

Fine-grained uncertainty relation and non-locality of tripartite systems

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Motivation and Aim of the present Work

- ▶ It is possible, to get complete information about a classical system via a number of measurements.
- ▶ In quantum physics, Uncertainty relation prohibits all the observables to have definite value in any quantum state.
- ▶ Classical physics obey local hidden variable theory, where as quantum physics is non-local.
- ▶ The non-locality is captured by violation of Bell's Inequality.
- ▶ Now the question is : Can uncertainty relation capture the above property ?
- ▶ First, J. Oppenheim and S. Wehner ([Science 330, 1072 \(2010\)](#)) proposed a new form of uncertainty relation (fine-grained) which can capture such strength of non-locality in bipartite system.
- ▶ In ([arXiv:1108.3818](#)), we categorize the different no-signaling theory under the strength of non-locality for tripartite system.

Uncertainty of a measurement

- ▶ Uncertainty of a measurement occur due to spread of measurement outcome for a given state of the system.
- ▶ The spread of measured values can be found from the probability distribution of outcomes of the observable.
- ▶ In classical physics, the spread of measurement outcome occurs due to lack of knowledge about the complete state of the system and inaccuracy of the measuring apparatus.
- ▶ In quantum physics, the spread of measurement outcome is intrinsic property of the quantum system.
- ▶ There are two different measure of spread which are
 - i Standard deviation
 - ii Entropy

Uncertainty measured by standard deviation

- ▶ In quantum mechanics, according to Heisenberg Uncertainty relation

$$\Delta A \Delta B \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle|,$$

two non-commuting observable would not be measured with arbitrary precision.

- ▶ Here, Uncertainty of an observable (say, t) is measured by standard deviation which is of the given form

$$\Delta t = \sqrt{\langle t^2 \rangle_\rho - \langle t \rangle_\rho^2}, \text{ where}$$

$$\langle t \rangle_\rho = \sum_{x^{(t)}} p(x^{(t)}|t) x^{(t)} \text{ and}$$

$$\langle t^2 \rangle_\rho = \sum_{x^{(t)}} p(x^{(t)}|t) (x^{(t)})^2$$

- ▶ Furthermore, it will be observed that the uncertainty measured by entropy is more fundamental.

Iwo Bialynicki-Birula and Lukasz Rudnicki, arXiv:1001.4668

Entropic uncertainty relation

- ▶ Entropic uncertainty relation is given by

$$\sum_t p(t) H(t)_\rho \geq c_{\mathcal{T}, \mathcal{D}},$$

where $p(t)$ any set of probability distribution $\mathcal{D} (= \{p(t)\})$ over set of measurements \mathcal{T} and $H(t)$ is given by

$$H(t) = - \sum_{x^{(t)}} p(x^{(t)}|t)_\rho \ln p(x^{(t)}|t)_\rho.$$

- ▶ Whenever $c_{\mathcal{T}, \mathcal{D}} > 0$, we are unable to predict the outcome of at least one of the measurements t with certainty.
- ▶ H. Maassen and J. B. M. Uffink, already, proved that

$$H(P) + H(Q) \geq -\ln c, \quad \text{PRL, 60, 1103 (1988)}$$

where $c = \max_{j,k} |\langle a_j | b_k \rangle|^2$

Fine-grained uncertainty relation (FUR)

- ▶ J. Oppenheim and S. Wehner has proposed a fine-grained uncertainty relation (FUR) which measure the uncertainty obtained from any combination of outcomes $x^{(t)}$ for different measurement t . For the combination of outcomes represented by string $\mathbf{x} = (x^{(1)}, \dots, x^{(n)})$ with $n = |\mathcal{T}|$ and distribution $\mathcal{D} (= \{p(t)\})$, the fine grained uncertainty relation is given by

$$\sum_{t=1}^n p(t) p(x^{(t)}|t)_\sigma \leq \zeta_{\mathbf{x}}(\mathcal{T}, \mathcal{D}),$$

Science 330, 1072 (2010)

where σ represents the state of the general physical system.

- ▶ $\zeta_{\mathbf{x}} < 1$, represents the uncertainty of all measurements simultaneously.

- ▶ The upper bound, $\zeta_{\mathbf{x}} = \max_{\sigma} \sum_{t=1}^n p(t) p(x^{(t)}|t)_\sigma$

characterizes the amount of uncertainty in a particular physical theory (represented by the state σ of the system).

