Geometrical representation of entanglement invariants of symmetric N-qubit states

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- The problem of enumeration of local invariants of quantum state described by a density matrix ρ is important in the context of quantum entanglement.
- Non-local correlations in quantum systems reflect entanglement between its parts.
- Genuine non- local properties should be described in a form invariant under local unitary operations.

Two N-qubit states are said to be locally equivalent if one can be transformed into the other by local operations. i.e., $\rho' = U \rho U^{\dagger}$

where $U \in SU(2)^{\times N}$ and the two quantum states ρ and ρ' are said to be equally entangled.

Symmetric states

Symmetric N-particle states remain unchanged by permutations of individual particles.
Symmetric states offer elegant mathematical analysis as the dimension of the Hilbert space reduces drastically from 2^N to (N $+$ 1).
Such a Hilbert space is spanned by the eigen states $\{|j, m\rangle; -j \leq m \leq +j\}$ of angular momentum operators J^2 and J_z , where $j = \frac{N}{2}$.

A large number of experimentally relevant states possesses symmetry under particle exchange and this property allows us to significantly reduce the computational complexity.

Completely symmetric systems are experimentally interesting, largely because it is often easier to nonselectively address an entire ensemble of particles rather than individually address each member and it is possible to express the dynamics of these systems using only symmetry preserving operators. Specifically, if we have N two level atoms, each atom may be represented as a spin- $\frac{1}{2}$ system and theoretical analysis can be carried out in terms of collective spin operator $\vec{J} = \frac{1}{2}\Sigma_{\alpha=1}^{N}\vec{\sigma}_{\alpha}$. Here $\vec{\sigma}_{\alpha}$ denote the Pauli spin operator of the αth qubit.

Spherical tensor representation of density matrix

The most general spin-j density matrix

$$\rho(\vec{J}) = \frac{Tr(\rho)}{(2j+1)} \sum_{k=0}^{2j} \sum_{q=-k}^{+k} t_q^k \, \tau_q^{k^\dagger}(\vec{J}) \; , \tag{1}$$

τ_q^k (with $\tau_0^0 = I$,the identity operator) are irreducible tensor operators of rank ‘k’.
τ_q^k satisfy the orthogonality relations

$$Tr(\tau_q^{k^\dagger} \tau_{q'}^{k'}) = (2j+1)\, \delta_{kk'} \delta_{qq'} \tag{2}$$

and

$$t_q^k = \frac{Tr(\rho\, \tau_q^k)}{Tr\rho} \tag{3}$$

ρ is Hermitian and $\tau_q^{k\dagger} = (-1)^q \tau_{-q}^k$ and hence

$$t_q^{k^*} = (-1)^q \, t_{-q}^k \tag{4}$$

Spherical tensor parameters $t_q^k\,'s$ have simple transformation properties under co-ordinate rotation.

In the rotated frame $t_q^k\,'s$ are given by

$$(t_q^k)^R = \sum_{q'=-k}^{+k} D^k_{q'\,q}(\phi, \theta, \psi) \, t_{q'}^k \ , \tag{5}$$

$D^k_{q'\,q}(\phi, \theta, \psi)$ denote Wigner-D matrix,

(ϕ, θ, ψ) Euler angles

Multiaxial representation of density matrix[1]

In general $t^k_{\pm k}$ can be made zero for any k by suitable rotation. i.e.,

$$(t^k_{\pm k})^R = 0 = \sum_{q'=-k}^{+k} D^k_{q',\pm k}(\phi,\theta,\psi)t^k_{q'} \tag{6}$$

Using Wigner expression for the rotation matrix D^k,

$$(t^k_{\pm k})^R = 0 = [\pm^{sin}_{cos}(\theta/2)]^{2k} \, exp[i(\phi+\psi)] \sum_{r=0}^{2k} C_r Z^r \tag{7}$$

Z is the complex variable

$$Z = cot(\theta/2)e^{-i\phi}$$

in the case of $(t^k_{+k})^R = 0$
and

$$Z = tan(\theta/2)e^{-i(\phi+\pi)}$$

in the case of $(t^k_{-k})^R = 0$.

