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Lecture Series on Relativistic Quantum Information

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Contents

Abstract

The insight that the world is fundamentally quantum mechanical inspired the development of quantum information theory. However, the world is not only quantum but also relativistic, and indeed many implementations of quantum information tasks involve truly relativistic systems. In this lecture series I consider relativistic effects on entanglement in flat and curved spacetimes. I will emphasize the qualitative differences to a non-relativistic treatment, and demonstrate that a thorough understanding of quantum information theory requires taking relativity into account. The exploitation of such relativistic effects will likely play an increasing role in the future development of quantum information theory. The relevance of these results extends beyond pure quantum information theory, and applications to foundational questions in cosmology and black hole physics will be presented.

Introduction

The main purpose of the research in the field of relativistic quantum information is developing quantum information theory that is compatible with the relativistic structure of spacetime. An ultimate aim is to exploit relativity in order to improve quantum information tasks. The vantage point of these investigations is that the world is fundamentally both quantum and relativistic. Impressive technological achievements and promises have already been derived from taking seriously solely the quantum aspects of matter: quantum cryptography and communication have become a technical reality in recent years, but the practical construction of a quantum computer still requires to understand how to efficiently store, manipulate and read information, without prohibitively large disturbances from the environment. Throwing relativity into the equation fundamentally changes the entire game, as I intend to show in this series of lectures. Hopefully, we will be able to push this exciting line of theoretical research to the point where relativistic effects in quantum information theory can be exploited technologically.

Far from yielding only quantitative corrections, relativity plays a dominant role in the qualitative behavior of many physical systems used to implement quantum information tasks in the laboratory. The prototypical example is provided by any system involving photons, be it for the transmission or manipulation of quantum information. There is no such thing as a non-relativistic approximation to photons, since these always travel at the speed of light. While relativistic quantum theory, commonly known as quantum field theory, is a very well studied subject in foundational particle physics, research in quantum information theory selectively focused almost exclusively on those aspects one can study without relativity. Thus both unexpected obstacles (such as relativistic degradation of entanglement) and unimagined possibilities for quantum information theory (such as improved quantum cryptography and hypersensitive quantum measurement devices) have gone unnoticed. Moreover, the impact of the work done in the field of relativistic quantum information extends beyond pure quantum information theory, and applications to foundational questions in cosmology and black hole physics have been found.

BACKGROUND

Quantum information was first considered in a relativistic setting by Czachor in 1997 who analyzed a relativistic version of the Einstein-Podolsky-Rosen-Bohm experiment [2, 3]. Czachor's work showed that relativistic effects are relevant to the experiment where the degree of violation of Bell's inequalities depends on the velocity of the entangled particles. In 2002 A. Peres & D. Terno at Technion pointed out that most concepts in quantum information theory may require a reassessment [4]. Further interesting results on entanglement in flat spacetime were obtained by Adami, Bergou, & Gingrich at Caltech, and Solano & Pachos at the Max-Plank Institute [5]. Their work shows that although entanglement is overall conserved under a change of inertial frame, it may swap between spin and position degrees of freedom. For the physically interesting case of non-inertial frames, however, collaborators and I were able to show in a series of papers [6, 7, 8] that entanglement is observer-dependent, since it is degraded from the perspective of observers in uniform acceleration. This effect introduces errors in entanglement-based quantum information tasks in non-inertial frames such as the teleportation scheme studied by P. Alsing (University of New Mexico) & G. Milburn (University of Queensland) [9, 10]. Errors induced by relativistic effects were also found in a cryptographic protocol analyzed by M. Czachor and M. Wilczewski at Politechnika Gdańska [11]. However, J. Barrett (Universite Libre de Bruxelles), L. Hardy (Perimeter Institute), and A. Kent (University of Cambridge) showed that constraints imposed by relativity can be useful in ensuring security in quantum cryptography even if quantum theory is incorrect [12]. Considering quantum information in curved spacetime is more complicated since generically particle states are ill-defined if the background spacetime does not at least feature a timelike Killing vector field. In spacetimes with two asymptotically flat regions, however, we showed that measurements of entanglement may be used to learn about the history of spacetime [13]. This work was followed up by G. Ver Steeg (Caltech) & N. Menicucci (Princeton) [14]. Entanglement in spacetime has also been studied by Shi at Tsinghua University [15] and by P. Kok, U. Yurtsever, S. L. Braunstein, and J. P. Dowling at Caltech and Bangor University [16].

