The geometric measure of multipartite entanglement

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MULTIPARTITE ENTANGLEMENT

$|\Psi\rangle \in \mathcal{H}_1 \otimes \cdots \mathcal{H}_n$ is entangled if

$$|\Psi\rangle \neq |\phi_1\rangle \cdots |\phi_n\rangle \qquad |\phi_k\rangle \in \mathcal{H}_k$$

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Geometric measure of entanglement $1 - g(|\Psi\rangle)$

$$g(|\Psi\rangle) = \max_{\substack{|\phi_1\rangle\cdots|\phi_n\rangle \\ \langle\phi_k|\phi_k\rangle = 1}} \left|\langle\Psi|\Big(|\phi_1\rangle \cdots |\phi_n\rangle\Big)\right|^2$$

Shimony 1995, Wei & Goldbart 2003

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Extend to mixed states by the usual convex roof definition

APPLICATIONS

- Constructing optimal entanglement witnesses
- Experimental estimate of entanglement
- Quantifies difficulty of distinguishing multipartite states by local means
- Identifying multipartite states for perfect quantum teleportation and superdense coding
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- Detecting quantum phase transitions in spin models

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   $$G\big(|\Psi\rangle|\Phi\rangle\big) = G\big(|\Psi\rangle\big) + G\big(|\Phi\rangle\big)$$

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3. The geometric measure is a lower bound for the relative entropy relative to separable states:

   $$G\big(|\Psi\rangle\big) \leq \min_{\text{separable } \sigma} \mathrm{tr}[\rho \log \rho - \rho \log \sigma]$$

   $(\rho = |\psi\rangle\langle\psi|)$.

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SCHMIDT DECOMPOSITION

Schmidt decomposition of bipartite state ($n = 2$):

$$\text{Given } |\Psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2, \quad \exists \text{ bases } |1\rangle_1, \ldots, |d_1\rangle_1 \text{ of } \mathcal{H}_1$$
$$\text{and} |1\rangle_2, \ldots, |d_2\rangle_2 \text{ of } \mathcal{H}_2$$

such that

$$|\Psi\rangle = \sum_k \lambda_k |k\rangle_1 |k\rangle_2$$

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$$|\Psi\rangle = \sum_k \lambda_k |k\rangle_1 |k\rangle_2$$

$$\lambda_k = \text{singular values of } (c_{ij}) \text{ where } |\Psi\rangle = \sum c_{ij} |i\rangle |j\rangle$$

Geometric measure

$$g(|\Psi\rangle) = \max \lambda_k^2$$

GENERALISED SCHMIDT DECOMPOSITION

H. Carteret, A. Higuchi, AS

Given $|\Psi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, there are bases of $\mathcal{H}_1, \ldots, \mathcal{H}_n$ such that the expansion

$$|\Psi\rangle = \sum c_{i_1 \cdots i_n} |i_1\rangle \cdots |i_n\rangle$$

has the minimum number of terms, with coefficients satisfying:

1. $c_{ji\cdots i} = c_{iji\cdots i} = \cdots = c_{i\cdots ij} = 0$ if $1 \leq i < j \leq d$;

This reduces the number of non-zero coefficients by $nd(d+1)/2$.

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Three qubits

$$|\Psi\rangle = a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle + f|111\rangle$$

$$a, b, c, d \text{ real}, \qquad a \geq b \geq c \geq d \geq 0.$$

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Geometric measure

$$g(|\Psi\rangle) = \text{coeff. of } |0\rangle|0\rangle \cdots |0\rangle$$

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GENERALISED SINGULAR-VALUE EQUATIONS

Problem ($n = 3$): Given $|\Psi\rangle = \sum a_{ijk}|i\rangle|j\rangle|k\rangle$,

find $\quad |\theta\rangle = \sum x_i|i\rangle, \quad |\phi\rangle = \sum y_j|j\rangle, \quad |\psi\rangle = \sum z_k|k\rangle$

to maximise $$\left|\langle\Psi|\big(|\theta\rangle|\phi\rangle|\psi\rangle\big)\right|^2$$

subject to $$\langle\theta|\theta\rangle = \langle\phi|\phi\rangle = \langle\psi|\psi\rangle = 1.$$

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Lagrange multipliers $\lambda_1, \lambda_2, \lambda_3 \quad \Longrightarrow \quad$

$$\begin{aligned} a_{ijk}y_jz_k &= \lambda_1\overline{x_i} \\ a_{ijk}x_iz_k &= \lambda_2\overline{y_j} \\ a_{ijk}x_iy_j &= \lambda_3\overline{z_k} \end{aligned}$$

$\|\mathbf{x}\| = \|\mathbf{y}\| = \|\mathbf{z}\| = 1 \;\Rightarrow\; \lambda_1 = \lambda_2 = \lambda_3 =$ required maximum λ.

