Entanglement Witnesses

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arXiv: 1108.1493 [Phys. Rev. Lett. **107**, 270501 (2011)];
1101.0477

Entanglement Witnesses

- *Quantum entanglement is a useful resource for performing many tasks not achievable using laws of classical physics, e.g.,* ***teleportation, superdense coding, cryptography, error correction, computation…..***

- ***Q:*** *How does one detect entanglement in the lab ?*

  Given an unknown state, will it be useful for information processing ?

Methodology: *Construction of witness (hermitian) operators for entanglement; measurement of expectation value in the given unknown state tells us whether it is entangled or not.*

Measurability in terms of number of parameters vis-à-vis state tomography

PLAN

- *Entanglement witnesses: status and motivations*
- *Witness for teleportation: proof of existence*
- *Proposal for teleportation witness operator: examples*
- *Construction of common and optimal witnesses*
- *Measurability of a witness and its utility*

Entanglement Witnesses

- Existence of Hermitian operator with at least one negative eigenvalue *[Horodecki , M. P. & R. 1996; Terhal 2000]*

- As a consequence of Hahn-Banach theorem of functional analysis – provides a necessary and sufficient condition to detect entanglement

- Various methods for construction of entanglement witnesses *[Adhikari & Ganguly PRA 2009; Guhne & Toth, Phys. Rep. 2009]*

- Search for optimal witnesses *[Lewenstein et al. 2000; Sperling & Vogel 2009];* Common witnesses for different classes of states *[Wu & Guo, 2007];* Schmidt number witness *[Terhal & Horodecki, 2000]*

- Thermodynamic properties for macroscopic systems *[Vedral et al., PRL 2009]*

- Measurability with decomposition in terms of spin/polarization observables

Witness for teleportation

- ***Motivation:*** Teleportation is a prototypical information processing task; present challenge in pushing experimental frontiers, *c.f. Zeilinger et al. Nature (1997), Jin et al., Nat. Phot. (2010).* **Utility:** distributed quantum computation.

- ***Not all entangled states useful for teleportation,*** e.g., in 2 x 2, MEMS, and other classes of NMEMS not useful when entanglement less than a certain value *[Adhikari, Majumdar, Roy, Ghosh, Nayak, QIC 2010];* problem is more compounded in higher dimensions.

- ***Q:*** *How could we know if a given state is useful for teleportation ?*

- ***Hint:*** *For a known state, teleportation fidelity depends on its **fully entangled fraction (FEF)***

Witness of states useful for teleportation

N. Ganguly, S. Adhikari, A. S. Majumdar, J. Chatterjee

arXiv: 1108.1493 [quant-ph]; to appear in Phys. Rev. Lett.

- How to determine whether an unknown state is useful as a resource for teleportation ?
- Property of fully entangled fraction (FEF) of states is related to their ability for performing teleportation
- Threshold value of FEF enables us to prove existence of teleportation witness operators

- *Fully entangled fraction of a state* $\rho \; in \;\; d \otimes d$

$$F(\rho) = max_U \langle \psi^+ | U^\dagger \otimes I \rho U \otimes I | \psi^+ \rangle$$

- $$|\psi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$$

- *State acts as teleportation channel fidelity exceeds (classical)* *2/3*

if ***FEF > 1/d***

[Bennett et al, 1996; Horodecki (M, P, R), 1999]

(FEF interesting mathematical concept, but hard to calculate in practice) ***: computed example in higher dimensions, Zhao et al, J. Phys. A 2010)***