Single qubit case

- ▶ In [1] measurement of two different observable, Z (σ_z) and X (σ_x) on single qubit state was consider with probability $\rho(t) = \frac{1}{2}$.

- ▶ The upper bound FUR, in this case, is given by

$$\zeta_x = \frac{1}{2} + \frac{1}{2\sqrt{2}},$$

- ▶ The minimum uncertainty for the measurement outcome
 - ▶ **spin up** corresponding to the observables X and Z is $\frac{1}{2} + \frac{1}{2\sqrt{2}}$ for the eigenstate $\frac{X+Z}{\sqrt{2}}$.
 - ▶ **spin down** corresponding to the observables X and Z is $\frac{1}{2} + \frac{1}{2\sqrt{2}}$ for the eigenstate $\frac{X-Z}{\sqrt{2}}$.
- ▶ The eigenstate of $\frac{X\pm Z}{\sqrt{2}}$ are known as maximally certain states (for which the fine-grained uncertainty has maximum value) in quantum theory.

Application of FUR in Bipartite system

- ▶ The FUR, gives more interesting results in bipartite system shared by Alice and Bob.
- ▶ Now, one can ask
 - i why quantum correlations strong enough to be non-local, yet not as strong as they could be?
 - ii Is there any principle that determines degree of non-locality ?
- ▶ Here we consider
 - i no-signaling principle is valid
 - ii FUR gives another constraint the allowed probability distribution under physical theory.
- ▶ Using these two assumptions, Oppenheim and Wehner was able to discriminate different class of no-signaling theories for bipartite systems.

CHSH game

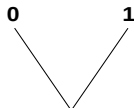
- ▶ To discriminate the allowed probability distribution under different classes of no-signaling theories in bipartite system, consider the following CHSH-game (a special class of non-local retrieval games).
- ▶ According to these games, Alice and Bob receive questions (measurement settings) 's' and 't' respectively, with some probability distribution $p(s, t)$ (for simplicity, $p(s, t) = p(s)p(t)$); and their answer (measurement outcome) 'a' or 'b' will be winning answers determined by the set of rules ($a \oplus b = s.t$), i.e., for every setting 's' and the corresponding outcome 'a' of Alice, there is a string $\mathbf{x}_{s,a} = (x_{s,a}^{(1)}, \dots, x_{s,a}^{(n)})$ of length $n = |\mathcal{I}|$ that determines the correct answer $b = x_{s,a}^t$ for the question 't' for Bob.

CHSH game

CHSH-Game

Question :

Measurement Settings



Probability Distribution :

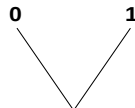
s



a

Answer :

Measurement Outcomes



$$p(s,t) = p(s) p(t)$$

t



b



$$\text{Winning condition : } a \oplus b = s \cdot t$$

FUR gives the different maximum winning probability under different physical theory in bipartite system.

FUR for bipartite system

- ▶ In the initial stage of the game, Alice and Bob discuss their strategy (i.e., choice of shared bipartite state and also measurement). They are not allowed to communicate with each other once the game has started.
- ▶ The probability of winning the game for a physical theory described by bipartite state (σ_{AB}) is given by

$$P^{game}(\mathcal{S}, \mathcal{T}, \sigma_{AB}) = \sum_{s,t} p(s, t) \sum_a p(a, b = x_{s,a}^t | s, t)_{\sigma_{AB}}$$

where, $p(a, b = x_{s,a}^t | s, t)_{\sigma_{AB}}$ is given by

$$p(a, b = x_{s,a}^t | s, t)_{\sigma_{AB}} = \sum_b V(a, b | s, t) \langle (A_s^a \otimes B_t^b) \rangle_{\sigma_{AB}}$$

- ▶ $V(a, b | s, t)$, the winning condition is given by

$$V(a, b | s, t) = 1 \quad \text{iff } a \oplus b = s.t$$

$$V(a, b | s, t) = 0 \quad \text{otherwise}$$

Discrimination of different no-signaling theory in bipartite system

- ▶ Using $A_s^a = \frac{\mathcal{I} + (-1)^a A_s}{2}$ and $B_t^b = \frac{\mathcal{I} + (-1)^b B_t}{2}$, the form of P^{game} is given below

$$P^{game}(S, T, \sigma_{AB}) = \frac{1}{2} \left(1 + \frac{\langle \mathcal{B}_{CHSH} \rangle \sigma_{AB}}{4} \right), \quad \mathcal{B}_{CHSH} = A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1$$

