Global vs local disorder and separability in terms of generalized conditional entropies

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Outline

- Separability issue and global vs local order
- Generalized Entropies:
  - -- Quantum Re´nyi Entropy
  - -- Tsallis Entropy and Abe-Rajagopal (AR) conditional Tsallis entropies
- Characterizing entanglement in (a) multiqubit states and (b) multimode Gaussian states using AR conditional entropies
- Majorization, limitations of spectral criteria, global vs local disorder in bound entangled states

von Neumann Entropy:

$$S\left(\rho\right) = -Tr\ \rho \log \rho$$

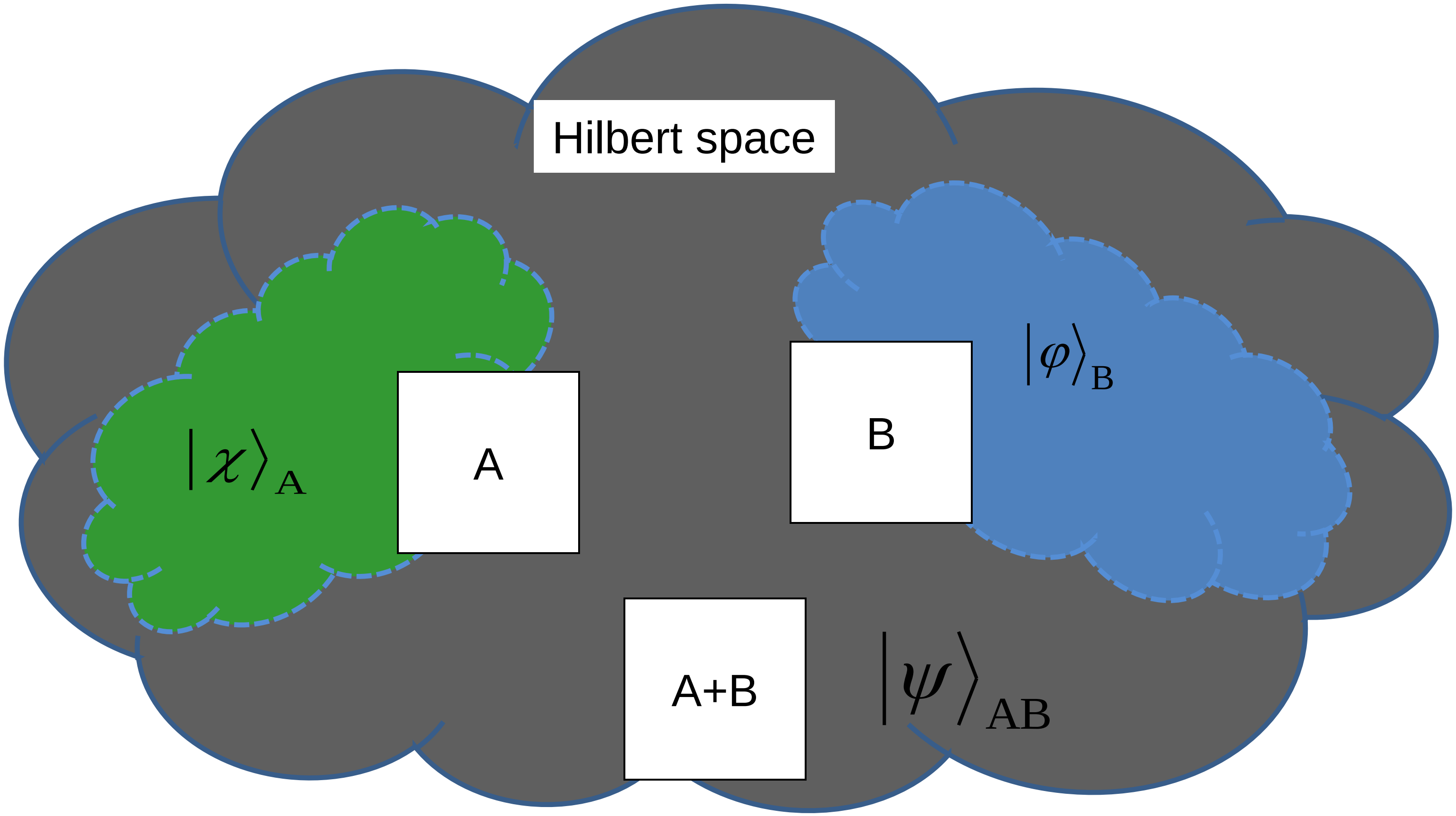
Mutual Entropy:

$$S\left(\rho_{AB} \parallel \rho_A \otimes \rho_B\right) = -Tr\ \rho_A \log \rho_A - Tr\ \rho_B \log \rho_B$$
$$+ Tr\ \rho_{AB} \log \rho_{AB} = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