SINGULAR VALUES OF A TENSOR

$$\text{Lagrange multipliers } \lambda_1, \lambda_2, \lambda_3 \quad \Longrightarrow \quad \begin{aligned} a_{ijk} y_j z_k &= \lambda_1 \overline{x_i} \\ a_{ijk} x_i z_k &= \lambda_2 \overline{y_j} \\ a_{ijk} x_i y_j &= \lambda_3 \overline{z_k} \end{aligned}$$

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$$\text{Compare } n = 2: \qquad \begin{aligned} a_{ij} y_j &= \lambda \overline{x_i} \\ a_{ij} x_i &= \lambda \overline{y_j} \end{aligned} \qquad\qquad \begin{aligned} A\mathbf{y} &= \lambda \overline{\mathbf{x}} \\ A^\dagger \overline{\mathbf{x}} &= \overline{\lambda} \mathbf{y} \end{aligned}$$

Choose phases to make λ real and ≥ 0: λ is a singular value of A (λ^2 is an eigenvalue of $A^\dagger A$).

DISCRIMINANTS AND HYPERDETERMINANTS

Homogeneous polynomial $f(z_1, \ldots, z_N)$.

The **discriminant** Δ_f is a polynomial in the coefficients of f such that

$$\Delta_f = 0 \iff \exists\, \mathbf{z} \text{ s. t. } \frac{\partial f}{\partial z_k}(\mathbf{z}) = 0, \quad k = 1, \ldots, N.$$

Examples

1. $f(\mathbf{z}) = a_{ij} z_i z_j = \mathbf{z}^{\mathrm{T}} A \mathbf{z}$:

$$\Delta_f = \det A.$$

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Examples

2. $N = d_1 + d_2, \quad \mathbf{z} = (\mathbf{x}, \mathbf{y}), \quad f(\mathbf{x}, \mathbf{y}) = a_{ij} x_i x_j$:

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Examples

3. $N = d_1 + d_2 + d_3, \quad \mathbf{z} = (\mathbf{u}, \mathbf{v}, \mathbf{w}), \quad f = a_{ijk} u_i v_j w_k$:

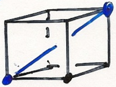
$\Delta_f =$ **hyperdeterminant** of the hypermatrix a_{ijk}
$= 0$ if a_{ijk} has singular value 0.

$d_1 = d_2 = d_3$ (three qubits): $|\Delta_f|^2 =$ 3-tangle (al. et Wootters)

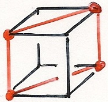
THE 3 x 3 HYPERDETERMINANT

Cayley's hyperdeterminant of $T = (t_{ijk})$

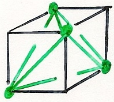
$$\Delta(T) = t_{000}^2\, t_{111}^2 + t_{001}^2\, t_{110}^2 + t_{010}^2\, t_{101}^2 + t_{011}^2\, t_{100}^2$$



$$-2\left(t_{000}\, t_{001}\, t_{110}\, t_{111} + \cdots\right)$$



$$+4\left(t_{000}\, t_{011}\, t_{101}\, t_{110} + t_{001}\, t_{010}\, t_{100}\, t_{111}\right)$$



CHARACTERISTIC POLYNOMIAL OF A TENSOR

Theorem (Hilling, AS) $\alpha : \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_n} \to \mathbb{C}$ multilinear:

$$\alpha\left(\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(d_n)}\right) = a_{i_1 \cdots i_n} u^{(1)}_{i_1} \cdots u^{(n)}_{i_n}$$

