimes \mathbf{I} \otimes \mathbf{I})(\mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I})(\mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I})(\sigma_x \otimes \mathbf{I} \otimes \sigma_x)(\mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I})) \otimes \mathbf{I} \otimes \mathbf{I})|\psi_5\rangle = (|\psi_5\rangle_{100010}). \quad (3)$$

  and then sends the three qubits to Bob.

- Bob performs a joint five partite von-Neumann measurement in $|\psi_5\rangle_{m_1m_2m_3m_4m_5}$ basis and distinguishes these states, thereby obtaining the message encoded by Alice.

- Alice first prepares an ordered sequence of N copies of the five qubit Brown *et al.*, state

  $|\psi_5\rangle$: $[(q_1^1,\ q_2^1,\ q_3^1,\ q_4^1,\ q_5^1)),\ (q_1^2,\ q_2^2, \ldots\ldots,\ q_5^N)\ ]$.

- Alice then takes the same one qubit from each $|\psi_5\rangle$ to form five ordered sequences corresponding to the five qubits

  $S_k = [q_k^1, q_k^2, q_k^3 \ldots . q_k^N]$.

- she keeps the particle sequences S_1, S_2, S_3 and transmits the sequences S_4 and S_5 to Bob.

Measurement basis of Alice and Bob

- $|\psi_5\rangle$ can be written down in two different forms:

$$
\begin{aligned}
|\psi_5\rangle_{12345} &= \frac{1}{2}(|001\rangle|\phi_-\rangle + |010\rangle|\psi_-\rangle + |100\rangle|\phi_+\rangle + |111\rangle|\psi_+\rangle)_{12345},\\
|\psi_5\rangle_{13245} &= ([|\phi_+\rangle|1\rangle + |\psi_+\rangle|0\rangle]|0+\rangle + [|\phi_+\rangle|1\rangle - |\psi_+\rangle|0\rangle]|0-\rangle)_{13245}\\
&\quad -[|\phi_-\rangle|1\rangle + |\psi_-\rangle|0\rangle]|1+\rangle + [|\phi_-\rangle|1\rangle - |\psi_-\rangle|0\rangle]|1-\rangle. \qquad (4)
\end{aligned}
$$

  $\rightarrow$ to check the presence of an eavesdropper.

- Measuring the qubits 4 and 5 by projection on the Bell basis, Alice's first three qubits collapse into one of the computational basis:

  $(|001\rangle, |010\rangle, |100\rangle, |111\rangle)_{123}$, $\rightarrow$ Basis of Alice's measurement

- For the other choice of $|\psi_5\rangle$, if Bob measures his two qubits in the basis $|0+\rangle_{45}, |0-\rangle_{45}, |1+\rangle_{45}, |1-\rangle_{45}$, corresponding Alice's measurement basis would be:

  $(|\phi_+\rangle|1\rangle+|\psi_+\rangle|0\rangle)_{132}$, $(|\phi_+\rangle|1\rangle-|\psi_+\rangle|0\rangle)_{132}$, $(-|\phi_-\rangle|1\rangle$ - $|\psi_-\rangle|0\rangle)_{132}$ and $(|\phi_-\rangle|1\rangle-|\psi_-\rangle|0\rangle)_{132}$.

- Thus, Alice and Bob can guard against eavesdropping.

Dense coding, NDD and quantum conversation

- The secret message "$m_1m_2m_3m_4m_5$" is encoded by Alice in 3 qubits [dense coding] and sent to Bob.

- Bob applies NDD on the 5 qubit Brown et al. state and decodes the message without disturbing the entanglement.

- The circuit outcome with input $|\psi_5\rangle_{\mathbf{m_1m_2m_3m_4m_5}}$ in general is $|\psi_5\rangle_{\mathbf{x}}|\mathbf{x}\rangle$ where $|\mathbf{x}\rangle$ is given by, $|\mathbf{m_1m_2m_3m_4m_5}\rangle$ and $|\psi_5\rangle_{\mathbf{x}}$ is given by, $|\psi\rangle_{\mathbf{m_1m_2m_3m_4m_5}}$.

- Bob now measures the ancillas in the product basis and thereby obtains the classical message.

- Bob encodes his message in the first 3 qubits unitarily, and sends the 5 qubits to Alice in two sets of (4,5) and (1,2,3).

- The cycle is established.