Conditional Entropy:

$$S\left(B \mid A\right) = S\left(\rho_{AB}\right) - S\left(\rho_B\right)$$

Composite quantum systems



Product state: →

- ❑ System **A** is in the state $|\chi\rangle_A$ regardless of **B**
- ❑ Measurements on **A** and **B** will be uncorrelated
- ❑ *Entanglement:* Superposition of product states, like

$$\boxed{|\psi\rangle_{AB} = |\chi\rangle_A \otimes |\varphi\rangle_B}$$

$$|\psi'\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|\chi\rangle_A \otimes |\varphi\rangle_B + |\varphi'\rangle_A \otimes |\chi'\rangle_B\right)$$

von Neumann entropy of subsystems
- A measure of entanglement

- Density operator of a pure entangled quantum state:

$$\rho_{\mathrm{AB}} = |\psi\rangle_{\mathrm{AB}} \langle\psi|_{\mathrm{AB}}$$

- Subsystem density operators:

$$\rho_{\mathrm{A}} = \mathrm{Tr}_B[\rho_{\mathrm{AB}}], \quad \rho_{\mathrm{B}} = \mathrm{Tr}_A[\rho_{\mathrm{AB}}]$$

- von Neumann entropy of subsystems

$$\mathrm{S} = -\mathrm{Tr}[\rho_{\mathrm{A}} \log \rho_{\mathrm{A}}] = -\mathrm{Tr}[\rho_{\mathrm{B}} \log \rho_{\mathrm{B}}]$$

is a good measure of entanglement for pure bipartite states:

✓ $\mathrm{S} \neq 0$ necessarily implies that the pure state $|\psi\rangle_{\mathrm{AB}}$ is quantum correlated (entangled).

✓ $\mathrm{S} = 0$ for product states $|\psi\rangle_{\mathrm{AB}} = |\chi\rangle_{\mathrm{A}} \otimes |\varphi\rangle_{\mathrm{B}}$

▪ *Subsystem density operators of entangled pure states are mixed and von Neumann conditional entropies*

$$S(B|A) = S(\rho_{AB}) - S(\rho_B)$$
$$= -S(\rho_B)$$

are ***negative.***

Or $$\boxed{S(\rho_B) > S(\rho_{AB})}$$

In general, an entangled state of two parties, Alice (*A*) and Bob (*B*), may be more disordered locally than globally.

Negative conditional entropies are unheard of in classical probability theory!

Classically the probabilities are always more disordered globally than locally and so, the behavior $H(X), H(Y) \leq H(X,Y)$ for Shannon Entropies is impossible. It is shown that separable states too obey a classical behavior $S(\rho_A), S(\rho_B) \leq S(\rho_{AB})$

- Negative conditional entropies (implied by the inequality provide sufficient - but not necessary - criterion to characterize mixed entangled states.

- In the case of two qubit Werner state,

$$\hat{\rho}_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|\,x + I_4\,(1-x)/4$$

where $0 \leq x \leq 1$, $|\psi_{AB}\rangle = \frac{1}{2}\,(|0_A\,1_B\rangle - |1_A\,0_B\rangle)$, the conditional entropic criterion leads to $0 \ \leq \ x \ \leq \ 0.747$ as the range of separability (the von Neumann conditional entropy is positive in this range of the parameter x), which is clearly weaker compared to that obtained through Peres' partial transpose criterion: $0 \ \leq \ x \ \leq \ \frac{1}{3}$.

Generalized entropic measures offer more sophisticated tools to explore global vs local disorder in mixed states and lead to stringent limitation on separability than that obtained using positivity of von Neumann conditional entropy.