[1] V.Ravishankar

The expansion coefficients

$$C_r = \left(\begin{array}{c} 2k \\ k+q \end{array} \right)^{\frac{1}{2}} t^k_q \; = \left(\begin{array}{c} 2k \\ r \end{array} \right)^{\frac{1}{2}} t^k_{r-k} \; .$$

Any arbitrary t^k_qcan be written as a spherical tensor product of the form

$$t^k_q = r_k(...((\hat{Q}_1 \otimes \hat{Q}_2)^2 \otimes \hat{Q}_3)^3 \otimes ....)^{k-1} \otimes \hat{Q}_k)^k_q \, , \qquad (8)$$

where

$$(\hat{Q}_1 \otimes \hat{Q}_2)^2_q = \sum_{q_1} C(11k; q_1 q_2 q)(\hat{Q}_1)_{q_1}(\hat{Q}_2)_{q_2} \; ;$$

$$(\hat{Q})_q = \sqrt{\frac{4\pi}{3}} \; Y_{1q}(\theta, \phi).$$

Here $C(11k; q_1 q_2 q)$ is the Clebsch Gordan Co-efficient and $Y_{1q}(\theta, \phi)$ are the well known spherical harmonics.

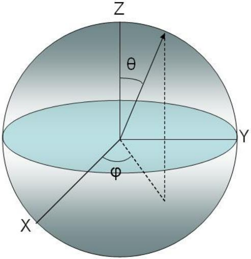
The state of a spin-j assembly can be represented geometrically by a set of 2j spheres, one corresponding to each value of k, k=1....2j, the kth sphere having k vectors specified on its surface.
Since

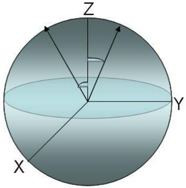
$$(\hat{Q}_i(\theta_i, \phi_i) \otimes \hat{Q}_j(\theta_j, \phi_j))^0_0 \tag{9}$$

is an invariant ($i \neq j$), one can construct in general

$$\left(\begin{array}{c} j(2j+1) \\ 2 \end{array} \right) \tag{10}$$

invariants from j(2j+1) axes. Together with 2j real positive scalars, there are $\left(\begin{array}{c} j(2j+1) \\ 2 \end{array} \right)$ + 2j invariants characterizing spin-j density matrix.


Z
θ
Y
φ
X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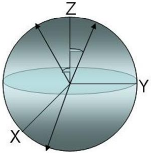

Z
Y
X

Pure spin-1 state

Direct product $|\psi_1\rangle \otimes |\psi_2\rangle$ of two spinors

$$|\psi_{12}\rangle = \begin{pmatrix} cos\frac{\theta_1}{2} \\ sin\frac{\theta_1}{2}e^{i\phi_1} \end{pmatrix} \otimes \begin{pmatrix} cos\frac{\theta_2}{2} \\ sin\frac{\theta_2}{2}e^{i\phi_2} \end{pmatrix} = \begin{pmatrix} cos\frac{\theta_1}{2}cos\frac{\theta_2}{2} \\ cos\frac{\theta_1}{2}sin\frac{\theta_2}{2}e^{i\phi_2} \\ sin\frac{\theta_1}{2}cos\frac{\theta_2}{2}e^{i\phi_1} \\ sin\frac{\theta_1}{2}sin\frac{\theta_2}{2}e^{i(\phi_1+\phi_2)} \end{pmatrix} \quad (11)$$

$0 \le \theta_{1,2} \le \pi$, $0 \le \phi_{1,2} \le 2\pi$

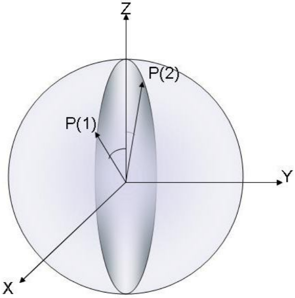
In the symmetric angular momentum subspace $|11\rangle$, $|10\rangle$, $|1-1\rangle$, the combined state is

$$|\psi_{12}\rangle_{sym} = \begin{pmatrix} cos\frac{\theta_1}{2}cos\frac{\theta_2}{2} \\ \frac{1}{\sqrt{2}}(cos\frac{\theta_1}{2}sin\frac{\theta_2}{2}e^{i\phi_2} + sin\frac{\theta_1}{2}cos\frac{\theta_2}{2}e^{i\phi_1}) \\ sin\frac{\theta_1}{2}sin\frac{\theta_2}{2}e^{i(\phi_1+\phi_2)} \end{pmatrix} \qquad (12)$$