Six Qubit State

- Borras et. al. [J. Phys. A. Math. Gen. 21]: Genuinely entangled 6-qubit state non-decomposable into pairs of Bell states.

  $|\psi_6\rangle = \frac{1}{4}[|000\rangle(|0\rangle|\phi_+\rangle + |1\rangle|\psi_+\rangle) + |001\rangle(|0\rangle|\psi_-\rangle - |1\rangle|\phi_-\rangle)$

  $+|010\rangle(|0\rangle|\psi_+\rangle - |1\rangle|\phi_+\rangle) + |011\rangle(|0\rangle|\phi_-\rangle + |1\rangle|\psi_-\rangle)+$

  $|100\rangle(-|0\rangle|\psi_-\rangle - |1\rangle|\phi_-\rangle) + |101\rangle(-|0\rangle|\phi_+\rangle + |1\rangle|\psi_-\rangle)$

  $|110\rangle(|0\rangle|\phi_-\rangle - |1\rangle|\psi_-\rangle) + |111\rangle(|0\rangle|\psi_+\rangle + |1\rangle|\phi_+\rangle)],$

  where $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\phi_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$.

- Reference: Choudhury, S., Muralidharan, S., Panigrahi, J. Phys. A. Math. Theor. 42, 115303 (2009).

Non-Standard Probabilistic Teleportation protocol

- A non-standard teleportation scheme is proposed, wherein probabilistic teleportation is achieved in conventionally non-teleporting channels.

- We make use of entanglement monogamy to incorporate an unknown state in a multipartite entangled channel, such that the receiver partially gets disentangled from the network.

- the sender performs local measurement based teleportation protocol in an appropriate measurement basis, which results with the receiver in the possession of an unknown state, connected by local unitary transformation with the state to be teleported.

- This procedure succeeds in a number of cases, like that of W and other non-maximally entangled four qubit states, where the conventional measurement based approach has failed.

Comparison between Conventional and non Conventional Approach

- Conventional Approach
  - Let us consider a Bell state of the form $[|\eta\rangle = \frac{1}{\sqrt{2}}[|0_A0_B\rangle + |1_A1_B\rangle]$.
  - Addition of single qubit unknown state to the Bell state at Alice's Side, $|\eta'\rangle = \frac{1}{\sqrt{2}}[\alpha|0_A0_A0_B\rangle + \alpha|0_A1_A1_B\rangle + \beta|1_A0_A0_B\rangle + \beta|1_A1_A1_B\rangle]$.
  - Now rearranging the above state,we get

$$\begin{aligned} |\eta'\rangle = |\phi_A^+\rangle[\alpha|0_B\rangle + \beta|1_B\rangle] + |\phi_A^-\rangle[\alpha|0_B\rangle - \beta|1_B\rangle] \\ +|\psi_A^+\rangle[\alpha|1_B\rangle + \beta|0_B\rangle] \\ +|\psi_A^+\rangle[\alpha|1_B\rangle - \beta|0_B\rangle]] \end{aligned} \quad (5)$$

    where, the $\{|\phi_A^{\pm}\rangle, |\psi_A^{\pm}\rangle\}$ forms a Bell-Basis.
  - Therefore a measurement in Bell Basis on Alice's side will lead to a state i.e. unitarily equivalent to the unknown state, she wanted to send.

Non - Conventional Approach

- In this Case also consider a Bell state of the form: $|\eta\rangle = \frac{1}{\sqrt{2}}[|0_A 0_B\rangle + |1_A 1_B\rangle]$.

- Similarly, addition of single qubit unknown state to the Bell state at Alice's Side, we get: $|\eta'\rangle = \frac{1}{\sqrt{2}}[\alpha|0_A 0_A 0_B\rangle + \alpha|0_A 1_A 1_B\rangle + \beta|1_A 0_A 0_B\rangle + \beta|1_A 1_A 1_B\rangle]$

- Now Alice applies C-NOT $1 \rightarrow 2$ and subsequently Hadamard on 1 qubit,we get

$$|\eta'\rangle = \frac{1}{2}[|00_A\rangle[\alpha|0_B\rangle + \beta|1_B\rangle] + |01_A\rangle[\alpha|0_B\rangle - \beta|1_B\rangle] + |10_A\rangle[\alpha|1_B\rangle + \beta|0_B\rangle] + |11_A\rangle[\alpha|1_B\rangle - \beta|0_B\rangle]] \quad (6)$$

where, the $\{|00_A\rangle, |01_A\rangle, |10_A\rangle, |11_A\rangle\}$ forms a two qubit computational basis.