R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A **210**, 377 (1996)
R. Horodecki, and M. Horodecki, PRA **54**, 1838 (1996)
S. Abe, and A. K. Rajagopal, Physica A **289**, 157 (2001)
C. Tsallis, S. Lloyd, and M. Baranger, PRA **63**, 042104 (2001)
S. Abe, PRA **65**, 052323 (2002)

- Quantum Rényi Entropy:

$$S_q^{(R)}(\rho) = \frac{1}{1-q}\ \log \mathrm{Tr}\left[\rho^q\right]$$

- In the limit $\lim_{q\to 1}\ S_q^{(R)}(\rho)$ Rényi Entropy reduces to the von Neumann entropy.
- Horodecki et. al. recognized that

$$S_q^{(R)}(\rho_{AB}^{(\mathrm{sep})}) \geq S_q^{(R)}(\rho_A), S_q^{(R)}(\rho_B)$$

  for separable states.
  In other words the conditional Rényi entropy

$$S_q^{(R)}(B|A) = S_q^{(R)}(\rho_{AB}) - S_q^{(R)}(\rho_A)$$

  is positive for all separable states and thus negative values of the conditional Rényi entropy is a signature of quantum entanglement.

Tsallis Entropy:

$$S_q(\hat{\rho}) = \frac{Tr[\hat{\rho}^q]-1}{1-q}$$

In the limit q→1 Tsallis Entropy $S_q(\rho)$ reduce to the von Neumann entropy

Abe-Rajagopal (AR) q-Conditional entropy:

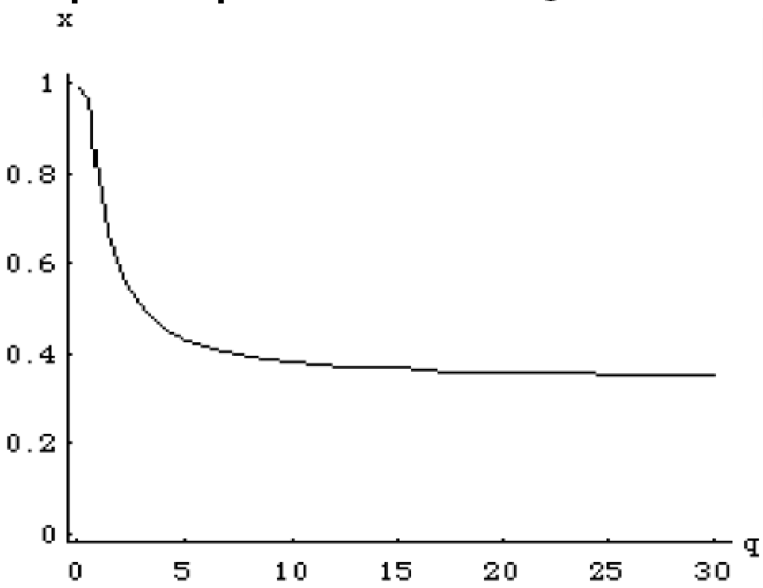
$$\begin{aligned} S_q(B|A) &= \frac{1}{1-q}\left[1-\frac{\mathrm{Tr}(\rho^q(A,B))}{\mathrm{Tr}(\rho^q(A))}\right] \\ &= \frac{1}{1-q}\left[1-\frac{\sum_n \lambda_n^q(A,B)}{\sum_m \lambda_m^q(A)}\right] \end{aligned}$$

$S_q(B|A) < 0$ is a signature of quantum entanglement

For the two qubit Werner State, the AR q-conditional entropy is given by

$$S_q(B|A) = S_q(A|B) = \frac{1}{1-q}\left[\frac{3}{2}\left(\frac{1-x}{2}\right)^q + \frac{1}{2}\left(\frac{1+3x}{2}\right)^q - 1\right]$$

An implicit plot of $S_q(B|A)=0$ with respect to q and $x \in [0,1]$



S. Abe, A.K. Rajagopal / Physica A 289 (2001) 157–164

x<0.748 is obtained from the conditional von Neumann entropy (q →1)