Existence of teleportation witness

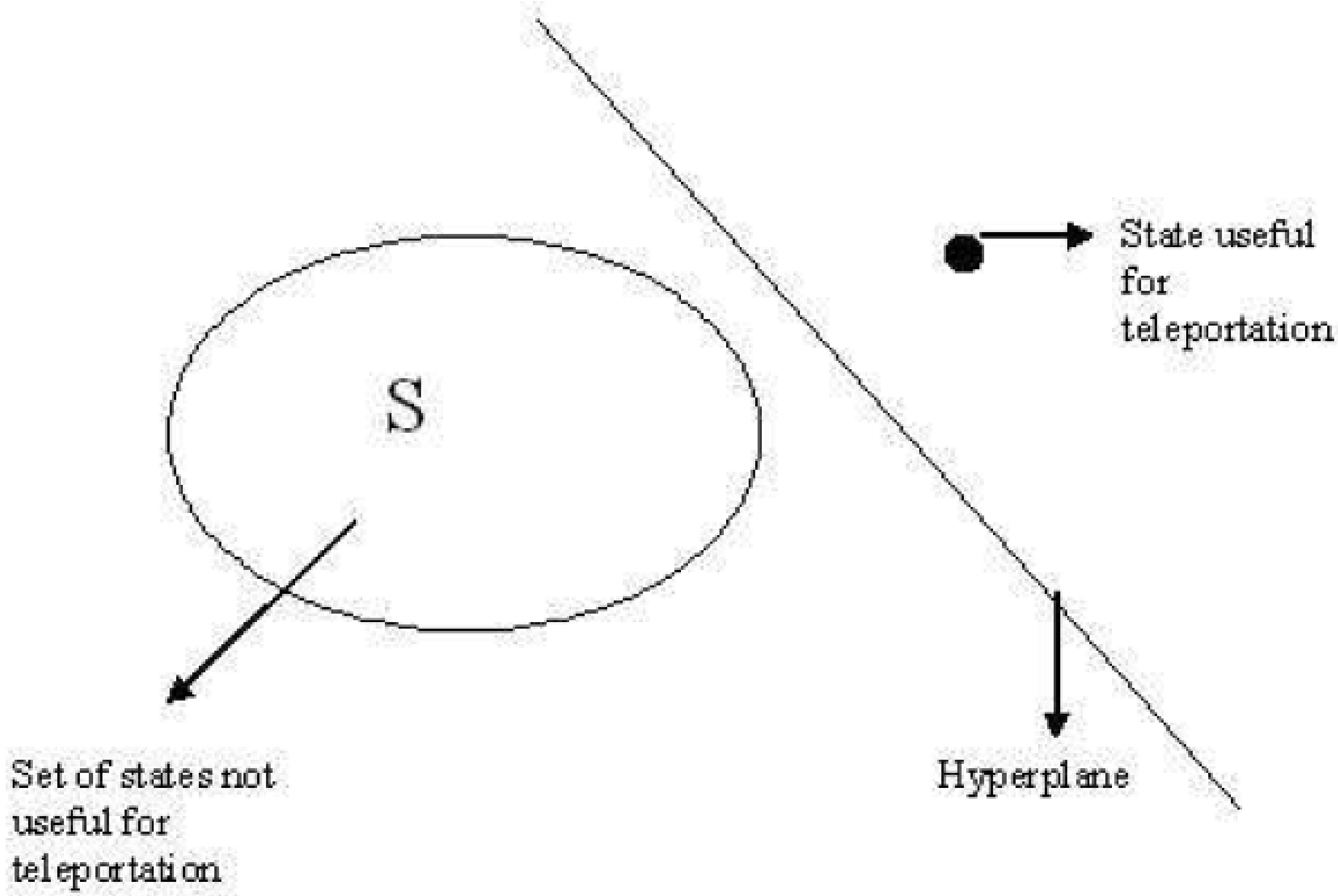
- ***Goal:*** *To show that the set of all states with* $F \leq 1/d$ *not useful for teleportation is separable from other states useful for teleportation*

- ***Proposition:*** *The set* $S = \{\rho : F(\rho) \leq \frac{1}{d}\}$ *is* ***convex*** *and* ***compact****.*

- *Any point lying outside S can be separated from it by a hyperplane*

- *Makes possible for Hermitian operators with at least one negative eigenvalue to be able to distinguish states useful for teleportation*