Employing quantum information to address open questions in other fields, gravity being a particularly fruitful example, has become important to quantum information scientists. J. Preskill (Caltech), D. Gottesman (Perimeter Institute), D. Ahn (University of Soul), S. Lloyd (MIT), G. Adesso and myself, among others, have shown that quantum information is a useful tool in the understanding of the information loss problem in black holes [17]. Quantum information has also been recently employed by D. Terno (Macquarie University), E. Livine (ENS-Lyon) and F. Markopoulou (Perimeter Institute) to make progress in quantum gravity [18]. Relativistic quantum information starts to be a topic of interest in the scientific community. However, technological applications of relativistic quantum information are yet to be proposed.

Chapter 1

Motivation and technical tools in quantum information

1.1 Why relativistic qunatum information?

In information theory we study how to send messages, how to transmit information in secure ways, how to compute and in general, how to process information. In order to process information, information must be stored in physical systems. Therefore, it is the underlying physical theory which sets the rules for which information tasks can be preformed and on their efficiency.

The field of information theory made great progress last century considering that the world is classical. Thanks to the seminal work by Turing [1], it was possible to theoretically predict what problems could, in principle, be solved. This before the first calculator was actually built!

Last century we also learned that there is a more fundamental theory of nature: quantum mechanics. It then became interesting to revise information theory in the light of this new physical theory. This is what quantum information is. We have been learning to exploit quantum properties, such as entanglement, to improve information task. A good example of this is quantum teleportation.

However, something very important is missing in the picture. Las century we also learned that the world is not only quantum but also relativistic. In fact, before quantum information was conceived some relativistic considerations in information theory were considered: we all know that it is not possible to send messages faster than the speed of light.

The field of information theory is going through interesting times. The way we process information is being revolutionized by incorporating quantum theory. However, if we want to push this revolution further, as far as we possibly can, we need to incorporate relativity. It is important to do so now since most, if not all, implementations of quantum information in fact employ relativistic systems. Such is the case in Cavity QED and quantum information based protocols employing photons. The research of relativistic quantum field aims at understanding how to process information in the overlap of quantum theory and relativity: were real life experiments take place. And in the same way as we have been able to exploit quantum resources to improve information tasks we might be able to learn how to make use of relativistic effects as well.

1.2 Abstract quantum information

The main aim of quantum information theory is to learn how to store, process and read information using quantum systems. In this section we will briefly revise basic concepts of quantum information theory. Namely, quantum entanglement for pure and mixed states. Entanglement is a quantum property which is a consequence of the superposition principle and the tensor product structure of the Hilbert space. It plays a central role in the field of quantum information since it is at the heart of many quantum information tasks such as quantum teleportation and quantum cryptography.

Pure States

Entanglement for pure bi-partite systems is well understood. In what follows we will define and learn how to quantifying entanglement in this case.

In quantum theory the state of a quantum particle is a vector in a d-dimensional Hilbert space $\mathcal{H}$ where $\mathcal{H}$ is an inner product space over C. A state in the Hilbert space $\mathcal{H}$ is denoted $|\psi\rangle \in \mathcal{H}$. It is interesting to consider how we introduce a second particle to the description of our system. The state of two particles A and B is a vector in a $(d \times d')$-dimensional Hilbert space $\mathcal{H}_{ab} = \mathcal{H}_a \otimes \mathcal{H}_b$. The space $\mathcal{H}_{ab}$ is the tensor product of the subspaces $\mathcal{H}_a$ and $\mathcal{H}_b$ of each particle. An element of the space $\mathcal{H}_{ab}$ is written as $|\psi_{ab}\rangle = \sum_{i,j} A_{ij} |i\rangle_a \otimes |j\rangle_b$.