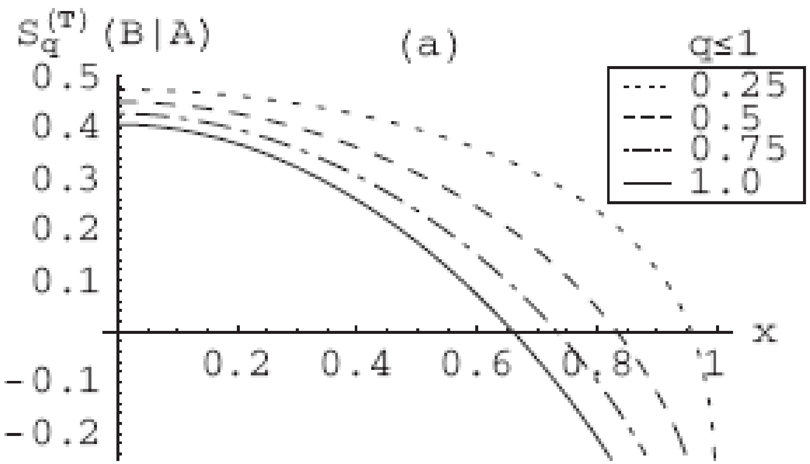
Peres' criterion

$S_q(B|A) \to 0$ in the limit $q \to \infty$ if and only if $x < 1/3$.

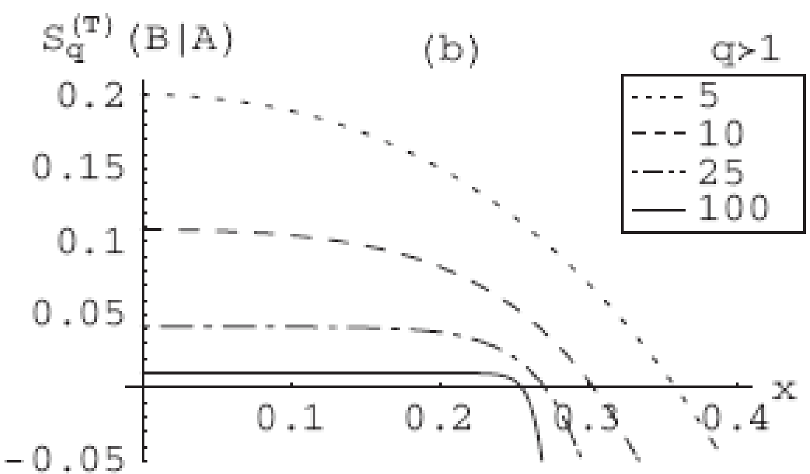
- As a further extension, Abe (PRA, **65**, 052323 (2002)) showed that the negativity of q-conditional entropy gives the correct range of inseparability for generalized Werner states of N-qudits.

Separability of one parameter symmetric multiqubit W and GHZ states using the AR q-conditional entropies

R Prabhu, A R Usha Devi and G Padmanabha, PRA **76**, 042337, 2007



The strongest limitation on separability, is obtained in the *q→ infinity* limit, and is found to agree with the Peres' criterion only for two and three qubit states of the GHZ family**.**



Vanishing von Neumann conditional entropy, leads to *x*=0.6593 -- a weaker domain of separability.

The AR q-entropy approach relies on finding the global and local spectra of the density matrices, which is not straightforward in the case of continuous variable systems. However, for n-mode Gaussian states, one can evaluate finite number (n) of symplectic eigenvalues of the corresponding 2n x 2n variance matrix (which completely characterizes the Gaussian state) -- in terms of which the eigenvalues of the density matrix may be expressed readily. It is thus possible to address the issue of separability based on conditional q-entropy approach in the context of Gaussian states

Review of Symplectic Transformations and Gaussian states

R. Simon, N. Mukunda, and B. Dutta, Phys. Rev. A **49**, 1567 (1994); Arvind, B. Dutta, N. Mukunda, and R. Simon, *ibid.* **52**, 1609 (1995).

- Consider a n-mode CV system – with a density operator ρ_n
- The $2n$ component operator column $\xi^T = (q_1, p_1; q_2, p_2; \ldots; q_n, p_n)$ of canonical quadrature operators $q_k =$ $(a_l + a_l^\dagger)/\sqrt{2},\ p_k = -i\,(a_l - a_l^\dagger)/\sqrt{2}$ (where $a_k^\dagger, a_k$ denote creation and annihilation operators of kth mode relations:

$$[\xi_\alpha, \xi_\beta] = i\,\Omega_{\alpha\beta};\quad \alpha, \beta = 1, 2, \ldots, 2n,$$

$$\text{where}\qquad \Omega = \bigoplus_{l=1}^{n} J,\quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- $2n \times 2n$ symplectic transformation $S \in \mathrm{Sp}(2n, \mathrm{R})$ preserves the canoncial commutation relations.