Separability of states using the Hahn-Banach theorem



Proof of the ***Proposition:***

In two steps:

First, the set $S = \{\rho : F(\rho) \le \frac{1}{d}\}$ *is convex*

Let $\rho_1, \rho_2 \in S$ thus, $F(\rho_1) \le \frac{1}{d}, \quad F(\rho_2) \le \frac{1}{d}.$

Now consider $\rho_c = \lambda\rho_1 + (1-\lambda)\rho_2 \qquad \lambda \in [0,1]$

$$F(\rho_c) \;=\; \langle\psi^+|U_c^\dagger \otimes I\rho_c U_c \otimes I|\psi^+\rangle = \lambda\langle\psi^+|U_c^\dagger \otimes I\rho_1 U_c \otimes I|\psi^+\rangle + (1-\lambda)\langle\psi^+|U_c^\dagger \otimes I\rho_2 U_c \otimes I|\psi^+\rangle$$

Let $F(\rho_i) = \langle\psi^+|U_i^\dagger \otimes I\rho_i U_i \otimes I|\psi^+\rangle, \quad (i = 1, 2)$ (U is compact)

Hence, $F(\rho_c) \le \lambda F(\rho_1) + (1-\lambda)F(\rho_2)$ and $F(\rho_c) \le \frac{1}{d}$ or $\rho_c \in S$

Proof of the ***Proposition:***

- ***Proof:*** *(ii) S is compact*

*for finite d Hilbert space, suffices to show S is **closed** and **bounded**.*

(every physical density matrix has a bounded spectrum: eigenvalues lying between 0 & 1; hence bounded)

Closure shown using properties of norm

$$|F(\rho_a) - F(\rho_b)| \leq K \|\rho_a - \rho_b\|$$

Proof of S being closed

For any two density matrices, let maximum of FEF be attained for U_a and U_b .

$$\text{i.e., } F(\rho_a) = \langle\psi^+|U_a^\dagger \otimes I \rho_a U_a \otimes I|\psi^+\rangle \text{ and } \quad F(\rho_b) = \langle\psi^+|U_b^\dagger \otimes I \rho_b U_b \otimes I|\psi^+\rangle.$$

Hence, $F(\rho_a) - F(\rho_b) \leq \langle\psi^+|U_a^\dagger \otimes I \rho_a U_a \otimes I|\psi^+\rangle - \langle\psi^+|U_a^\dagger \otimes I \rho_b U_a \otimes I|\psi^+\rangle$

Or, $F(\rho_a) - F(\rho_b) \leq |\langle\psi^+|U_a^\dagger \otimes I(\rho_a - \rho_b)U_a \otimes I|\psi^+\rangle|$

Lemma: Let A and B be two matrices of size $m \times n$ and $n \times r$ respectively.

Then $\|AB\| \leq \|A\|\|B\|$, $\quad \|A\| = \sqrt{TrA^\dagger A}$

$$F(\rho_a) - F(\rho_b) \leq \|\langle\psi^+|\|\|U_a^\dagger \otimes I\|\|(\rho_a - \rho_b)\|\|U_a \otimes I\|\||\psi^+\rangle\| \leq C^2K_1^2\|\rho_a - \rho_b\|$$

(Set of all unitary operators is compact, it is bounded: for any U, $\|U \otimes I\| \leq K_1$. $\|\langle\psi^+|\| = C$)

Similarly, $F(\rho_b) - F(\rho_a) \leq C^2K_1^2\|\rho_b - \rho_a\| = C^2K_1^2\|\rho_a - \rho_b\|$

$$|F(\rho_a) - F(\rho_b)| \leq C^2K_1^2\|\rho_a - \rho_b\|$$

(Hence, F is a continuous function).

