Alternating two dimensional quantum walks constructed using a single qubit coin

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Overview

Joint work with...Carlo Di Franco and Thomas Busch (University College Cork) and Takuya Machida (Meiji University, Japan)

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2. Quantum Random Walks
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   - The Hadamard Walk
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A classical random walk can be defined by specifying

1. States: $|n\rangle$ for $n \in \mathbb{Z}$
2. Allowed transitions:

$$|n\rangle \to \begin{cases} |n-1\rangle & \text{a left move} \\ \text{or} & \\ |n+1\rangle & \text{a right move} \end{cases}$$

3. An initial state: $|0\rangle$
4. Rule(s) to carry out the transitions: In our case we toss a (fair) coin , and move left or right with equal probability (0.5).

This defines a 1 dimensional random walk, sometimes known as the "drunk man's walk", which is well understood in mathematics and computer science. Let us denote by $p_c(n, k)$

the probability of finding the particle at position k in an n step walk ($-n \leq k \leq n$). Some of its properties are

1. The probability distribution is Gaussian (plot of $p_c(n,k)$ against k).
2. For an odd (even) number of steps, the particle can only finish at an odd (even) integer position: $p_c(n,k) = 0$ unless $(n-k) \mod 2 = 0$
3. The maximum probability is always at the origin (for an even number of steps in the walk): $p_c(n,0) > p_c(n,k)$ for even n and even $k \neq 0$.
4. (Non-localization) For an infinitely long walk, the probability of finding the particle at any fixed point goes to zero: $\lim_{n\to\infty} p_c(n,k) = 0$.
5. On average, after n steps the walker will be at distance $\sqrt{n}$ from the origin.

Example: A short classical walk

In a 3 step walk, starting at $|0\rangle$, what is the probability of finding the particle at the point $|-1\rangle$?
We denote by $L(R)$ a left (right) step respectively.

- Possible paths that terminate at $|-1\rangle$ are LLR, LRL and RLL, i.e. those paths with precisely 1 right step and 2 left steps. So there are 3 possible paths.
- Total number of possible paths is $2^3 = 8$.

So the probability is $3/8 = 0.375$.

So, how do we calculate the probabilities?

Physicist	(Feynman) Path Integral.
Statistician	Summation over Outcomes.

Let us denote by N_L the number of left steps and N_R the number of right steps. Of course, fixing N_L and N_R fixes k and n, and vice versa: The equations are

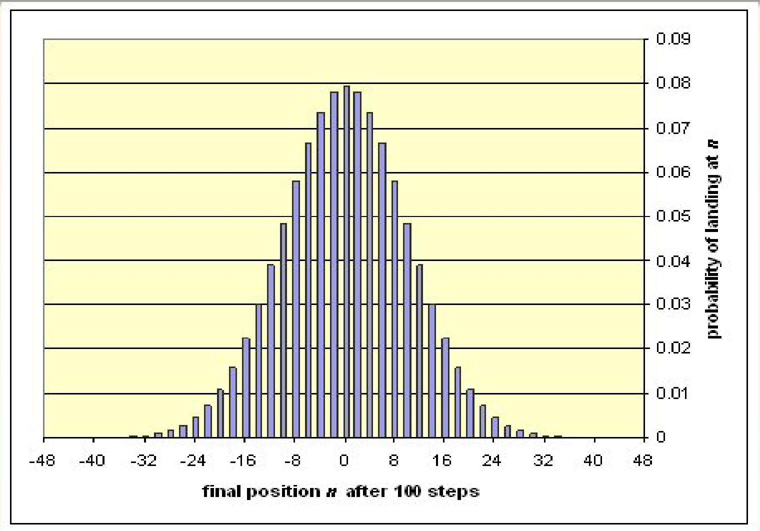
$$\begin{aligned} N_R + N_L &= n \\ N_R - N_L &= k \end{aligned}$$

We have an easy closed form solution for the probabilities $p_c(n,k)$: For a fixed N_L and N_R,

$$\frac{n!}{N_L! N_R! 2^n} = \frac{n!}{((n-k)/2)!((n+k)/2)! 2^n} = p_c(n,k)$$

We can tabulate the probabilities for the first few iterations:

	Number of steps				
position	0	1	2	3	4
4	0	0	0	0	0.0625
3	0	0	0	0.125	0
2	0	0	0.25	0	0.0625
1	0	0.5	0	0.375	0
0	1	0	0.5	0	0.0375
-1	0	0.5	0	0.375	0
-2	0	0	0.25	0	0.0625
-3	0	0	0	0.125	0
-4	0	0	0	0	0.0625


0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
probability of landing at n
-48
-40
-32
-24
-16
-8
0
8
16
24
32
40
48
final position n after 100 steps

Quantum Random Walks: Definitions

One Dimensional Discrete Quantum Walks (also known as Quantum Markov Chains) take place on the State Space spanned by vectors

$$|n, p\rangle \tag{1}$$

where $n \in Z$ (the integers) and $p \in \{0, 1\}$ is a boolean variable. p is often called the 'coin' state or the chirality, with

$0 \equiv$ spin up

$1 \equiv$ spin down

We can view p as the "quantum part" of the walk, while n is the "classical part".
One step of the walk is given by the transitions

$$|n, 0\rangle \longrightarrow a\,|n-1, 0\rangle + b\,|n+1, 1\rangle \tag{2}$$

$$|n, 1\rangle \longrightarrow c\,|n-1, 0\rangle + d\,|n+1, 1\rangle \tag{3}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2), \tag{4}$$

the group of 2×2 unitary matrices of determinant 1.

Aside

We can view the transitions as consisting of 2 distinct steps, a "coin flip" operation C followed by a shift operation S:

$$C: \qquad |n,0\rangle \longrightarrow a\,|n,0\rangle + b\,|n,1\rangle \tag{5}$$
$$C: \qquad |n,1\rangle \longrightarrow c\,|n,0\rangle + d\,|n,1\rangle \tag{6}$$
$$S: \qquad |n,p\rangle \longrightarrow |n \pm 1, p\rangle \tag{7}$$

These walks have also been well studied: See Kempe (arXiv:quant−ph/0303081) for a thorough review.

The final probability distribution depends on

- The initial (coin) state
- The coin flip matrix used

The Hadamard Walk

We choose for our coin flip operation the Hadamard matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{8}$$

If we start at initial state $|0,0\rangle$, the first few steps of a standard quantum (Hadamard) random walk would be

$$|0,0\rangle \longrightarrow \frac{1}{\sqrt{2}}(|-1,0\rangle + |1,1\rangle) \longrightarrow \tag{9}$$

$$\frac{1}{2}(|-2,0\rangle + |0,1\rangle + |0,0\rangle - |2,1\rangle) \longrightarrow \tag{10}$$

$$\frac{1}{2\sqrt{2}}(|-3,0\rangle + |-1,1\rangle + |-1,0\rangle - |1,1\rangle + |-1,0\rangle + |1,1\rangle - |1,0\rangle + |3,1\rangle). \tag{11}$$

Thus after the third step of the walk we see
destructive interference (cancellation of 4th. and 6th. terms)
constructive interference (addition of 3rd. and 5th. terms)
which are features that do not exist in the classical case.
The calculation of $p_q(n,k)$ proceeds by again looking at all paths that lead to a particular position k, but this time, since we are in the quantum domain:

- We calculate firstly amplitudes $-$ so we have a $1/\sqrt{2}$ factor added at every step, and final probabilities are amplitudes squared (in our examples, with the Hadamard walk there are no imaginary numbers).
- There are also phases that we must take account of in our amplitude calculations.

In particular, note that the phase -1 from the Hadamard matrix arises every time, in a particular path, we follow a right step by another right step.

Asymptotic Properties

Brun, Carteret & Ambainis (arXiv:quant$-$ph/0210161) have calculated explicitly (using combinatorial techniques) the amplitudes:

Amplitude for final state $|k, 0\rangle$

$$\frac{1}{\sqrt{2^n}} \sum_{C=1}^{M} (-1)^{N_L - C} \binom{N_L - 1}{C - 1} \binom{N_R}{C - 1}$$

Amplitude for final state $|k, 1\rangle$

$$\frac{1}{\sqrt{2^n}} \sum_{C=1}^{M} (-1)^{N_L - C} \binom{N_L - 1}{C - 1} \binom{N_R}{C}$$

where M has value N_L for $k \geq 0$ and value $N_R + 1$ otherwise.
The analyses of Kempe (and others) show that for the quantum (Hadamard) random walk with initial state $|0, 0\rangle$