Note: This is very different to classical physics where if a particle has three degrees of freedom then two identical particles have $3 \oplus 3 = 6$ degrees of freedom where $\oplus$ denotes the direct sum. In quantum mechanics the vector state describing the state of the two 3-dimensional particles is $3 \times 3 = 9$ dimensional.

Given the tensor product structure of the Hilbert space for two particles we can consider the following definitions:

Definition 1 *A state $|\psi_{ab}\rangle \in \mathcal{H}_a \otimes \mathcal{H}_b$ is seperable if $|\psi_{ab}\rangle = |\psi\rangle_a \otimes |\psi\rangle_b$.*

Comments (physics)

1. A separable state can be prepared by local operations and classical communication. This means that observers manipulate each particle independently by making measurements or applying unitary transformations of the form $|\psi_{ab}\rangle = \hat{U}_a \otimes \hat{U}_b |\psi_i\rangle$ where U_a and U_b are unitaries acting on

particle A and B, respectively. The observers are also allowed to exchange classical information.

2. By measurements on particle A one learns nothing about particle B.

Definition 2 *If the state is not seperable then it is entangled.*

Comments (physics):

1. An entangled state cannot be prepared by local operations and classical communication. Observers must make global operations on the systems. An example are interactions between A and B.

2. By measurements on particle A one can learn infomation about B.

Example 1 ***Two qubits***

A state $|\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_a |1\rangle_b + |1\rangle_a |1\rangle_b \right)$ *in the Hilbert space* $\mathcal{H}^2 \otimes \mathcal{H}^2$ *can be written as the product*

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_a + |1\rangle_a \right) \otimes |1\rangle_b$$

and is thus, a seperable state. If we measure subsystem B then we cannot learn anything about subsystem A.

The state $|\phi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_a |0\rangle_b + |1\rangle_a |1\rangle_b \right)$ *is not a seperable state since we cannot write it as the product of two states that belong seperatlety to the individual subsytems* $\mathcal{H}_a$ *and* $\mathcal{H}_b$ *i.e.*

$$|\phi\rangle \neq |\psi_a\rangle \otimes |\psi_b\rangle$$

.

Question: How do we know if a general state $|\psi_{ab}\rangle = \sum A_{ij} |i\rangle_a |j\rangle_b$ is entangled or not? To answer this consider the following theorem:

Theorem 2 *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be d-dimensional Hilbert spaces. For any vector* $|\psi_{ab}\rangle \in \mathcal{H}_a \otimes \mathcal{H}_b$ *there exists a set of orthonormal vectors*

$$\{|j\rangle_a\} \subset \mathcal{H}_a \text{ and } \{|l\rangle_b\} \subset \mathcal{H}_b$$

such that we can write

$$|\psi_{ab}\rangle = \sum_i \lambda_i |i\rangle_a |i\rangle_b$$

where the Schmidt coefficients λ_i *are non-negative scalars.*

This special basis is called the Schmidt basis. For simplicity we assumed the Hilbert spaces for each system to have the same dimension. However, this statement can be generalized to a $d \times d'$ system. Note that the correlations between systems A and B are now made explicit. Therefore, if $\lambda_{i\neq j} = 0$ and

$\lambda_j = 1$ then the state is seperable. In the case that all the λ_i's are equal and $\lambda_i = \frac{1}{\sqrt{d}}$ then the state is maximally entangled.