Now, for any density matrix ρ, with $F(\rho) \in [\frac{1}{d^2}, 1]$ ([maximally mixed , max. ent. pure]

For S, $F(\rho) \in [\frac{1}{d^2}, \frac{1}{d}]$ Hence, $S = \{\rho : F(\rho) \leq \frac{1}{d}\} = F^{-1}([\frac{1}{d^2}, \frac{1}{d}])$ is ***<u>Closed.</u>*** *[QED]*

Summary of proof of existence of teleportation witness

- *The set* $S = \{\rho : F(\rho) \leq \frac{1}{d}\}$ *is* ***convex*** *and* ***compact***.

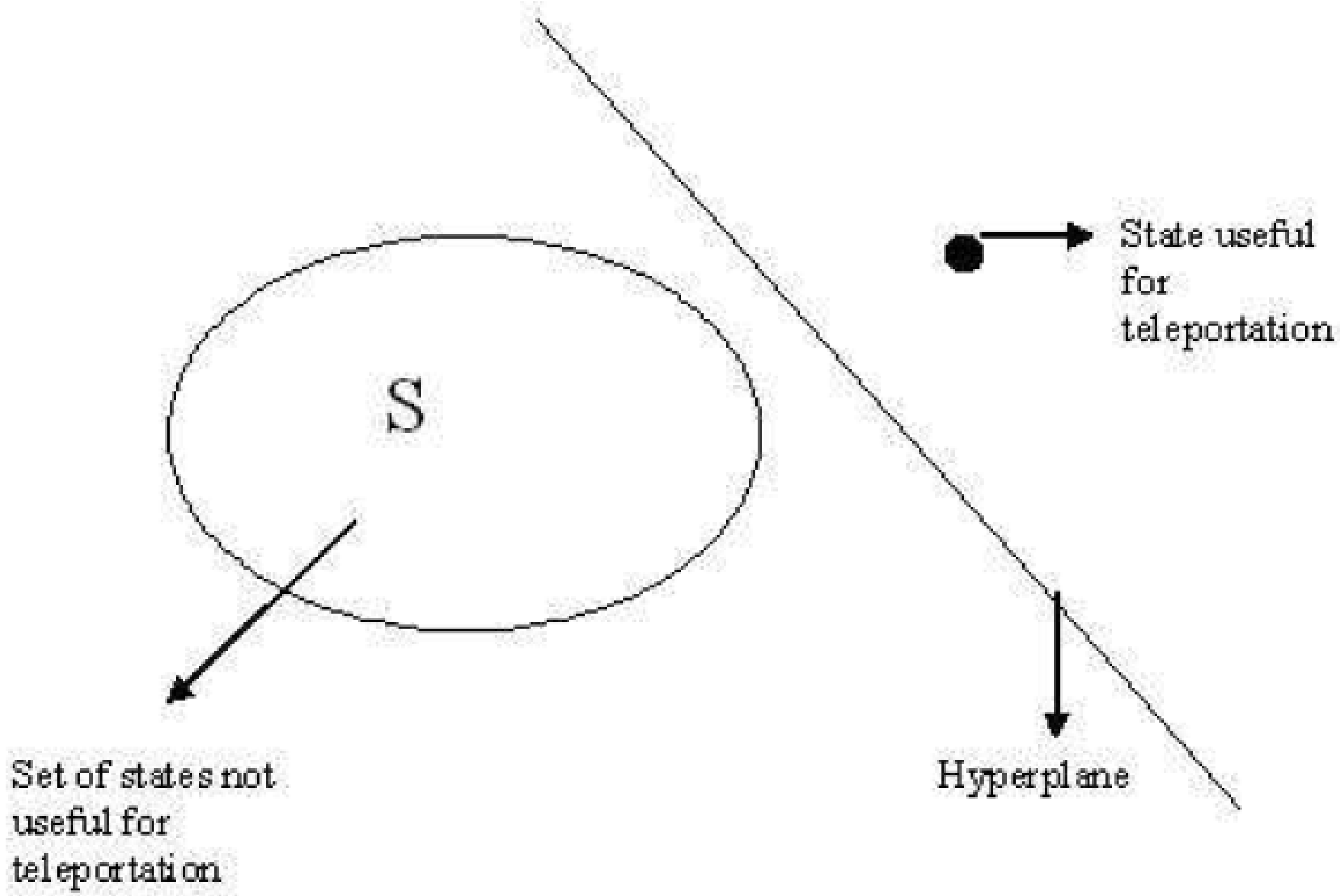
- *Any point lying outside S can be separated from it by a hyperplane*