Probabilistic Teleportation of single qubit unknown state through $|W\rangle$ state

- Consider a W-state (particle 2 and 3 belongs to Alice, 4 belongs to Bob) - $|W\rangle_{234} = \frac{1}{\sqrt{3}}[|100\rangle_{234} + |010\rangle_{234} + |001\rangle_{234}]$

- Addition of unknown state at Alice's end:

$$|\Psi\rangle_1 \otimes |W\rangle_{234} = \frac{1}{\sqrt{3}}[\alpha|0100\rangle_{1234} + \alpha|0010\rangle_{1234} + \alpha|0001\rangle_{1234} \tag{7}$$
$$+ \beta|1100\rangle_{1234}$$
$$+\beta|1010\rangle_{1234} + \beta|1001\rangle_{1234}]. \tag{8}$$

- Initially, Alice applies a control-NOT $1 \rightarrow 2$ and subsequently a Hadamard gate on the first qubit :

$$\begin{aligned}|\Psi''\rangle_{1234} = \frac{1}{\sqrt{6}}[&|010\rangle_{123}(\alpha|0\rangle_4 + \beta|1\rangle_4) + |000\rangle_{123}(\alpha|1\rangle_4 + \beta|0\rangle_4) \\ +&|110\rangle_{123}(\alpha|0\rangle_4 - \beta|1\rangle_4) + |100\rangle_{123}(\alpha|1\rangle_4 - \beta|0\rangle_4) + \alpha|001\rangle_{123}|0\rangle_4 \\ +&\alpha|101\rangle_{123}|0\rangle_4 + \beta|011\rangle_{123}|0\rangle_4 - \beta|111\rangle_{123}|0\rangle_4].\end{aligned} \tag{9}$$

Probabilistic Teleportation of single qubit unknown state through $|W\rangle$ state

- At last ,Alice performs a three particle measurement on her qubits in the computational basis.
- And communicates the obtained result to Bob, on the other hand he carries out the required operations on his qubit to get the desired state.
- Note:- It is evident that teleportation is possible, only if the measurement outcomes are $|010\rangle_{123}$, $|000\rangle_{123}$, $|110\rangle_{123}$ and $|100\rangle_{123}$ and fails completely if the outcomes are $|001\rangle_{123}$, $|101\rangle_{123}$, $|011\rangle_{123}$ and $|111\rangle_{123}$. Hence probability of teleportation is half.

2-qubit non-maximally entangled unknown state through $|W\rangle$

- Probabilistic teleportation of unknown two qubit entangled state can be done through W-state as the entangled channel: Bob measures one of his particles before unitary transformation [Cao et al.]

- Teleportation occurs without the interaction of Alice's qubits with Bob's qubits. Although general unknown 2-qubit state cannot be teleported using W-state [Muralidharan et al.], probabilistic teleportation of non-maximally entangled two qubit state: $|\Psi\rangle_{12} = \alpha|00\rangle + \beta[|01\rangle + |10\rangle]$ is possible ($|\alpha|^2 + 2|\beta|^2 = 1$).

- A 4-qubit W-state of the type (particle 3 with Alice and 4,5 with Bob):

$$
\begin{aligned}
|\Psi\rangle_{12} \otimes |W\rangle_{345} = \frac{1}{\sqrt{3}}[&\alpha|00001\rangle_{12345} + \alpha|00010\rangle_{12345} + \alpha|00100\rangle_{12345} \\
&+ \beta|10001\rangle_{12345} + \beta|10010\rangle_{12345} + \beta|10100\rangle_{12345} \\
&+ \beta|01001\rangle_{12345} + \beta|01010\rangle_{12345} + \beta|01100\rangle_{12345}]. \quad (10)
\end{aligned}
$$

2-qubit non-maximally entangled unknown state through $|W\rangle$

- Alice applies a control-NOTs $2 \rightarrow 3$ and $1 \rightarrow 3$ subsequently and measures the qubit 3 in computational basis:

$$\langle 1_3|\Psi\prime\rangle_{12345} = \frac{1}{\sqrt{\alpha^2+4\beta^2}}[\alpha|0000\rangle_{1245} + \beta|1010\rangle_{1245} + \beta|0100\rangle_{1245} + \beta|0110\rangle_{1245} + \beta|1001\rangle_{1245}], \quad (11)$$

or

$$\langle 0_3|\Psi\prime\rangle_{12345} = \frac{1}{\sqrt{2\alpha^2+2\beta^2}}[\alpha|0001\rangle_{1245} + \alpha|0010\rangle_{1245} + \beta|1000\rangle_{1245} + \beta|0100\rangle_{1245}]. \quad (12)$$

- It can be seen that, teleportation is possible if the measurement outcome is $|1_3\rangle$.