1. The probability distribution $p_q(n, k)$ is not gaussian $-$ it oscillates with many peaks.

2. For large n, the place at which the particle is most likely to be found is not at the origin: rather it is most likely to be at distance $n/\sqrt{2}$ from the origin.
3. In some general sense, the particle "travels further": The probability distribution is spread fairly evenly between $-n/\sqrt{2}$ and $n/\sqrt{2}$, and only decreases rapidly outside these limits.
4. It is not even symmetric. The asymmetric nature of $p_q(n,k)$ is a figment of the initial state chosen: We can choose a more symmetric initial state to give a symmetric probability distribution.
5. (Non-localization) For an infinitely long walk, the probability of finding the particle at any fixed point goes to zero: $\lim_{n\to\infty} p_q(n,k) = 0$.

2D Quantum Random Walk

In general, to move on a 2D square lattice, we toss a "coin with four sides". Our setup is

1. States: $|m, n, p\rangle$ for $m, n \in \mathbb{Z}$ and $p \in \{0, 1, 2, 3\}$.
2. Transitions:

$$
\begin{aligned}
|m,n,0\rangle &\longrightarrow a_{11}\,|m+1,n+1,0\rangle + a_{12}\,|m+1,n-1,1\rangle \\
&\quad + a_{13}\,|m-1,n+1,2\rangle + a_{14}\,|m-1,n-1,3\rangle \\
|m,n,1\rangle &\longrightarrow a_{21}\,|m+1,n+1,0\rangle + a_{22}\,|m+1,n-1,1\rangle \\
&\quad + a_{23}\,|m-1,n+1,2\rangle + a_{24}\,|m-1,n-1,3\rangle \\
|m,n,2\rangle &\longrightarrow a_{31}\,|m+1,n+1,0\rangle + a_{32}\,|m+1,n-1,1\rangle \\
&\quad + a_{33}\,|m-1,n+1,2\rangle + a_{34}\,|m-1,n-1,3\rangle \\
|m,n,3\rangle &\longrightarrow a_{41}\,|m+1,n+1,0\rangle + a_{42}\,|m+1,n-1,1\rangle \\
&\quad + a_{43}\,|m-1,n+1,2\rangle + a_{44}\,|m-1,n-1,3\rangle
\end{aligned}
$$

where

$$A = (a_{ij})$$

is a 4x4 unitary matrix.

3. An initial state: $|0, 0, p\rangle$

The Grover Walk

This is a 2D quantum walk using the "coin"

$$G = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

which is the 4x4 Grover "diffusion" matrix $G = -2\,|s\rangle\,\langle s| + I$, where $s = \sum_x |x\rangle\,/\sqrt{n}$.

Observation

G is equivalent to the 4x4 Sylvester matrix $H \otimes H$, where H is the standard 2x2 Hadamard.

The remarkable feature of the Grover walk is we get localization at the origin ($\lim_{n\to\infty} p_q(n,(0,0)) \neq 0$) for *all initial states* except the particular initial state

$$(|0,0,0\rangle - |0,0,1\rangle - |0,0,2\rangle + |0,0,3\rangle)/2.$$

The Alternating Walk

Our setup is

1. States: $|m,n,c\rangle$ for $m,n \in \mathbb{Z}$ and $c \in \{0,1\}$.
2. Transitions:

1st. step (horizontal move)

$$\begin{aligned} |m,n,0\rangle &\longrightarrow a\,|m-1,n,0\rangle + b\,|m+1,n,1\rangle \\ |m,n,1\rangle &\longrightarrow c\,|m-1,n,0\rangle + d\,|m+1,n,1\rangle \end{aligned}$$

2nd. step (vertical move)

$$\begin{aligned} |m,n,0\rangle &\longrightarrow a\,|m,n-1,0\rangle + b\,|m,n+1,1\rangle \\ |m,n,1\rangle &\longrightarrow c\,|m,n-1,0\rangle + d\,|m,n+1,1\rangle \end{aligned}$$

3. An initial state: $|0,0,c\rangle$

This produces a probability distribution on the same (square lattice) nodes as in the standard 2D quantum walks.

Question

When do the two probability distributions (from the standard 2D quantum walk and from our alternating walk) coincide?