- *The set of all states with* $F \leq {}^{1}\!/_{d}$
  not useful for teleportation is separable from other states useful for teleportation

- ***Makes possible for Hermitian operators with at least one negative eigenvalue to be able to distinguish states useful for teleportation***

Separability of states using the Hahn-Banach theorem

(S is convex and compact)



Construction of witness operator

Properties of the witness operator: $Tr(W\sigma) \geq 0,$ for all states σ which are not useful for teleportation, and $Tr(W\chi) < 0,$ for at least one state χ which is useful for teleportation.

Proposed witness operator:

$$W = \frac{1}{d}I - |\psi^+\rangle\langle\psi^+|$$

$$|\psi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$$

$$Tr(W\sigma) = \frac{1}{d} - \langle\psi^+|\sigma|\psi^+\rangle$$

$$Tr(W\sigma) \geq \frac{1}{d} - max_U \langle\psi^+|U^\dagger \otimes I \sigma U \otimes I|\psi^+\rangle$$

$$Tr(W\sigma) \geq \frac{1}{d} - F(\sigma)$$

Now, for a separable state $\sigma \in S$ $\quad Tr(W\sigma) \geq 0$

Application of Witness: examples

(i) Werner State: $\chi_{wer} = (1-v)\frac{I}{d^2} + v|\psi_d\rangle\langle\psi_d|$ $\quad |\psi_d\rangle = \sum_{i=0}^{d-1} \alpha_i |ii\rangle$

$$Tr(W\chi_{wer}) = \frac{1}{d} - \frac{1-v}{d^2} - \frac{v}{d}\sum_{i=0}^{d-1}\alpha_i \sum_{i=0}^{d-1}\alpha_i^*$$

In $2\otimes 2$ dimensions $\quad Tr(W\chi_{wer}) = \frac{1-3v}{4} < 0$, when $v > \frac{1}{3}$.

All $2\otimes 2$ *entangled Werner states are useful for teleportation*

(ii) MEMS *(Munro, et al, 2001)*

$$\chi_{MEMS} = \begin{pmatrix} h(C) & 0 & 0 & C/2 \\ 0 & 1-2h(C) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C/2 & 0 & 0 & h(C) \end{pmatrix}$$