Special Lakin frame

Let the azimuths of the above two directions (θ_1, ϕ_1), (θ_2, ϕ_2) with respect to x_0 are respectively 0 and π.
If the angular separation between the two directions is 2θ, then the state $|\psi\rangle$ has the explicit form

$$|\psi\rangle = \frac{\sqrt{2}}{\sqrt{1 + cos^2\theta}}[cos^2\frac{\theta}{2}|11\rangle_{\hat{Z}_0} - sin^2\frac{\theta}{2}|1-1\rangle_{\hat{Z}_0}] \qquad (13)$$


Z
P(2)
P(1)
Y
X

The density matrix corresponding to the above state is

$$\rho_s = \frac{2}{(1+cos^2\theta)} \begin{pmatrix} cos^4\frac{\theta}{2} & 0 & -sin^2\frac{\theta}{2}cos^2\frac{\theta}{2} \\ 0 & 0 & 0 \\ -sin^2\frac{\theta}{2}cos^2\frac{\theta}{2} & 0 & sin^4\frac{\theta}{2} \end{pmatrix} \tag{14}$$

The standard representation of the density matrix in terms of $t_q^{k\prime}s$

$$\rho_s = \frac{Tr(\rho)}{3} \begin{pmatrix} 1+\sqrt{\frac{3}{2}}\,t_0^1+\frac{t_0^2}{\sqrt{2}} & \sqrt{\frac{3}{2}}\,(t^1_{-1}+t^2_{-1}) & \sqrt{3}\,t^2_{-2} \\ -\sqrt{\frac{3}{2}}\,(t_1^1+t_1^2) & 1-\sqrt{2}\,t_0^2 & \sqrt{\frac{3}{2}}\,(t^1_{-1}-t^2_{-1}) \\ \sqrt{3}\,t_2^2 & -\sqrt{\frac{3}{2}}\,(t_1^1-t_1^2) & 1-\sqrt{\frac{3}{2}}\,t_0^1+\frac{t_0^2}{\sqrt{2}} \end{pmatrix} \tag{15}$$

The non-zero $t_q^k\,'s$ are

$$t_0^1 = \frac{\sqrt{6}cos\theta}{1+cos^2\theta}\ ,\ t_0^2 = \frac{1}{\sqrt{2}}\ ,$$

$$t_2^2 = t_{-2}^2 = \frac{\sqrt{3}sin^2\theta}{2(1+cos^2\theta)}$$

As $t^1_0 = r_1(\hat{Q}_1)^1_0$,

$$r_1 = \frac{t^1_0}{(\hat{Q}_1)^1_0} \quad . \tag{16}$$

Solving for the polynomial equation (7) for t^2, we get

$$(\hat{Q}_2)^1_q = \sqrt{\frac{4\pi}{3}} Y^1_q(\theta, 0)$$

and

$$(\hat{Q}_3)^1_q = \sqrt{\frac{4\pi}{3}} Y^1_q(\theta, \pi)$$

. Hence

$$r_2 = \frac{t^2_0}{(\hat{Q}_2 \otimes \hat{Q}_3)^2_0} = \frac{t^2_2}{(\hat{Q}_2 \otimes \hat{Q}_3)^2_2} \tag{17}$$