Answer (so far)

We have a (1 parameter) family of cases where coincidence occurs.

Result 1

The following two models coincide:

model **(a)**

Standard 2D quantum walk with

- Initial state $(|0,0,0\rangle - |0,0,1\rangle - |0,0,2\rangle + |0,0,3\rangle)/2$.
- Coin

$$G = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

model **(b)**

Alternating quantum walk with

- Initial state $(|0,0,0\rangle + i\,|0,0,1\rangle)/\sqrt{2}$.
- Coin

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Specifically in the wavefunction, if we denote by

$\alpha_{m,n,p}(t)$ the coefficient of the state $|m,n,p\rangle$ at time t in the standard 2D quantum walk;

$\beta_{m,n,c}(t)$ the coefficient of the state $|m,n,c\rangle$ at time t in the alternating walk

then the explicit correspondence is

$$\begin{pmatrix} \beta_{m,n,o}(t) \\ \beta_{m,n,1}(t) \end{pmatrix} = (-1)^t e^{i\pi/4} \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & -1 & 0 & i \end{pmatrix} \begin{pmatrix} \alpha_{m,n,0}(t) \\ \alpha_{m,n,1}(t) \\ \alpha_{m,n,2}(t) \\ \alpha_{m,n,3}(t) \end{pmatrix}$$

Our Understanding: From the following table - the complex amplitudes of the single qubit store twice as many numbers as the strictly real amplitudes of the 4-state Grover coin.

	Entries in the initial state	Entries in the coin matrix	Entries in arbitrary state
Alternate Walk	Complex values	Real Values	Complex values
2D Grover Walk	Real Values	Real Values	Real Values

We asked the question - if we use other (real) 2x2 matrices (as a coin flip) in an alternating walk, what do they correspond to in terms of the 2D picture?

Result 2

We have found correspondences (in 2D quantum walks) for alternating walks using any possible real single qubit coin operator. All these coin operators are simply the 2x2 orthogonal matrices which are classified as

rotations	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	(which includes the identity)
reflections	$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$	(which includes the Hadamard, and the “swap” (permutation) matrix)

Entanglement generation

Let us consider the entanglement between x and y positions of the state at time t. We find that our alternating walk, with the Hadamard matrix and initial state $(|0,0,0\rangle + i\,|0,0,1\rangle)/\sqrt{2}$ generates more of this entanglement than its corresponding Grover walk: Even though the probability amplitudes correspond at every point in the lattice, the spatial entanglement does not.

We have investigated the dependence of this on an arbitrary initial state,

$$|\psi\rangle = \cos(\frac{\theta}{2})\,|0,0,0\rangle + e^{i\phi}\sin(\frac{\theta}{2})\,|0,0,1\rangle$$

and we find maximal entanglement uniquely when $\theta = \phi = \pi/2$.

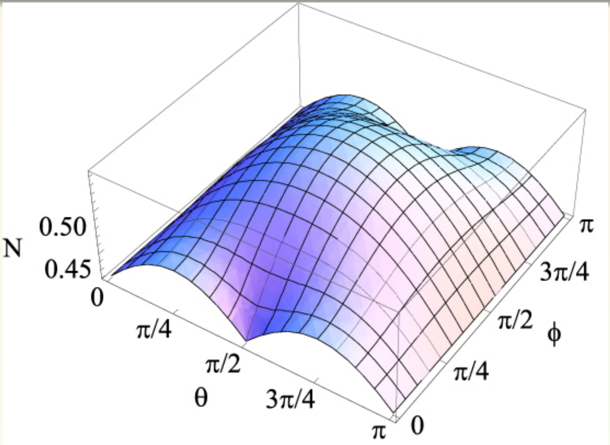


Figure: Entanglement (Negativity) between x and y positions as a function of the initial state $(\cos(\theta/2)\,|0,0,0\rangle + e^{i\phi}\sin(\theta/2)\,|0,0,1\rangle)$ for the alternating walk

Conclusions

- We have shown a new correspondence between certain alternating walks and certain 2D quantum walks - the advance here is physically for the alternating walk, it is easier experimentally.
- We show the spatial entanglement generated in the alternating walk is superior to the 2D one.
- (Time permitting) 1D quantum random walk with memory displays localization at the origin.

References

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