- ▶ To characterize the allowed probability distribution under the theory, we need to know the maximum winning probability, maximized over all possible strategies for Alice and Bob, i.e.,

$$P_{\max}^{game} = \max_{S, T, \sigma_{AB}} P^{game}(S, T, \sigma_{AB})$$

- ▶ The value of P^{game} allowed by
 - ▶ classical physics is $\frac{3}{4}$ (as classically, the Bell-CHSH inequality is bounded by 2).
 - ▶ quantum mechanics is $\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)$ (due to the maximum violation of Bell inequality, $\langle \mathcal{B}_{CHSH} \rangle = 2\sqrt{2}$).
 - ▶ no-signaling theories with maximum Bell violation ($\langle \mathcal{B}_{CHSH} \rangle = 4$, that occurs for the PR-box) is 1.

Non-local retrieval game for tripartite system

- ▶ In recent work [2] we generalize the fine-grained uncertainty relation for the case of tripartite systems, classifying different nosignaling theories on the basis of their degree of nonlocality.
- ▶ Here, we consider a non-local retrieval game similar to CHSH-game for bipartite systems. Here, Alice, Bob and Charlie receive respective binary questions 's', 't', and 'u' $\in \{0, 1\}$, and they win the game if their respective outcomes (binary) 'a', 'b', and 'c' $\in \{0, 1\}$ satisfy certain rules.
- ▶ We limit our analysis to the three kinds of no-signaling boxes, known as full-correlation boxes, for which all one-party and two-party correlation in the system vanishes [3]. They win the game if their answers satisfy certain set of rules, either

$$a \oplus b \oplus c = s.t \oplus t.u \oplus u.s$$

or $a \oplus b \oplus c = s.t \oplus s.u$

or else $a \oplus b \oplus c = s.t.u$

²T. Pramanik, A. S. Majumdar, arXiv:1108.3818,

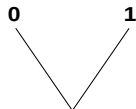
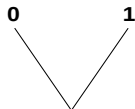
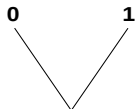
³J. Bancal and V. Scarani, J. Phys. A: Math. Theor. 44, 065303 (2011)

Non-local retrieval game for tripartite system

Nonlocal retrieval game in tripartite system, similar to CHSH game

Question :

Measurement Settings



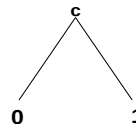
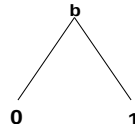
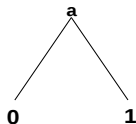
Probability Distribution :

$$p(s,t) = p(s) p(t) p(u)$$



Answer :

Measurement Outcomes



Winning condition : $a \oplus b \oplus c = s.t \oplus t.u \oplus s.u$ or $s.t \oplus t.u$ or $s.t.u$

FUR gives the different maximum winning probability under different physical theory in tripartite system for different winning condition.

FUR for tripartite system

- ▶ The winning probability of our prescribed game under a no-signaling theory described by a shared tripartite state σ_{ABC} (among Alice, Bob and Charlie) is given by

$$p_{\text{game}}(\mathcal{S}, \mathcal{T}, \mathcal{U}, \sigma_{ABC}) = \sum_{s,t,u} p(s, t, u) \sum_{a,b} p(a, b, c = x_{s,t,a,b}^{(u)} | s, t, u) \sigma_{ABC}$$

- ▶ $p(a, b, c | s, t, u) \sigma_{ABC}$ the joint probability of getting outcomes 'a', 'b' and 'c' for corresponding settings 's', 't' and 'u' given by

$$p(a, b, c = x_{s,t,a,b}^{(u)} | s, t, u) \sigma_{ABC} = \sum_c V(a, b, c | s, t, u) \langle A_s^a \otimes B_t^b \otimes C_u^c \rangle_{\sigma_{ABC}}$$

- ▶ The winning condition, $V(a, b, c | s, t, u)$ is non zero ($= 1$) only when the input-output are correlated one of the full correlation box (non-local) and zero otherwise.