It is clear that the distribution of the Schmidt coefficients determine how entangled the state is. Therefore, to quantify entanglement in the pure bi-partite case we need a monotonous and continuous function of the λ_i's such that

1. $S(\lambda_i) = 0$ for separable states
2. $S(\lambda_i) = \log(d)$ for maximally entangled states

Considering the density matrix $\rho_{ab} = |\psi_{ab}\rangle \langle\psi_{ab}|$ and it's reduced density matrix $\rho_b = Tr_a(\rho_{ab})$, we find that the Von-Neuman entropy

$$\begin{aligned} S(\rho_b) &= -Tr\left(\rho_b \log_2 \rho_b\right) \\ &= -\sum_i |\lambda_i|^2 \log_2 |\lambda_i|^2 \end{aligned}$$

quantifies the entanglement between system A and B. We observe from the Schmidt decomposition that it is equivalent to trace over either system A or B and therefore, $S(\rho_a) = S(\rho_b)$. The Von-Neuman entropy of a pure state is $S(\rho_{ab}) = 0$.

Example 3 *Calculate the Von-Neumann entropy of the state* $|\psi\rangle_{ab} = \frac{1}{\sqrt{2}}\left(|0,0\rangle + |1,1\rangle\right)$. *The density matrix is given by*

$$\rho_{ab} = \frac{1}{2}(|0,0\rangle\langle 0,0| + |0,0\rangle\langle 1,1| + |1,1\rangle\langle 0,0| + |1,1\rangle\langle 0,0| + |1,1\rangle\langle 1,1|) \quad (1.1)$$

Tracing over the first system we obtain

$$\rho_a = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \quad (1.2)$$

which is already in diagonal form. The matrix has two degenerate eigenvalues equal to $1/2$. *Therefore, the Von-Neumann entropy is* $S(\rho_a) = 2(1/2\log_2(1/2)) = 1$ *which shows that the state is maximally entangled.*

Mixed States

Quantifying entanglement in the mixed case is more involved since there is no analog to the Schmidt decomposition in this case. However, it is possible to define what a separable mixed state is. A mixed state is separable if we can write its density matrix as

$$\rho_{ab} = \sum_i \omega_i \rho_a^i \otimes \rho_b^i$$

where $\sum_i \omega_i = 1$.

To find out if a general mixed state is entangled or not it is convenient to define the partial transpose of a density matrix. Consider the general mixed state

$$\rho_{ab} = \sum_{ijkl} C_{ijkl} |i\rangle_a |j\rangle_{bb} \langle k|_a \langle l|. \tag{1.3}$$

The partial transpose ρ_{ab}^{PT} of ρ_{ab} is

$$\rho_{ab}^{PT} = \sum_{ijkl} C_{ljki} |i\rangle_a |j\rangle_{bb} \langle k|_a \langle l|. \tag{1.4}$$

or equivalentely

$$\rho_{ab}^{PT} = \sum_{ijkl} C_{ikjl} |i\rangle_a |j\rangle_{bb} \langle k|_a \langle l|. \tag{1.5}$$

Since the partial transpose of a separable state has positive eigenvalues it is possible to construct a separability criterion.

Necessary Condition for Seperability (Peres): If the eigenvalues of $\rho_{ab}^{PT} \geqslant 0$ then ρ_{ab} is separable.

However, this criterion is only sufficient for 2×2 and 2×3 systems. For systems of higher dimension the criterion is only necessary meaning that there are entangled states with positive partial transpose. Such states are known as bound entangled states.

Adding up the negative eigenvalues gives an estimate of how entangled a state is. Therefore, we will now define the negativity and logarithmic negativity which are two entanglement monotones.

Definition 3 *The negativity of a density matrix ρ_{ab} is defined as the sum of the negative eigenvalues of the partial transpose ρ_{ab}^{PT}*

$$N(\rho_{ab}) := \frac{\|\rho_{ab}^{PT}\| - 1}{2}$$

Where $\|\cdot\|$ denotes the trace norm $\|X\| := Tr\left[\sqrt{X^\dagger X}\right]$.