2-qubit non-maximally entangled unknown state through $|W\rangle$

- Alice applies a Hadamard on qubit 1 and then measures qubit 2 in computational basis, leading to two outcomes:

$$\langle 0_2|\Psi''\rangle_{1245} = \frac{1}{\sqrt{2}}[(|0\rangle_1)(\beta|01\rangle_{45} + \beta|10\rangle_{45}) - (|1\rangle_1)(\beta|01\rangle_{45} + \beta|10\rangle_{45}) + (|0\rangle_1)(\alpha|00\rangle_{45}) + (|1\rangle_1)(\alpha|00\rangle_{45})] \quad (13)$$

or

$$\langle 1_2|\Psi''\rangle_{1245} = \frac{1}{2\beta^2}[(|0\rangle_1)(\beta|00\rangle_{45} + \beta|10\rangle_{45}) + (|1\rangle_1)(\beta|00\rangle_{45} + \beta|10\rangle_{45})]. \quad (14)$$

- Teleportation is possible if measurement outcome is $|0_2\rangle$ and fails if it is $|1_2\rangle$. First outcome: Alice measures qubit 1 in computational basis and sends the results to Bob, who makes local operations to get the state.

- If the outcome of Alice's measurement is $|1\rangle$, then Bob applies a unitary transformation to obtain the desired state.

2-qubit non-maximally entangled unknown state through $|W\rangle$

- Probabilistic teleportation of partially entangled two qubits have been implemented probabilistically.

- If one considers $|W\rangle_{234} = \frac{1}{\sqrt{3}}[|101\rangle_{234} + |110\rangle_{234} + |011\rangle_{234}]$ as entangled channel, then the state that can be teleported is $|\Psi\rangle_{12} = \alpha|11\rangle_{12} + \beta[|01\rangle_{12} + |10\rangle_{12}]$.

- **W-State Recovery** :- It is interesting to note that, even if she fails to teleport through non-maximally entangled unknown two qubit state, Alice can regain this unknown two qubit state, with certain probability, by classical communication to Bob.

- Bob then applies a Hadamard operation on particle 5, followed by a measurement of his two particles in computational basis.

- With the help of Bob, we can check that Alice is able to recover the w-state again probabilistically.

Teleportation Through Quadripartite State

- On SLOCC basis classification, 4 particle states are classified into different types. a subset of which is investigated for possibility of implementing the present protocol.

- These states are:

$$|P_1\rangle \equiv |W\rangle = \frac{1}{2}[|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle],$$
$$|P_2\rangle = \frac{1}{\sqrt{5}}[|0000\rangle + |1111\rangle + |0011\rangle + |0101\rangle + |0110\rangle], \quad (15)$$

  Teleportation of unknown 1 qubit state through $|P_1\rangle$ and $|P_2\rangle$ and 2 qubit non-maximally entangled state through $|P_1\rangle$ and $|P_2\rangle$, conventional measurement based approach has failed, whereas the present protocol results in a probabilistic teleportation.

- We start with,

$$|P_1\rangle_{3456} = [\frac{1}{2}][|0001\rangle_{3456} + |0010\rangle_{3456} + |0100\rangle_{3456} + |1000\rangle_{3456}], \quad (16)$$

  where the particles 3, 4 and 5 belong to Alice and the particle 6 belongs to Bob.