Discrimination of no-signaling theory in tripartite system

- ▶ Here, we discriminate among different no-signaling theories using the upper bound of the probability of winning the game (P_{\max}^{game} , i.e., maximum winning probability of the game given by FUR) under three by three different non-local boxes.
- ▶ P_{\max}^{game} is obtained by maximizing over all possible strategies (i.e., the choice of the shared tripartite state and measurement settings by the three parties) for Alice, Bob and Charlie.

Fine-grained uncertainty relation using

$$a \otimes b \otimes c = s.t + u.s$$

- ▶ In this case the expression of $P^{game}(\mathcal{S}, \mathcal{T}, \mathcal{U}, \sigma_{ABC})$ is given by

$$P^{game}(\mathcal{S}, \mathcal{T}, \mathcal{U}, \sigma_{ABC}) = \frac{1}{2} \left[1 + \frac{\langle \mathbf{S}_2 \rangle_{\sigma_{ABC}}}{8} \right], \text{ where}$$

$$\mathbf{S}_2 = A_0 \otimes B_0 \otimes C_0 + A_0 \otimes B_0 \otimes C_1 + A_0 \otimes B_1 \otimes C_0 + A_1 \otimes B_0 \otimes C_0 \\ + A_0 \otimes B_1 \otimes C_1 - A_1 \otimes B_0 \otimes C_1 - A_1 \otimes B_1 \otimes C_0 + A_1 \otimes B_1 \otimes C_1$$

- ▶ According to local hidden variable theory (where all the variables A_i , B_i and C_i take either '+1' or '-1') the maximum value of $\langle \mathbf{S}_2 \rangle_{\sigma_{ABC}}$ is 4.
- ▶ The value of P_{\max}^{game} for classical physics is $\frac{3}{4}$.

Fine-grained uncertainty relation using

$$a \otimes b \otimes c = s.t + u.s \text{ (contd.)}$$

- ▶ In quantum physics, we maximize the $\langle \mathbf{S}_2 \rangle_{\sigma_{ABC}} = \text{Tr}[\mathbf{S}_2 \cdot \sigma_{ABC}]$ over all possible projective spin measurements on both the GHZ-state and the W-state.
- ▶ The choice of two observables by each party are
$$\Pi_{\alpha_0} = \sin(\theta_{\alpha_0}) \cos(\phi_{\alpha_0}) \sigma_x + \sin(\theta_{\alpha_0}) \sin(\phi_{\alpha_0}) \sigma_y + \cos(\theta_{\alpha_0}) \sigma_z$$
$$\Pi_{\alpha_1} = \sin(\theta_{\alpha_1}) \cos(\phi_{\alpha_1}) \sigma_x + \sin(\theta_{\alpha_1}) \sin(\phi_{\alpha_1}) \sigma_y + \cos(\theta_{\alpha_1}) \sigma_z$$
- ▶ It is found numerically (using **Mathematica**) that the maximum value of $\langle \mathbf{S}_2 \rangle_{\sigma_{ABC}}$ for the GHZ-state as well as the W-state is 4.
- ▶ For example, the P_{\max}^{game} occurs for the GHZ-state for the projective measurements either along the direction $\{3.1149, 2.5271\}$ or along $\{1.5708, 0.4608\}$ by Alice; along the direction $\{1.5708, 1.7282\}$ or along $\{4.7124, 1.7282\}$ by Bob; and along the direction $\{4.7124, 0.9526\}$ or along $\{4.7124, 4.0942\}$ by Charlie.

Fine-grained uncertainty relation using

$$a \otimes b \otimes c = s.t + u.s \text{ (contd.)}$$

- ▶ The maximum winning probability of the non-local retrieval game in quantum physics is $\frac{3}{4}$ which is same as the value obtained in classical physics.
- ▶ The value of P_{\max}^{game} under the considered no-signaling theory is 1.
- ▶ In this case, we **unable to discriminate** quantum theory from classical theory with the help of the maximum winning probability (i.e., the **upper bound** of **FUR**) of our prescribed non-local game.