Definition 4 *The logarithmic negativity of a density matrix ρ_{ab} is defined as*

$$E_N(\rho_{ab}) := \log_2 \|\rho_{ab}^{PT}\|$$

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Chapter 2

Technical tools in relativity

We are interested in understanding entanglement when the underlying spacetime is considered in the description of states. In this lecture we will learn some basic notions of spacetime which is a 4-dimentional manifold $\mathcal{M}$ (space) with a Lorentizian metric. For our purpose it is enough to consider that $\mathcal{M}$ is a collection of points which locally looks like $\mathbb{R}^4$. We will introduce the basic mathematical structures we need to define a Lorentzian metric, a light cone and a world line. We will start with the following (strong) remarks:

There is <u>no</u> time

There is <u>no</u> space

There are only cones.

2.1 Necessary mathematical structures

In order to clarify these remarks we will introduce the following mathematical structures:

1. **Coordinates** are 4-functions χ^μ such that $\chi^\mu : p \in M_a \to \mathbb{R}$ where p is a point in the region $\mathcal{M}_a$ of the manifold $\mathcal{M}$. Since $\mu = \{0, 1, 2, 3\}$, the functions $\chi^\mu = (\chi^0, \chi^1, \chi^2, \chi^3)$. A choice of coordinates is not unique, therefore, it is possible to choose other coordinates in M_a, $\bar{\chi}^\mu = \bar{\chi}^\mu(\chi^0, \chi^1, \chi^2, \chi^3)$ such that

$$\begin{aligned} d\bar{\chi}^\mu &= \frac{\partial \bar{\chi}^0}{\partial \chi^0} d\chi^0 + \frac{\partial \bar{\chi}^1}{\partial \chi^1} d\chi^1 + \frac{\partial \bar{\chi}^2}{\partial \chi^2} d\chi^2 + \frac{\partial \bar{\chi}^3}{\partial \chi^3} d\chi^3 \\ &= \frac{\partial \bar{\chi}^\mu}{\partial \chi^\nu} d\chi^\nu \end{aligned}$$

   Note that we sum over the repeated indices (Einstein's convention). The transformation is well defined were the Jacobian is non-zero i.e.

$$det \left(\frac{\partial \bar{\chi}^\mu}{\partial \chi^\nu} \right) \neq 0$$

2. **Contravariant vectors** are a set of quantities that transform according to
$$\bar{\chi}^\mu = \frac{\partial \bar{\chi}^\mu}{\partial \chi^\nu} \chi^\nu \tag{2.1}$$
where $\bar{\chi}^\mu$ and χ^ν are coordinates associated with the point p. In the same fashion we define contravariant tensors as the set of quantities that transform as
$$\bar{T}^{\mu\nu} = \frac{\partial \bar{\chi}^\mu}{\partial \chi^\alpha} \frac{\partial \bar{\chi}^\nu}{\partial \chi^\beta} T^{\alpha\beta} \tag{2.2}$$

3. **Covariant tensors** transform according to
$$\bar{T}_{ab} = \frac{\partial \chi^c}{\partial \bar{\chi}^a} \frac{\partial \chi^d}{\partial \bar{\chi}^b} T_{cd}$$
We say that quantities with a single index are of rank 1 (vectors) and that quantities that have two indices are of rank 2.

4. **Metric tensors** are rank 2 covariant tensors which are symmetric i.e.
$$g_{\alpha\beta} = g_{\beta\alpha}$$
If $det(g) \neq 0$ then the metric is non-singular and we can define the inverse metric tensor $g^{\alpha\beta}$ such that
$$g_{ab} g^{bd} = \delta_a{}^d.$$

2.2 Lorentzian metrics, light cones and world lines

We are now in position to define the basic notions of spacetime.

1. A **Lorentzian metric** is a metric with signature $\{+ - - -\}$. That is if at any given point in spacetime we can find coordinates such that
$$g_{ab} = \eta_{ab} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$
Example 4 *In flat spacetime the metric $g_{ab} = \eta_{ab}$ everywhere.*

Metrics are used to define distances and length vectors.

2. A **line element** is the infinitesimal distance between two neighbouring points χ^a and $\chi^a + d\chi^a$ defined as
$$ds^2 = g_{ab}(\chi) d\chi^a d\chi^b$$
.