Teleportation Through quadripartite states $|P_1\rangle$

- Addition of unknown state of 1 qubit at the Alice's end yields,

$$\begin{aligned}|\Psi\rangle_1 \otimes |P_1\rangle_{3456} = \frac{1}{2}[&\alpha|00001\rangle_{13456} + \alpha|00010\rangle_{13456} + \alpha|00100\rangle_{13456}\\ &+\alpha|01000\rangle_{13456} + \beta|10001\rangle_{13456} + \beta|10010\rangle_{13456}\\ &+\beta|10100\rangle_{13456} + \beta|11000\rangle_{13456}].\end{aligned} \quad (17)$$

- The above state is not expressible in orthogonal measurement basis for Alice's end: unsuitable for standard teleportation. Probabilistic teleportation can be done by using local operations. Alice applies control-NOT with 1 as a control qubit and 4 as a target qubit and then another with 2 as a control qubit and 3 as a target:

$$\begin{aligned}|P_1\prime\rangle_{13456} = \frac{1}{2}[&\alpha|00001\rangle_{13456} + \alpha|00010\rangle_{13456} + \alpha|00100\rangle_{13456}\\ &+\alpha|01100\rangle_{13456} + \beta|10011\rangle_{13456} + \beta|10000\rangle_{13456}\\ &+\beta|10110\rangle_{13456} + \beta|11110\rangle_{13456}],\end{aligned} \quad (18)$$

  Now Alice measures 1, 2 and 3, 4 successively in Bell basis to get the probabilistic result.

Teleportation Through $|P_2\rangle$ state

- We consider $|P_2\rangle$ state as the entangled channel:

$$|P_2\rangle = \frac{1}{\sqrt{5}}[|0000\rangle_{3456} + |1111\rangle_{3456} + |0011\rangle_{3456} + |0101\rangle_{3456} + |0110\rangle_{3456}]. \tag{19}$$

  Here the particles 3,4 and 5 belong to Alice and the particle 6 belongs to Bob.

- After the addition of unknown state of 1 qubit at the Alice's end and application of a control-NOT with 4 as target qubit and 1 as control qubit and a Hadamard on 1. Subsequently, Alice measures qubit 3 through von Neumann measurement.

- Teleportation is possible if the result is $|0_3\rangle$ and fails if it is $|1_3\rangle$. Alice performs a measurement of qubit 4 in computational basis, and measures qubit 1 and 5 together in computational basis ($|00_{15}\rangle$, $|10_{15}\rangle$, $|01_{15}\rangle$ and $|11_{15}\rangle$) and sends the information to Bob. Bob performs unitary transformation to get the desired state.

Quantum Information Splitting (**QIS**)

- Was first demonstrated using 3 qubit GHZ state: $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ as a shared entangled resource.

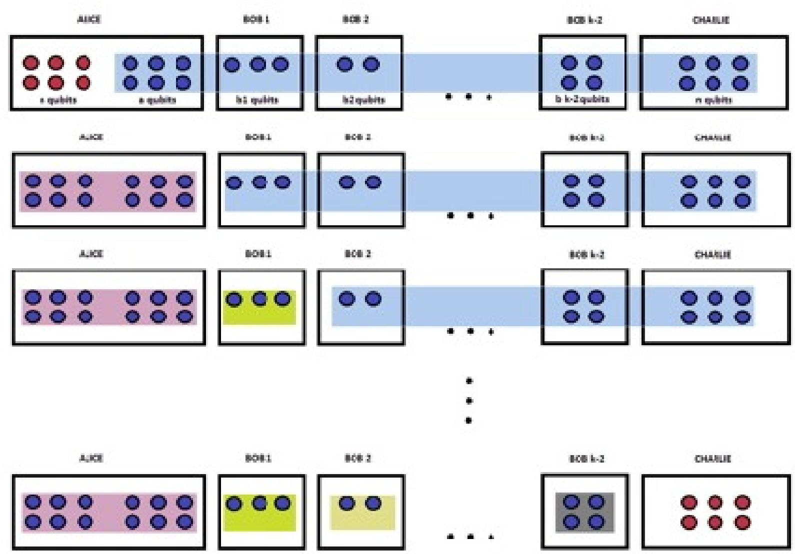
- The 4-qubit state can be used for QIS with only one protocol. The 5-qubit state can be used for single qubit QIS through 3 different protocols, whereas only one protocol is feasible for two-qubit QIS.

- Splitting quantum information among participants using N qubit entangled channel.

- Characterizing entangled states for quantum networking protocols.

- Reference: S. Muralidharan, S. Karumanchi, S. Narayanaswamy. R. Srikanth, and P. K. Panigrahi, eprint quant-ph/0907.3532v2.