The invariants associated with the most general pure spin-1 state are

$$\mathcal{I}_1 = r_1 \ , \mathcal{I}_2 = r_2 \ , \tag{18}$$

$$\mathcal{I}_3 = (\hat{Q}_1 \otimes \hat{Q}_2)^0_0 \, , \tag{19}$$

$$\mathcal{I}_4 = (\hat{Q}_1 \otimes \hat{Q}_3)^0_0, \tag{20}$$

$$\mathcal{I}_5 = (\hat{Q}_2 \otimes \hat{Q}_3)^0_0. \tag{21}$$

Explicitly,

$$\mathcal{I}_1 = \frac{\sqrt{6}|cos\theta|}{1 + cos^2\theta} \ ,$$

$$\mathcal{I}_2 = \frac{\sqrt{3}}{1 + cos^2\theta} \ ,$$

$$\mathcal{I}_3 = \mathcal{I}_4 = -\frac{cos\theta}{\sqrt{3}} \ ,$$

$$\mathcal{I}_5 = -\frac{cos2\theta}{\sqrt{3}}.$$

The state $|\psi\rangle$ is separable for $\theta = 0$ and π .
Hence the invariants in the case of pure spin-1 separable states are

$$\mathcal{I}_1 = \sqrt{\frac{3}{2}} \, , \mathcal{I}_2 = \frac{\sqrt{3}}{2} \, ,$$

$$\mathcal{I}_3 = \mathcal{I}_4 = \mp\frac{1}{\sqrt{3}} \, , \mathcal{I}_5 = -\frac{1}{\sqrt{3}} \, .$$

Bell state

$|\psi\rangle = \frac{|11\rangle + |1-1\rangle}{\sqrt{2}}$

Non zero $t_q^k{}'s$: $t_0^2 = \frac{1}{\sqrt{2}}$,$t_2^2 = \frac{\sqrt{3}}{2}$, $t_{-2}^2 = \frac{\sqrt{3}}{2}$

Polynomial equation: $Z^2 =$ -1 ,

$(\theta_1, \phi_1) = (\frac{\pi}{2}, \frac{\pi}{2})$, $(\theta_2, \phi_2) = (\frac{\pi}{2}, \frac{3\pi}{2})$, $r_2 = \sqrt{3}$

For $|\psi\rangle = \frac{|11\rangle - |1-1\rangle}{\sqrt{2}}$

Non zero $t_q^k{}'s$: $t_0^2 = \frac{1}{\sqrt{2}}$,$t_2^2 = -\frac{\sqrt{3}}{2}$, $t_{-2}^2 = -\frac{\sqrt{3}}{2}$

Polynomial equation: $Z^2 = +1$,

$(\theta_1, \phi_1) = (\frac{\pi}{2}, 0)$, $(\theta_2, \phi_2) = (\frac{\pi}{2}, \pi)$, $r_2 = \sqrt{3}$

GHZ state

$|\psi\rangle = \frac{|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle}{\sqrt{2}}$

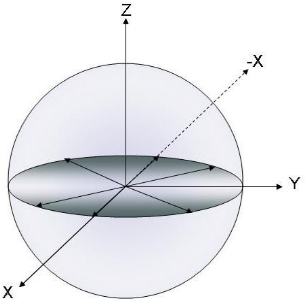
Non zero $t_q^{k\prime}s$ are $t_0^2 = 1$,$t_3^3 = -1$, $t_{-3}^3 = 1$
$Z^6 = 1$,

$(\theta_1, \phi_1) = (\frac{\pi}{2}, 0)$, $(\theta_2, \phi_2) = (\frac{\pi}{2}, \frac{\pi}{3})$, $(\theta_3, \phi_3) = (\frac{\pi}{2}, \frac{2\pi}{3})$, $r_3 = \frac{1}{2\sqrt{2}}$

For $|\psi\rangle = \frac{|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle}{\sqrt{2}}$

Non zero $t_q^{k\prime}s$ are $t_0^2 = 1$,$t_3^3 = 1$, $t_{-3}^3 = -1$

$Z^6 = 1$,


Z
-X
Y
X

Mixed spin-1 state

$$\rho(i) = \frac{1}{2}\left[\, I + \vec{\sigma}(i) \cdot \vec{p}(i) \,\right] = \frac{1}{2} \sum_{k,q} t^k_q(i)\, \tau^{k^\dagger}_q(i); \quad i = 1, 2. \tag{22}$$

where $\vec{p}(i)$ — the polarization vectors and

$\vec{\sigma}(i)$ — the Pauli spin matrices.