Example 5 *In flat spacetime in* $(1+1)$*-dim the metric is* $\eta_{ab} = \{+-\}$ *and so*

$$\begin{aligned} ds^2 &= \eta_{ab} d\chi^a d\chi^b \\ &= \eta_{00}(d\chi^0)^2 + \eta_{11}(d\chi^1)^2 \\ &= dt^2 - dx^2 \end{aligned}$$

3. The **norm of a contravariant vector** χ^a is defined as

$$\begin{aligned} g(\chi,\chi) &:= \chi^2 \\ &= g_{ab}\chi^a\chi^b \end{aligned}$$

   We can therefore define three types of vector as

   (a) timelike $g_{ab}\chi^a\chi^b > 0$

   (b) spacelike $g_{ab}\chi^a\chi^b < 0$

   (c) null or lightlike $g_{ab}\chi^a\chi^b = 0$

4. **Light cone** The set of all null vectors at p define a light cone. $\eta_{ab}\chi^a\chi^b = 0$ is the equation of a double light cone.

5. A **timelike world line** is a curve $\chi^\mu(u)$ whose tangent vector is everywhere timelike. These are the tracks where particles and observers can travel.

   Using the definition of the line element we can derive the interval between two spacetime points p_1 and p_2. Taking the line element and dividing by du^2 we see that

$$\left(\frac{ds}{du}\right)^2 = g_{ab}\left(\frac{d\chi^a}{du}\right)\left(\frac{d\chi^b}{du}\right)$$

   where u parametrizes the trajectory. Thus integrating over u we find the proper distance between two points is

$$L[\chi] := \int \sqrt{g_{ab}\left(\frac{d\chi^a}{du}\right)\left(\frac{d\chi^b}{du}\right)}du$$

   REMARK This implies that there is no global time and no global space. There is a well defined notion of time for each observer given by the length $L[\chi] = \zeta$.

Example 6 *We will find the line element* ds^2 *of the* $(1+1)$*-dim flat spacetime in the following coordinates*
a) $u = t - x$ *and* $v = t + x$
b) $x = \frac{e^{a\xi}}{a}\cosh(a\zeta)$ *and* $t = \frac{e^{a\xi}}{a}\sinh(a\zeta)$ *where* a *is a constant. These coordinates are known as Rindler coordinates.*

a) The line element

$$ds^2 = g_{ab}(X)dX^a dX^b$$

in flat spacetime is given by $ds^2 = dt^2 - dx^2$. Thus, to find the line element in the new coordinates we need to calculate the derivatives

$$\begin{aligned} du &= dt - dx \\ dv &= dt + dx \end{aligned}$$

and thus

$$\begin{aligned} dt &= \frac{1}{2}(du + dx) \\ dx &= \frac{1}{2}(du - dx) \end{aligned}$$

therefore,

$$\begin{aligned} ds^2 &= dt^2 - dx^2 \\ &= \frac{1}{4}((du + dv)^2 - (du - dv)^2) \\ &= \frac{1}{4}((du^2 + dv^2 + 2dudv) - (du^2 + dv^2 - 2dudv)) \\ &= dudv \end{aligned}$$

b) Given the coordiates $x = \frac{1}{a}e^{a\eta}\sinh(a\zeta)$ and $t = \frac{1}{a}e^{a\eta}\cosh(a\zeta)$, where a is a constant, we find that

$$\begin{aligned} dt &= d\eta e^{a\eta}\cosh(a\zeta) + d\zeta e^{a\eta}\sinh a\zeta \\ dx &= d\eta e^{a\eta}\sinh(a\zeta) + d\zeta e^{a\eta}\cosh a\zeta \end{aligned}$$

therefore,

$$\begin{aligned} ds^2 &= dt^2 - dx^2 \\ &= e^{2a\eta}(\cosh(a\zeta)d\eta + \sinh(a\zeta)d\zeta)^2 - e^{2a\eta}(\sinh(a\zeta)d\eta + \cosh(a\zeta)d\zeta)^2 \\ &= e^{2a\eta}(\cosh^2(a\zeta) - \sinh^2(a\zeta))d\eta^2 + e^{2a\eta}(\sinh^2(a\zeta) - \cosh^2(a\zeta))d\zeta^2 \\ &= e^{2a\eta}(d\eta^2 - d\zeta^2) \end{aligned}$$