Quantum Information Splitting



Pictorial representation of QIS among k parties

Quantum Information Splitting

- Multipartite entangled system is complex $\longrightarrow$ More than one way of splitting and sharing.

$$|\psi_1\rangle \otimes |\phi_{2345}\rangle = \sum_{i=1}^{4}\sum_{j=1}^{2}(|\psi_{12}\rangle_i \otimes |\phi_{34}\rangle_j \otimes |\phi_5\rangle_j) \qquad (20)$$

or,

$$|\psi_1\rangle \otimes |\phi_{2345}\rangle = \sum_{i=1}^{4}\sum_{j=1}^{2}(|\psi_{123}\rangle_i \otimes |\phi_4\rangle_j \otimes |\phi_5\rangle_j) \qquad (21)$$

Quantum Information Splitting

- First case:

  $|\psi_{12}\rangle_i \longrightarrow$ 4 orthogonal 2-qubit measurement outcomes of dealer

  $|\phi_{34}\rangle_j \longrightarrow$ 2 2-qubit orthogonal measurement outcomes of intermediate party

- Second case:

  $|\psi_{123}\rangle_i \longrightarrow$ 4 orthogonal 3-qubit measurement outcomes of dealer

  $|\phi_4\rangle_j \longrightarrow$ 2 1-qubit measurement outcomes of intermediate party

Quantum Information Splitting

- All distributions of qubits don't yield successful QIS protocols.

- Four-qubit linear cluster state cannot be used for the QIS of an unknown two-qubit state

- Five qubit linear cluster state can be used for the same.

- Entanglement properties of physical system mediating QIS protocol: Quantify the number of ways of splitting quantum information.

Quantum Information Splitting

- Protocol count for QIS of unknown n-qubit state:
  - $|\psi_n\rangle = \sum_{i_1,\ldots i_n=0}^{1} \alpha_{i_1,\ldots i_n} |i_1,\ldots i_n\rangle$
  - $\alpha_{i_1,\ldots i_n} \in C$
  - $\sum |\alpha_{i_1,\ldots i_n}|^2 = 1$

Quantum Information Splitting

- Theorem 1:

If Alice, Bob(s) and Charlie share an N qubit entangled state and Alice has a arbitrary n qubit state $|\psi_n\rangle$ that she wants the Bobs and Charlie to share, then Alice needs to possess a minimum of n qubits for this purpose.

Quantum Information Splitting

- Proof:
  - Conflate all Bobs and Charlie into Dolly $\longrightarrow \mathcal{H}_D$ (tensor product of Bobs and Charlie).
  - Quantum teleportation from Alice to Dolly (Information splitting): Maximal entanglement exists.
  - Schmidt decomposition: Dolly's density operator will be maximally mixed in a 2^n-dimensional *subspace* of $\mathcal{H}_D$.
  - If Alice possess m qubits in the entangled quantum network then $m \geq n$ from a quantum encryption perspective.

Quantum Information Splitting

- After Alice's joint measurement in $\mathcal{H}_x \otimes \mathcal{H}_A$, but before her classical communication to Dolly $\longrightarrow$ Dolly's density operator remains maximally mixed (No-signaling theorem)

- Dolly's state:

$$\mathbf{T} : |\psi\rangle \longrightarrow \sum_{j=1}^{P} U_j |\psi\rangle\langle\psi| U_j^{\dagger} \tag{22}$$

Quantum Information Splitting

- Alice's classical communication (j) $\longrightarrow U_j$ (restores $|\psi\rangle$).
- Minimal number P in Eq.(22) $\longrightarrow$ For arbitrary input state $|\psi\rangle$, $\mathbf{T}(|\psi\rangle) = \mathbf{I}/D$.
- $P = D^2 \longrightarrow$ Alice's classical communication $\geq \log(D^2) = 2n$ bits $\longrightarrow$ Alice's measurement must satisfy $m + n \geq 2n$, or, $m \geq n$.

- Theorem 2:

It is necessary for the recipient's system to be in a maximally mixed state, but not for that of any intermediate party **P**.