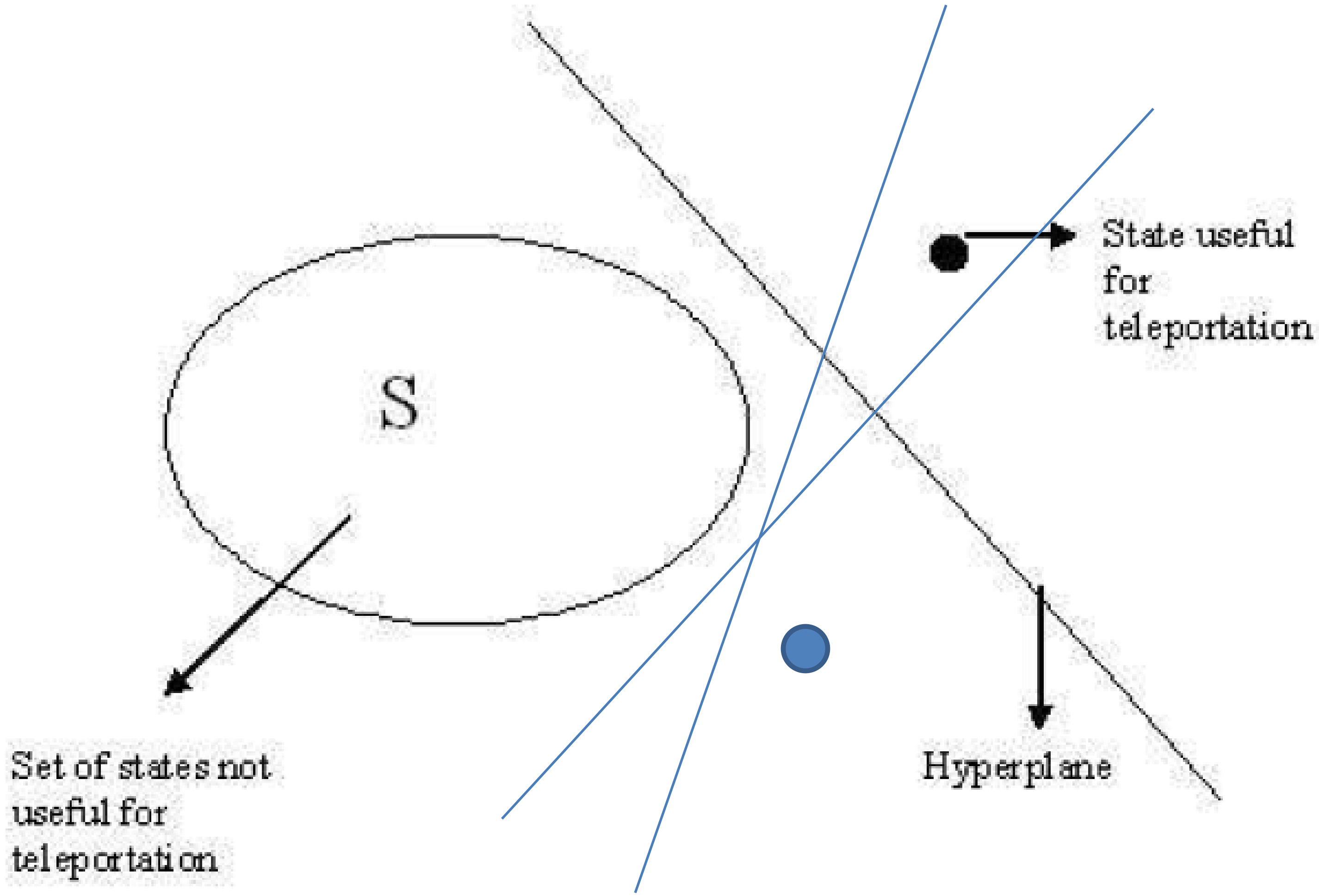
$Tr(W\rho_{MEMS}) \geq 0$ when $0 \leq C \leq \frac{1}{3}$

*Non-vanishing entanglement, but **not** useful for teleportation*

(confirms earlier results [*Lee, Kim, 2000, Adhikari et al QIC 2010]* in $2\otimes 2$

Utility for higher dimensions where FEF is hard to compute.

Witnesses are not universal



Finding common witnesses

[N. Ganguly, S. Adhikari, PRA 2009; N. Ganguly, S. Adhikari, A. S. Majumdar, arXiv: 1101.0477]

Motivations: Witness not universal or optimal; *fails for certain states, e.g.,*

$$\rho_\phi = a|\phi\rangle\langle\phi| + (1-a)|11\rangle\langle 11| \qquad |\phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \text{ and } 0 \leq a \leq 1$$

State useful for teleportation, but *witness W is unable to detect it, as* $Tr(W\rho_\phi) = \frac{a}{2} \geq 0$

(similar to what happens in the case of entanglement witnesses)

Goal: *Given two classes of states, to find a common witness operator*

(studied here in the context of entanglement witnesses; to be extended for teleportation witnesses)

Criterion for existence of common entanglement witnesses: For a pair of entangled states,