Example 7 *In a black hole spacetime the line element is given by*

$$ds^2 = (1 - (2m/R))dT^2 - (1 - (2m/R))^{-1}dR^2$$

where m is the mass of the black hole and R the radius. Note how, far from the Schwarzschild radius $R = 2m$, spacetime is flat. As we come closer to the Schwarzschild radius the light cones tilt. At the Schwarzschild radius the light cones are tilted such that particles following world lines can fall into the black hole but cannot escape.

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Chapter 3

Technical tools from quantum field theory I

We are interested in understanding quantum information in a relativistic quantum world. For this we must consider the most general situation: quantum field theory in curved spacetime. The most important lesson we have learned from quantum field theory is that fields are fundamental notions, and not particles. As we will see, particles are derived notions (if at all possible). In this lecture we will therefore consider fields on a spacetime. We will learn how to quantize a field in curved spacetime and under what circumstances particles can be defined. This is important for information theory in which all concepts are based on the notions of subsystems (particles). Throught our lectures we will work in natural units $c = \hbar = 1$.

3.1 The Klein-Gordon equation

We will consider the simplest field which is the Klein-Gordon un-charged scalar field ϕ. The field ϕ satisfies the Klein-Gordon equation $(\Box + m^2)\phi = 0$, where the operator $\Box$ is known as the d'Alambertian and is defined as

$$\Box\phi := \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)$$

where $g = det(g^{ab})$ and $\partial_\mu = \frac{\partial}{\partial\chi^\mu}$.

Example 8 *In flat spacetime* $(1+1)$*-dim we have* $g_{\nu\mu} = \eta_{\nu\mu} = \{+-\}$ *and thus*

$$\begin{aligned} \Box\phi &= \partial_0(\eta^{00}\partial_0) + \partial_1(\eta_1^1\partial_1) \\ &= \partial_t^2 - \partial_x^2 \end{aligned}$$

For $m = 0$ *the solutions to* $\Box\phi = 0$ *in 2-dim are plane waves of the form*

$$u_k = \frac{1}{\sqrt{2\pi\omega}}e^{i(kx-wt)}$$

with $\omega = |k|$ *and* $-\infty < k < \infty$.

We would like to quantize this field. For this we note that the solutions of the equation form a vector space over $\mathbb{C}$. We can therefore, consider the following (bad!) idea:

By supplying an inner product we can construct a Hilbert space and possibly consider $\hat{u}_k$ as operators. However, the inner product must be Lorentz invariant. The only possible choice is

$$(\phi, \psi) = -i \int_\Sigma (\psi^* \partial \mu \phi - (\partial_\mu \psi^*)\phi) d\Sigma^\mu$$

where Σ is a spacelike hypersurface.

We then note that (ϕ, ψ) can be negative! Therefore, we cannot define a probability density. This means that it is not possible to construct a one-particle relativistic quantum theory.

3.2 Time-like Killing vector fields

To understand the deeper reasons why this naive program does not work, let us introduce the notion of a time-like Killing vector field. We are interested in finding a transformation $\chi^a \to \bar{\chi}^a$ which leaves the metric $g_{ab}(\chi)$ invariant. That means that the transformed metric $\bar{g}^{ab}(\bar{x})$ is the same function of its argument $\bar{\chi}^a$ as the original metric $g_{ab}(\chi)$ is of its argument χ^a. Since the metric is a covariant tensor it transforms according to

$$g_{ab}(\chi) = \frac{\partial \bar{\chi}^c}{\partial \chi^a} \frac{\partial \bar{\chi}^d}{\partial \chi^b} \bar{g}_{cd}(\bar{\chi})$$