- As a consequence of the Stone-Neumann theorem) a corresponding unitary operator $U(S)$ on the Hilbert space on which the operators ξ act

$$U^{\dagger}(S)\xi_{\alpha}U(S) = \xi'_{\alpha} = \sum_{\alpha'} S_{\alpha\alpha'}\xi_{\alpha'}$$

$$\text{such that } [\xi'_{\alpha}, \xi'_{\beta}] = i\,\Omega_{\alpha\beta} \Rightarrow S\Omega S^{T} = \Omega.$$

- Our focus is on Gaussian states, which are completely determined by the $2n \times 2n$ covariance matrix

$$V_{\alpha\beta} = \frac{1}{2}\langle\{\Delta\xi_{\alpha}, \Delta\xi_{\beta}\}\rangle, \quad \alpha, \beta = 1, 2, \ldots, 2n,$$

where $\Delta\xi = \xi - \langle\xi\rangle$, $\{O_1, O_2\} = O_1\,O_2 + O_2\,O_1$ and $\langle O\rangle = \mathrm{Tr}[\rho O]$ denotes the expectation value of the operator O.

(First statistical moments $\langle\xi_{\alpha}\rangle$ can be arbitrarily adjusted by local unitary operations, and they do not play any role in the discussion of entanglement property of the state. We can set them as zero without any loss of generality.)

- A Gaussian state is mapped to another Gaussian state under symplectic transformations

$$\Rightarrow \quad V' = SVS^T.$$

- Williamson theorem: For every covariance matrix V there exists a symplectic matrix S such that

$$SVS^T = \mathrm{diag}(\nu_1, \nu_1; \nu_2, \nu_2; \ldots; \nu_n, \nu_n)$$

and $\nu_k, k = 0, 1, \ldots, n$ denote the symplectic eigenvalues.

- Gaussian density matrix is expressed as a tensor product of n thermal states of oscillators:

$$\rho_n \to \rho'_n = U(S)\, \rho_n\, U^\dagger(S) = \bigotimes_{k=1}^{n} \rho(\nu_k)$$

$$\text{where } \rho(\nu_k) = \frac{1}{\nu_k + \frac{1}{2}} \sum_{j=0}^{\infty} \left(\frac{\nu_k - \frac{1}{2}}{\nu_k + \frac{1}{2}} \right)^j \, |j\rangle_k \langle j|.$$

(Here $\{|j\rangle_k,\ j = 0, 1, \ldots, \infty\}$ denote the number states of the kth mode).

- An arbitrary positive power $\mathrm{Tr}[\rho^q]$, $0 < q \leq \infty$ of the n-mode Gaussian density operator may thus be readily expressed in terms of the symplectic eigenvalues as follows

$$\begin{aligned} \mathrm{Tr}[\rho_n^q] &= \prod_{k=1}^{n} \mathrm{Tr}\left[\rho^q(\nu_k)\right] \\ &= \prod_{k=1}^{n} \frac{1}{(\nu_k + \frac{1}{2})^q - (\nu_k - \frac{1}{2})^q}. \end{aligned}$$

- AR q-conditional entropy for a bipartite division of a n mode Gaussian system $\rho_n(A,B)$, with marginals $\mathrm{Tr}_B[\rho_n(A,B)] = \rho_N(A)$, $\mathrm{Tr}_A[\rho_N(A,B)] = \rho_{(n-N)}(B)$ (with $A \to N$ modes, $B \to (n-N)$ modes, $N < n$), in terms of symplectic eigenvalues of $\rho_n(A,B)$ and $\rho_N(A)$, using Eq. (2):

$$S_q(B|A) = \frac{1}{q-1}\left[1 - \frac{\prod_{l=1}^{N}\left[\left(\nu_l^{(A)}+\frac{1}{2}\right)^q - \left(\nu_l^{(A)}-\frac{1}{2}\right)^q\right]}{\prod_{k=1}^{n}\left[\left(\nu_k^{(AB)}+\frac{1}{2}\right)^q - \left(\nu_k^{(AB)}-\frac{1}{2}\right)^q\right]}\right]$$

- The q-conditional entropy is necessarily positive, when the modes A, B are separable. Negative values of $S_q(B|A)$ therefore imply entanglement between the modes A and B – offering a sufficient condition to characterize entanglement in Gaussian states.