Given $\lambda \in \mathbb{R}$, define $\tilde{\alpha}(\lambda) : \mathbb{R}^{2d_3} \times \cdots \times \mathbb{R}^{2d_n} \to \mathbb{R}$ by

$$\tilde{\alpha}(\lambda)\left(\mathbf{u}^{(3)}, \ldots, \mathbf{u}^{(n)}\right) = \det\left[A^\dagger A - \|\mathbf{u}^{(3)}\|^2 \cdots \|\mathbf{u}^{(n)}\|^2 \mathbf{I}, \right]$$

$$A_{i_1 i_2} = a_{i_1 i_2 i_3 \cdots i_n} u^{(3)}_{i_3} \cdots u^{(n)}_{i_n}.$$

If $\lambda \neq 0$, the equations

$$a_{i_1 \cdots i_n} u^{(1)}_{i_1} \cdots \widehat{u^{(r)}_{i_r}} \cdots u^{(n)}_{i_n} = \lambda \overline{u^{(r)}_{i_r}}$$

have a solution with all $\mathbf{u}^{(r)}$ non-zero if and only if $\tilde{\alpha}(\lambda)$ has a real critical point. If this so, λ satisfies the polynomial equation

$$\text{discriminant of } \tilde{\alpha}(\lambda) = 0.$$

THREE-QUBIT STATES

$$|\Psi\rangle = \sum a_{ijk}|i\rangle|j\rangle|k\rangle \qquad (i,j,k=0,1)$$

Geometric measure of entanglement $g(|\Psi\rangle) = \lambda$ given by

$$\begin{aligned} a_{ijk}y_j z_k &= \lambda\overline{x_i} \\ a_{ijk}x_i z_k &= \lambda\overline{y_j} \\ a_{ijk}x_i y_j &= \lambda\overline{z_k} \end{aligned}$$

Let $A(\mathbf{z}) = 2\times 2$ matrix $a_{ijk}z_k$; then

$$A(\mathbf{z})\mathbf{y} = \lambda\overline{\mathbf{x}}, \quad A(\mathbf{z})^\dagger\overline{x} = \lambda\mathbf{y},$$

$$\text{so} \quad \tilde{\alpha}(\lambda)(\mathbf{z},\overline{\mathbf{z}}) = \det\left[A(\mathbf{z})^\dagger A(\mathbf{z}) - \lambda^2\mathbf{z}^\dagger\mathbf{z}\right] = 0.$$

Characteristic equation of a_{ijk}:

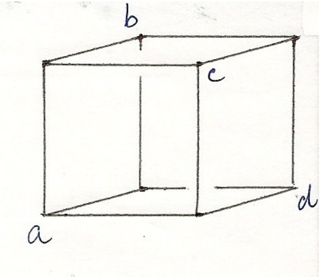
$$\Delta(\lambda) = \text{discriminant } \Delta_{z,\overline{z}}\tilde{\alpha}(\lambda) = \Delta_{\overline{z}}(\Delta_z\tilde{\alpha}(\lambda)) = 0$$

Degree 12 in λ^2.

THE ART OF THE SOLUBLE

Tetrahedral state

$$|\Psi\rangle = a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle$$



THE ART OF THE SOLUBLE

Tetrahedral state

$$|\Psi\rangle = a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle$$

$$\Delta(\lambda) = -16a^2b^2c^2d^2(\lambda^2 - a^2)(\lambda^2 - b^2)(\lambda^2 - c^2)(\lambda^2 - d^2)Q(\lambda)^2$$

$$Q(\lambda) = \lambda^4(4S^2\lambda^2 - L^2)(4S'^2\lambda^2 - L'^2)$$

$$L^2 = (ab + cd)(ac + bd)(ad + bc)$$
$$S^2 = (s - a)(s - b)(s - c)(s - d)$$
$$s = \tfrac{1}{2}(a + b + c + d)$$

S = area of cyclic quadrilateral with sides a, b, c, d (Brahmagupta)

S', L' obtained from S, L by changing sign of d.

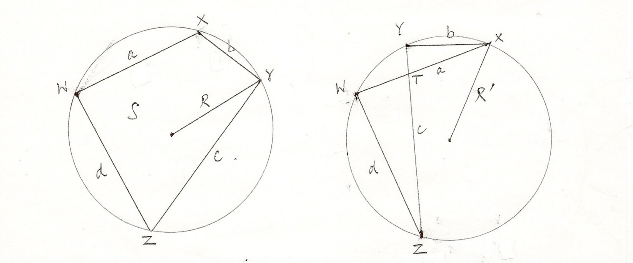
SOLUTIONS OF THE CHARACTERISTIC EQUATION

$$|\Psi\rangle = a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle$$

$$\lambda = 0 \text{ (twice)},\ a,\ b,\ c,\ d,\ \frac{L}{2S} \text{ (twice)},\ \frac{L'}{2S'} \text{ (twice)}.$$