The combined density matrix is the direct product of the individual density matrices

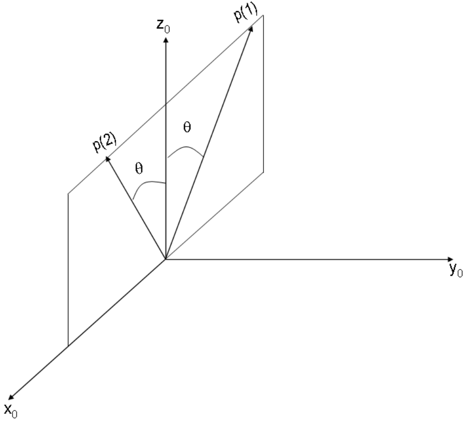
$$\rho_c = \rho(1) \otimes \rho(2)\ . \tag{23}$$

In this case, entanglement appears due to the projection of the

combined density matrix onto the desired spin-1 space.

Consider special Lakin frame (SLF)
then $t^1_{\pm 1} = 0$ and $t^2_2 = t^2_{-2}$

Choose a simple case of $|\vec{p}(1)| = |\vec{p}(2)| = p$, then we get $t^2_{\pm 1} = 0$



z_0
p(1)
p(2)
θ
θ
y_0
x_0

The density matrix for spin-1 mixed system in the symmetric subspace $|11\rangle$, $|10\rangle$ and $|1-1\rangle$ is

$$\rho_s = \frac{1}{(3+p^2cos2\theta)}\begin{pmatrix}(1+pcos\theta)^2 & 0 & -p^2sin^2\theta \\ 0 & 1-p^2 & 0 \\ -p^2sin^2\theta & 0 & (1-pcos\theta)^2\end{pmatrix}$$

Observe that when p=1, the mixed state density matrix is exactly the same as that of pure state density matrix

The non-zero $t_q^k\,'s$ are

$$t_0^1 = \frac{2\sqrt{6}pcos\theta}{(3+p^2cos2\theta)} \ ,$$

$$t_0^2 = \frac{\sqrt{2}p^2(1+cos^2\theta)}{(3+p^2cos2\theta)} \ ,$$

$$t_2^2 = \frac{\sqrt{3}p^2sin^2\theta}{(3+p^2cos2\theta)} \, .$$

Solving for the polynomial equation (7) for t^1, t^2, we get

$$\hat{Q}_1 = \hat{z}_0 \, ,$$

$$\hat{Q}_2 = \vec{p}(1) \, ,$$

and

$$\hat{Q}_3 = \vec{p}(2) \, .$$

Thus the invariants associated with the most general mixed spin-1 state are

$$\mathcal{I}_1 = \frac{2\sqrt{6}p|cos\theta|}{(3 + p^2 cos2\theta)} \ ,$$

$$\mathcal{I}_2 = \frac{2\sqrt{3}p^2}{(3 + p^2 cos2\theta)} \ ,$$

$$\mathcal{I}_3 = \mathcal{I}_4 = -\frac{cos\theta}{\sqrt{3}} \ , \ \mathcal{I}_5 = -\frac{cos2\theta}{\sqrt{3}}.$$

Note that in both pure as well as mixed state,

$$\mathcal{I}_3 = \mathcal{I}_4 = -\frac{cos\theta}{\sqrt{3}},$$

$$\mathcal{I}_5 = -\frac{cos2\theta}{\sqrt{3}}.$$

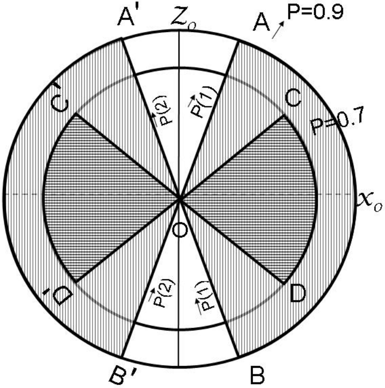


Figure: Range of θ for which the system is entangled for beam and target polarization p $=$ 0.9 (verical lines) and p $=$ 0.7 (horizontal lines). Extreme lines of the entangled regions represent the polarization vectors in SLF


Thank You