ρ_1, ρ_2 common EW exists iff

$\lambda\rho_1 + (1-\lambda)\rho_2$ is an entangled state $\forall\lambda \in [0,1].$

[Wu & Guo, PRA 2007]

Construction of common EW

Consider two NPT states ρ_1, ρ_2 Consider two sets S_1, S_2

consisting of all eigenvectors corresponding to negative eigenvalues of $\rho_1^{T_A}$ and $\rho_2^{T_A}$

Proposition: If $S_1 \cap S_2 \neq \phi$, *then there exists* ***a common EW***

Proof:

Let $S_1 \cap S_2 \neq \phi$. Then there exists a non-zero vector $|\eta\rangle \in S_1 \cap S_2$. Let $W = (|\eta\rangle\langle\eta|)^{T_A}$.

$$Tr(W\rho_1) = Tr((|\eta\rangle\langle\eta|)^{T_A}\rho_1) = Tr((|\eta\rangle\langle\eta|)\rho_1^{T_A}) < 0$$

Similarly, $Tr(W\rho_2) < 0$

If now we consider $\rho = \lambda\rho_1 + (1-\lambda)\rho_2, \lambda \in [0,1]$, then $Tr(W\rho) < 0$.

Example of a qutrit system

Consider two states $$\rho_1 = |\psi_1\rangle\langle\psi_1| \text{ and } \rho_2 = |\psi_2\rangle\langle\psi_2|$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |\psi_2\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$$

vector $|e_-\rangle = |01\rangle - |10\rangle$ common to $\rho_1^{T_A}$ and $\rho_2^{T_A}$ corresponding to their respective negative eigenvalues

$$W = U^{T_A} \qquad U = |e_-\rangle\langle e_-| \qquad Tr(W\rho_1) < 0 \text{ and } Tr(W\rho_2) < 0$$

Thus, W is a common witness. $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$ *Is entangled for all* $\lambda \in [0,1]$ *and can be detected by W.*

Above states are NPT. For PPTES, **nondecomposable** *witness operator*

$$W \neq P + Q^{T_A}$$

required for common EW.

[Ganguli et al arXiv: 1101.0477]

Measurability of Witness operator

- Hermitian witness operator: $$W = \frac{1}{d} I - |\psi^+\rangle\langle\psi^+|$$
- decomposed in 2 x 2: $$W = \frac{1}{4}[I \otimes I - \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z]$$

$\langle W \rangle = Tr(W\chi)$ *requires measurement of 3 unknown parameters.*

(Far less than 15 required for full state tomography !

Difference even larger in higher dimensions)

For implementation using polarized photons [c.f., Barbieri, et al, PRL 2003]

(decomposition in terms of locally measurable form)

W = ½(|HV><HV|+|VH><VH> - |DD><DD|- |FF><FF|+ |LL><LL> + |RR><RR>)

In terms of horizontal, vertical, diagonal, and left & right circular polarization states.

Entanglement Witness: Summary & Conclusions

[arXiv: 1108.1493 (to appear in Phys. Rev. Lett.); 1101.0477]

- *Teleportation is an important information processing task*

- *Not all entangled states are useful for performing teleportation. Connected to property of the fully entangled fraction*

- *We propose and prove existence of hermitian witness operators for distinguishing unknown states useful for teleportation using a geometric consequence of Hahn-Banach theorem*

- *Towards optimality – methods for constructing common witnesses*

- *Examples of Witness operator -- its utility and measurability*

Entanglement Witnesses: Future Directions

arXiv: 1108.1493 [Phys. Rev. Lett. **107**, 270501 (2011)]; 1101.0477

- *Common entanglement witnesses and optimality – witnesses for PPTES and edge states, common teleportation witness….*

- *Witnesses for other information processing tasks, e.g., dense coding, secure key generation, etc.*

- *Macroscopic entanglement witnesses – robustness against dissipative effects.*