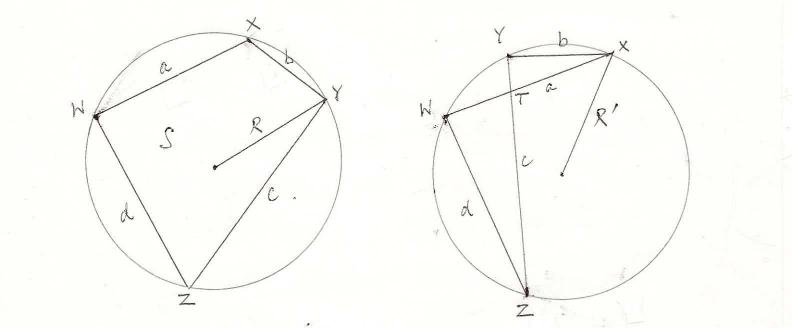
$$\frac{L}{2S} = 2R = \text{ diameter of circumcircle of cyclic quadrilateral}$$

$$\frac{L'}{2S'} = 2R' = \text{ diameter of circumcircle of self-intersecting quadrilateral}$$



GEOMETRY OF THE GEOMETRIC MEASURE

Tamaryan, Park, Son & Tamaryan



$$\left.\begin{matrix} S \\ S' \end{matrix}\right\} = |\vec{WX} \times \vec{YX} + \vec{WZ} \times \vec{YZ}| = \begin{cases} \text{area of } WXYZ \\ |\Delta TWZ - \Delta TXY| \end{cases}$$

GETTING THE RIGHT SOLUTION

$$S^2 = \frac{1}{16}(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)$$

$$S'^2 = S^2 - abcd$$

If $S^2 < 0$, there is no cyclic quadrilateral and no solution $2R$.
In this case, $g(|\Psi\rangle) = \max(a, b, c, d)$.

If $0 < S^2 < abcd$, $2R = L/2S$ is the largest solution.

If $S^2 > abcd$, the largest solution is either $2R$ or $2R'$.
But only $2R$ gives a real critical point.

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MATRICES vs. HYPERMATRICES

	Matrices	2×2×2 hypermatrix
Singular vectors span $\mathcal{H}$	✓	✓✓
Singular values are real	✓	✗
Every sol^{n} of the char. eq^{n} is a singular value	✓	✗
Singular vectors with λ = vector subspace	✓	✗
Multiplicity of λ = dimension	✓	
= number of sing. vectors		✓
Singular vectors with different λ are orthogonal	✓	✗

MANY QUBITS

Generalised W state

$$|\Psi\rangle = c_1|10\ldots0\rangle + c_2|010\ldots0\rangle + \cdots + c_n|0\ldots01\rangle$$

$(0 \leq c_1 \leq \cdots \leq c_n \in \mathbb{R})$

Slightly entangled $(\max|c_k|^2 > 1/2)$:

$$g(|\Psi\rangle) = c_n^2$$

$|\Psi\rangle$ has many different nearest product states.

Highly entangled ($c_n^2 \leq 1/2$):

$|\Psi\rangle$ has nearest product state $|u_1\rangle \ldots |u_n\rangle$ where

$$|u_k\rangle = \sin\theta_k|0\rangle + e^{i\phi}\cos\theta_k|1\rangle$$

and the geometric measure is

$$g(|\Psi\rangle = 2r\sin\theta_1\sin\theta_2\ldots\sin\theta_n$$

where r is the unique solution of

$$\sqrt{1-\frac{c_1^2}{r^2}} + \cdots + \sqrt{1-\frac{c_{n-1}^2}{r^2}} \pm \sqrt{1-\frac{c_n^2}{r^2}} = n-2$$

and
$$\sin 2\theta_k = \frac{c_k}{r}.$$

Then

$$\cos^2\theta_1 + \cdots + \cos^2\theta_n = 1,$$

so the set of highly entangled states has the form $S^{n-1} \times S^1$.

Highly entangled ($c_n^2 \leq 1/2$):

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($+$ if $c_n < r_0$; $-$ if $r_0 < c_n < \sqrt{c_1^2 + \cdots + c_{n-1}^2}$);
r_0 satisfies the same equation with the last term removed;
ϕ is arbitrary; and

$$\sin 2\theta_k = \frac{c_k}{r}.$$