Quantum Information Splitting

- Proof:
  - Agrawal-Pati theorem: Charli's en qubits are maximally mixed. But, the reduced density operator for any intermediate party **P**, need not be maximally mixed.
  - Example: Alice, Bob and Charlie share an entangled state:
    $|\zeta\rangle \equiv \cos\theta|+\rangle_B|\psi^-\rangle_{AC} + \sin\theta|-\rangle_B|\psi^+\rangle_{AC}$,
    $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ and $\theta \in [0, \pi]$

Quantum Information Splitting

- $\longrightarrow$ Manifestly non-maximally entangled for $\theta \neq \pi/4$.
- Alice's and Bob's measurements commute, Bob does not use Alice's classical communication, Bob might measure first. For outcome $|\pm\rangle$.
- $\longrightarrow$ Alice teleports to Charlie with fidelity 1
- $\longrightarrow$ Alice's and Charlie's reduced density operator for $|\zeta\rangle$ is maximally mixed

Quantum Information Splitting

- Lemma 1:

The maximum number of protocols one can construct is $(N-2n)$.

- Proof:
  - Charlie has the last n qubits $\longrightarrow$ Reconstructs unknown n qubit information $\longrightarrow$ $(N-n+1)^{\text{th}}$ qubit to the N^{th} qubit.
  - First $(N-n)$ qubits distributed among Alice and Bob $\longrightarrow$ $(N-n-1)$ protocols
  - But all the protocols with Alice having $< n$ qubits fail (Theorem 2) $\longrightarrow$ Total number of protocols $= (N-2n)$.
  - For at least one protocol to work out $\longrightarrow$ $N \geq (2n+1)$

Quantum Information Splitting

- Corollary:

  For $N = 4$ and $n = 2 \longrightarrow$ 4 qubit states cannot be used for the QIS of an unknown 2 qubit state $|\psi_2\rangle$.

- Illustration:
  - Alice: Unknown two qubit state $|\psi_2\rangle$ and qubit 1
  - Bob: Qubit 2 and
  - Charlie: 3,4 in the 4 qubit cluster state:
    $|C_4\rangle = \frac{1}{2}(|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle)_{1234}$

Quantum Information Splitting

- Alice: Three-qubit measurement and obtains $\frac{1}{2}(|000\rangle + |100\rangle + |011\rangle - |111\rangle)$

- Bob-Charlie system: $\alpha_{00}(|000\rangle + |110\rangle) + \alpha_{01}(|000\rangle + |110\rangle) + \alpha_{10}(|001\rangle - |111\rangle) + \alpha_{11}(|001\rangle - |111\rangle)$.

- One cannot obtain $|\psi_2\rangle$ by performing another measurement or transform into another state through LOCC and perform a measurement to get $|\psi\rangle_2$.

Quantum Information Splitting

- Theorem 3:

If k $(3 \leq k \leq N-2n+2)$ parties share an N qubit entangled state and the first party has an arbitrary n qubit state that he/she wants the remaining members to share, then the maximum number of protocols that can be constructed for this purpose is $\sum_{j=k-2}^{N-2n} \mathbf{P}_{k-2}(j)$ in the symmetric case, and bounded above by $\sum_{j=k-2}^{N-2n} \mathbf{Q}_{k-2}(j) = {}^{N-2n}C_{k-2}$ in the general case.

Quantum Information Splitting

- Proof:
  - The Bobs have minimum $k-2$ qubits and maximum $N-2n$ qubits, j in total.
  - Symmetric case: Number of protocols = Number of ways j can be partitioned into $k-2$ slots (at least 1 entry/slot) = which is $\mathbf{P}_{k-2}(j)$.
  - Sum of all j's = Total number of protocols in the symmetric case.
  - If each Bob is inequivalent to any other: Number of protocols = $\sum_{j=k-2}^{N-2n} \mathbf{Q}_{k-2}(j) = \sum_{j=k-2}^{N-2n} {}^{j-1}C_{k-3} = {}^{N-2n}C_{k-2}$, as $\mathbf{Q}_l(m) = {}^{m-1}C_{l-1}$ $\longrightarrow$ Upper bound on number of protocols(Partial symmetry among Bobs)

Quantum Information Splitting

- Theorem 4:

If Alice, Bob and Charlie share an N-qubit entangled state and Alice has an (entangled) n-qubit entangled state of the form $|\phi_n\rangle = \alpha|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n}$ that she wants Bob and Charlie to share, then Alice needs to possess only one qubit for this purpose.

Quantum Information Splitting

- Proof:
  - Alice, Bob and Charlie share a N-entangled GHZ state $\frac{1}{\sqrt{2}}(|0\rangle^N + |1\rangle^N)$, $N = 2 + n$.
  - First qubit with Alice, second with Bob, and the remaining with Charlie.
  - Alice: $(n + 1)$ particle measurement $\longrightarrow$ To Charlie (or Bob) using $(n + 1)$ classical bits.