- Peres' Partial Transpose Criterion: The lowest symplectic eigenvalue $\tilde{\nu}_{\min}$ of the variance matrix $\tilde{V}$ (where the canonical momenta p_l of the transposed modes reverse their sign of the partially transposed density matrix ρ^T satisfies $\tilde{\nu}_{\min} \geq \frac{1}{2}$ for all separable Gaussian states. And violation of this condition viz.,

$$\tilde{\nu}_{\min} < \frac{1}{2}$$

is a characteristic of entanglement. The PPT based characterization serves as a necessary and sufficient condition for separability in two mode Gaussian states.

$\boxed{\text{R. Simon, Phys. Rev. Lett. } \mathbf{84}\text{, 2726 (2000)}}$

Two mode squeezed thermal state

$$\rho(A,B) = U(S_r)\,\rho_{\rm th}(A)\otimes\rho_{\rm th}(B)\,U^\dagger(S_r)$$

$$U(S_r) \;=\; \exp\left[\frac{r}{2}(a_1^\dagger a_2^\dagger - a_1 a_2)\right]$$

$\rho_{\rm th}(A)$, $\rho_{\rm th}(B)$ denote single mode thermal states, both at same temperature T

Variance Matrix:

$$V(A,B) = \frac{\coth(\beta/2)}{2}\begin{pmatrix} \cosh r & 0 & \sinh r & 0 \\ 0 & \cosh r & 0 & -\sinh r \\ \sinh r & 0 & \cosh r & 0 \\ 0 & -\sinh r & 0 & \cosh r \end{pmatrix}$$

Symplectic eigenvalues --
Two mode state:

$$\nu^{(AB)}_{k=1,2} = \frac{\coth(\beta/2)}{2}$$

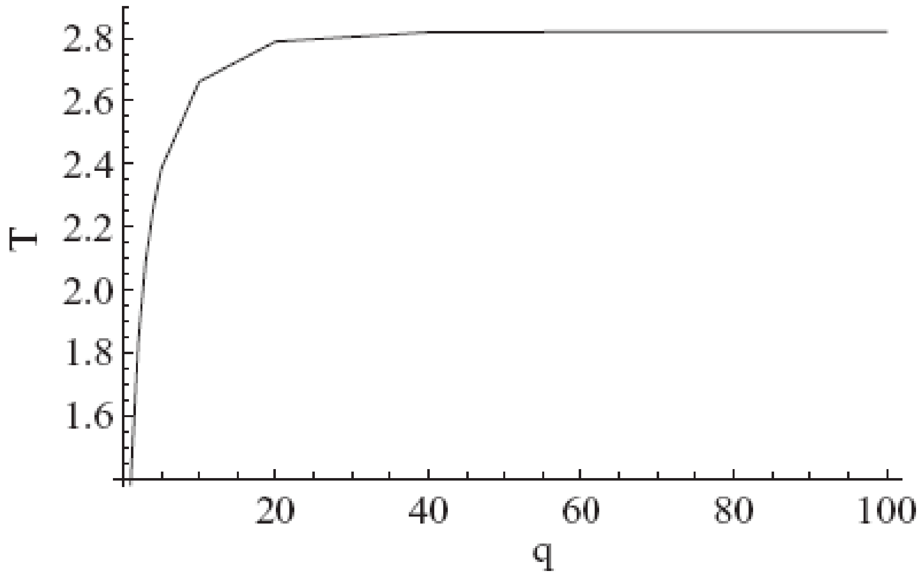
Single mode subsystem:

$$\nu^{(A)} = \frac{\coth(\beta/2)\,\cosh r}{2}$$

AR q-conditional entropy:

$$S_q(B|A) = \frac{1}{q-1}\left[\frac{(\coth(\beta/2)\,\cosh r + 1)^q - (\coth(\beta/2)\,\cosh r - 1)^q}{[(\coth(\beta/2) + 1)^q - (\coth(\beta/2) - 1)^q]^2}\right]$$

Implicit plot of $S_q(B|A) = 0$



$T_c^{(1)} \approx 1.381$

for q=1
(above this temperature von Neumann conditional entropy is negative)

$\boxed{T_c^{(\infty)} \to 2.82}$

squeezing parameter r=2,

$\boxed{T_c^{(1)} < T_c^{(\infty)} < T_c^{\mathrm{PPT}}}$

We studied the separability features of Gaussian states using AR q-entropic approach and compared the results with those obtained from conditional von-Neumann entropy (q=1 limit of AR q-entropy) and with the PPT method.