Fine-grained uncertainty relation using $a \otimes b \otimes c = s.t.u$

- ▶ In this case the expression of $P^{game}(\mathcal{S}, \mathcal{T}, \mathcal{U}, \sigma_{ABC})$ is given by

$$P^{game}(\mathcal{S}, \mathcal{T}, \mathcal{U}, \sigma_{ABC}) = \frac{1}{2} \left[1 + \frac{\langle \mathbf{S}_3 \rangle_{\sigma_{ABC}}}{8} \right], \text{ where}$$

$$\mathbf{S}_3 = A_0 \otimes B_0 \otimes C_0 + A_0 \otimes B_0 \otimes C_1 + A_0 \otimes B_1 \otimes C_0 + A_1 \otimes B_0 \otimes C_0 + A_0 \otimes B_1 \otimes C_1 + A_1 \otimes B_0 \otimes C_1 + A_1 \otimes B_1 \otimes C_0 - A_1 \otimes B_1 \otimes C_1$$

- ▶ According to local hidden variable theory, the maximum value of $\langle \mathbf{S}_2 \rangle_{\sigma_{ABC}}$ is 6.
- ▶ Hence, in classical physics, the maximum winning probability of the game (P_{max}^{game}) is $\frac{7}{8}$.
- ▶ Similar to the above case, the maximum value of $\langle \mathbf{S}_2 \rangle_{\sigma_{ABC}}$ over all possible projective spin measurements on both the GHZ-state and the W-state is 6.
- ▶ The value of P_{max}^{game} , allowed by quantum physics is $\frac{7}{8}$.
- ▶ The value of P_{max}^{game} , allowed by the considered no-signaling theory non-locality is 1.

Fine-grained uncertainty relation using

$$a \otimes b \otimes c = s.t + t.u + u.s$$

- ▶ The probability of winning the prescribed non-local game is given by

$$P^{game}(\mathcal{S}, \mathcal{T}, \mathcal{U}, \sigma_{ABC}) = \frac{1}{2} \left[1 + \frac{\langle \mathbf{S}_1 \rangle_{\sigma_{ABC}}}{8} \right], \text{ where}$$

$$\mathbf{S}_1 = A_0 \otimes B_0 \otimes C_0 + A_0 \otimes B_0 \otimes C_1 + A_0 \otimes B_1 \otimes C_0 + A_1 \otimes B_0 \otimes C_0 - A_0 \otimes B_1 \otimes C_1 - A_1 \otimes B_0 \otimes C_1 - A_1 \otimes B_1 \otimes C_0 - A_1 \otimes B_1 \otimes C_1$$

- ▶ The value of P_{\max}^{game} allowed in classical physics is $3/4$ which follows from the Svetlichny inequality [4]

$$\langle \mathbf{S}_1 \rangle_{\sigma_{ABC}} \leq 4$$

⁴ G. Svetlichny, Phys. Rev. D 35, 3066 (1987).

Fine-grained uncertainty relation using

$$a \otimes b \otimes c = s.t + t.u + u.s \text{ (cond.)}$$

- ▶ For the case of quantum physics, we consider the two classes of pure entangled states for tripartite systems, i.e., the GHZ state and the W state.
- ▶ The maximum violation of the Svetlichny inequality is $4\sqrt{2}$ which occurs for the GHZ-state [5], whereas the violation of the Svetlichny inequality by the W-state is given by 4.354 [6].
- ▶ the value of P_{\max}^{game} allowed in quantum physics is $(\frac{1}{2} + \frac{1}{2\sqrt{2}})$.
- ▶ For the case of the considered no-signaling theory the value of P_{\max}^{game} in this case is 1.
- ▶ Here, **FUR** is able to discriminate the different no-signaling theories, i.e., **classical theory**, **quantum theory** and **other no-signaling theory** and also different quantum state with the help of **degree of non-locality**.

⁵D. Collins et al., Phys. Rev. Lett. 88, 170405 (2002), ⁶J. L. Cereceda, Phys. Rev. A 66, 024102 (2002).

Discussion

- ▶ Fine-grained uncertainty relation discriminates different no-signaling theory with the help of the strength of non-locality of bipartite systems manifested by Bell-CHSH inequality.
- ▶ In tripartite system, The winning conditions given by the other two full correlation boxes which violate the Mermin inequality but not the Svetlichny inequality, the fine-grained uncertainty relation is unable to discriminate quantum physics from classical physics in terms of the degree of non-locality.
- ▶ The winning condition given by Svetlichny box discriminates different no-signaling theory with the help of of the strength of non-locality manifested by Svetlichny inequality.
- ▶ Further, within quantum theory it is clear from the upper bound of P_{\max}^{game} that the GHZ-state is more non-local than the W-state

Thank You