Quantum Information Splitting

Alice's Outcome	Bob-Charlie State
$\|\psi_1\rangle$	$\alpha\|0\rangle^{\otimes(n+1)} + \beta\|1\rangle^{\otimes(n+1)}$
$\|\psi_2\rangle$	$\alpha\|0\rangle^{\otimes(n+1)} - \beta\|1\rangle^{\otimes(n+1)}$
$\|\psi_3\rangle$	$\alpha\|1\rangle^{\otimes(n+1)} + \beta\|0\rangle^{\otimes(n+1)}$
$\|\psi_4\rangle$	$\alpha\|1\rangle^{\otimes(n+1)} - \beta\|0\rangle^{\otimes(n+1)}$

$\sqrt{2}|\psi_{1,2}\rangle = |0\rangle^{\otimes(n+1)} \pm |1\rangle^{\otimes(n+1)}$ and $\sqrt{2}|\psi_{3,4}\rangle = |0\rangle^{\otimes n} \otimes |1\rangle \pm |1\rangle^{\otimes n} \otimes |0\rangle$ are form mutual orthogonal measurement outcomes

Quantum Information Splitting

- Alice's 2-bit communication $\longrightarrow$ Charlie (or Bob): Single-qubit Pauli operation I, Z, X or Y $\longrightarrow$ $\alpha|0\rangle^{\otimes(n+1)} + \beta|1\rangle^{\otimes(n+1)}$.

- On measuring each of his qubit in a suitable basis ($|\pm\rangle$) $\longrightarrow$ 1-bit outcome to Charlie $\longrightarrow$ $|\phi_n\rangle$ can be reconstructed.

- If Alice's 2-bit outcome is known $\longrightarrow$ Bob/Charlie has partial information about α or β $\longrightarrow$ $|\phi_n\rangle$.

- Lemma 2:

If $k = 3$, the number of protocols one can construct is $N - n - 1$ and it is ${}^{N-2n}\mathrm{C}_{k-2} + n - 1$ for $k > 3$.

- Proof:
  - $k = 3$: Number of protocols one can construct $= N - n - 1$.
  - Theorem 1: Number of protocols is $N - 2n$ when Alice has $\geq n$ qubits and protocols with Alice having $\leq n$ fail.
  - Theorem 3: If $n > 2$, protocols with Alice having $\leq n$ qubits also work.
  - Charlie (retriever) with n qubits $\longrightarrow$ Remaining $N - n$ qubits are shared between remaining parties.
  - Theorem 1: Number of protocols by sharing $N - n$ between Alice and Bob $=$ $N - n - 1$ If $k > 3$, it is ${}^{N-2n}\mathrm{C}_{k-2} + n - 1$.

Quantum Information Splitting

- Efficiency of Information Splitting:
  - The theorems $\longrightarrow$ efficiency of a quantum channel for QIS.
  - Splitting efficiency (η) of a quantum channel:

$$\eta = \frac{n_0}{n_{\mathsf{max}}} \frac{\sum_{n=1}^{n_{\mathsf{max}}} n\zeta_n}{\sum_{n=1}^{n_{\mathsf{max}}} n\zeta'_n}, \tag{23}$$

  - ζ_n = Number of of protocols constructed by splitting $|\psi_n\rangle$ among k parties for a given entangled channel.

Quantum Information Splitting

- ζ'_n = Maximum number of protocols by splitting $|\psi_n\rangle$ among k parties (Theorem 3).

- $n_{\text{max}} = \lfloor \frac{N-k+2}{2} \rfloor$: Independent of particular channel, but on N and k only.

- n_0: Largest size of a secret (in qubit units) that can be split with N qubit entangled channel among k parties.

Conclusions

- Quantum communication can be used for information sharing in an efficient and secure manner.
- There can be multiple ways of carrying it out with different advantages.
- Quantum teleportation becomes most efficient through NDD, enabling secure quantum conversation.
- Non-conventional teleportation protocols are useful for quantum communication, which can be carried out through different channels.
- Quantum Information Splitting (QIS) is useful in quantum communication, for which one needs to know about the extensive properties of the channels.

Thanks

Thanks for such a patient listening.