Strongest limitation on separability is realized in the limit q $\rightarrow$ infinity, although the q-entropy approach leads to weaker domain of separability than the exact one obtained from PPT method.

Sudha, A. R. Usha Devi, A. K. Rajagopal, PRA **81,** 024303 (2010)

Theory of Majorization:

A relationship between separability with global and local spectra of composite systems follows from the theory of Majorization.

Given $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ probability vectors (i.e., the components are positive and they sum to 1) the symbol $x \prec y$ is read as "x is majorized by y" i.e., x is more disordered than y .

More specifically, arranging the vectors x and y *in decreasing order as* $x^{\downarrow} = (x_1^{\downarrow}, \ldots, x_d^{\downarrow})$, where $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \cdots \geq x_d^{\downarrow}$

$$\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow}, \quad \Longrightarrow \quad x \prec y$$

$k = 1, \ldots, d-1$

Disorder criterion of separability

M. A. Nielsen and J. Kempe, Phys. Rev. Lett. **86,** 5184 (2001)

Majorization is known to be a more stringent notion of disorder than entropy in the sense that if $x \prec y$ then it follows that $H(x) \geq H(y)$

If ρ_{AB} *is separable, then the global and local eigenvalues obey the majorization condition*

$$\lambda(\rho_{AB}) \prec \lambda(\rho_A) \quad \text{and} \quad \lambda(\rho_{AB}) \prec \lambda(\rho_B).$$

The majorization results in conditional entropies of separable states being non-negative.

Limitations of spectral criteria (disorder criterion)

Two qubit iso-spectral states: Nielsen and Kempe (Phys. Rev. Lett. **86**, 5184 (2001)) show that attempts to characterize separability based only upon the eigenvalue spectra of the state ρ_{AB} and that of its subsystems ρ_A, ρ_B fail. They illustrate it with the help of an example of two 2 qubit states which are isospectral - of which, one is an entangled state and the other seperable:

$$\rho_{AB}^{(1)} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_{AB}^{(2)} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

(in the standard basis $\{||0,0\rangle,\ ||0,1\rangle,\ ||1,0\rangle,\ ||1,1\rangle\}$) The state $\rho_{AB}^{(1)}$ is an entangled state (negative under partial transpose) whereas $\rho_{AB}^{(2)}$ is separable - and both the states have same global and local eigenvalue spectra.

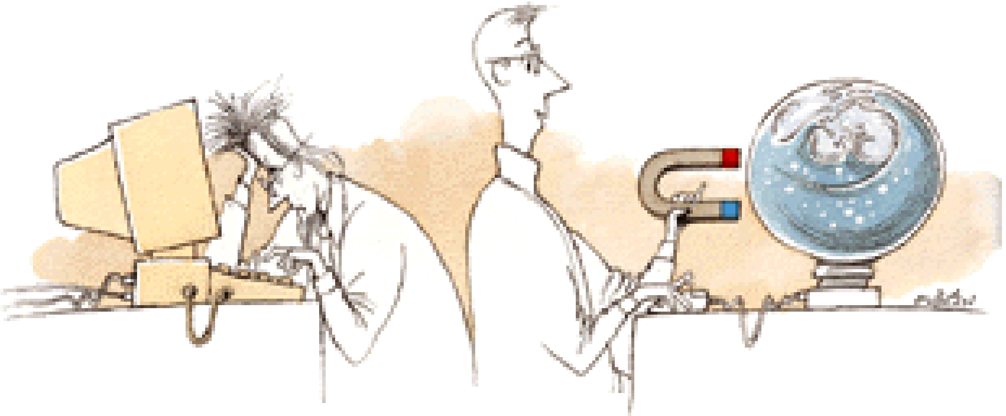
Distillability and global vs local disorder

T. Hiroshima, Phys. Rev. Lett. **91,** 057902 (2003)

If a bipartite quantum state satisfies the reduction criterion, i.e.,

$\rho_A \otimes I_B \geq \rho_{AB}$ and $I_A \otimes \rho_B \geq \rho_{AB}$, then it satisfies the majorization criterion too i.e., $\lambda(\rho_{AB}) \prec \lambda(\rho_A)$

In turn a bound entangled state, (which obeys reduction criterion and so is undistillable) obeys majorization criterion. This relates distillability with majorization criterion.



Thanks