Therefore, the metric is invariant under the transformation $\chi^a \to \bar{\chi^a}$ if

$$g_{ab}(\chi) = \frac{\partial \bar{\chi}^c}{\partial \chi^a} \frac{\partial \bar{\chi}^d}{\partial \chi^b} g_{cd}(\bar{\chi}).$$

This equation is complicated therefore, it is easier to consider the infinitesimal transformation $\chi^a \to \chi^a + \delta u X^a(\chi) = \bar{\chi}^a$ where δu is small and arbitrary and X is a vector field. Therefore we ask that in the limit $\delta u \to 0$

$$\lim_{\delta u \to 0} \frac{g_{ab}(\bar{\chi}) - \bar{g}_{ab}(\bar{\chi})}{\delta u} = 0.$$

Differentiating $\bar{\chi}^a = \chi^a + \delta u X^a(\chi)$

$$\begin{aligned} \frac{\partial \bar{\chi}^a}{\partial \chi^c} &= \delta^a{}_c + \delta u \partial_c X^a \\ \frac{\partial \bar{\chi}^b}{\partial \chi^d} &= \delta^b{}_d + \delta u \partial_d X^b \end{aligned}$$

We find that in a Taylor expansion of $g_{ab}(\bar{\chi})$

$$g_{ab}(\bar{\chi}) = g_{ab}(\chi^e + \delta u X^e) = g_{ab}(\chi) + \delta u X^e \partial_e g_{ab}(\chi) + ...) \tag{3.1}$$

therefore,

$$\begin{aligned} g_{ab}(\chi) &= (\delta^c{}_a + \delta u \partial_a X^c)(\delta^d{}_b + \delta u \partial_d X^b)(g_{cd}(\chi^e) + X^e \delta u \partial_e g_{cd}(\chi^e) + ...) \\ &= \delta^c{}_a \delta^d{}_b g_{cd}(\chi) + \delta u \left[\delta^c{}_a \partial_d X^b g_{cd}(\chi) + \delta^d{}_b \partial_a X^c g_{cd}(\chi) + X^e \partial_e g_{cd}(\chi)\right] + \theta(du^2) \\ &= g_{ab}(\chi) + \delta u \left[X^e \partial_e g_{ab}(\chi) + g_{ad} \partial_b X^d + g_{bd} \partial_a X^d\right] + \theta(du^2) \end{aligned}$$

Working to first order in δu and subtracting $g_{ab}(\chi)$ on both sides, it follows that the quantity in the bracket must vanish,

$$\left[X^e \partial_e g_{ab}(\chi) + g_{ad} \partial_b X^d + g_{bd} \partial_a X^d\right] = 0$$

We identify the object inside the bracket with the Lie deriviative of the metric tensor

$$\mathcal{L}_X g_{ab} = X^e \partial_e g_{ab}(\chi) + g_{ad} \partial_b X^d + g_{bd} \partial_a X^d$$

Therefore, we say that X is a Killing vector field if the equation $\mathcal{L}_X g^{ab} = 0$ is satisfied.

3.3 Quantizing the field and the definition of particles

Having defined a Killing vector field X, let us choose a special basis for the solutions of $\Box \phi = 0$ such that

$$\begin{aligned} i\mathcal{L}_X u_k &= iX^\mu \partial_\mu u_k \\ &= \omega u_k \end{aligned}$$

where we have considered the action of a Lie derivative on a function. Vectors lying within the light cone at each point are called time-like. Therefore, if X is a timelike vector field, the Lie deriviative corresponds to ∂_t. This implies that the eigenvalue equation above takes form of a Schrödinger-like equation where we can identify $\omega > 0$ with a frequency

$$\begin{aligned} i\partial_t u_k &= -i\omega u_k \\ i\partial_t u_k^* &= i\omega u_k^* \end{aligned}$$

The solutions to the Klein-Gordon equation are therefore classified in the following way

$$\begin{aligned} u_k &\rightarrow \text{positive frequence solutions} \\ u_k^* &\rightarrow \text{negative frequence solutions} \end{aligned}$$

Interestingly, we observe that the inner product is positive for positive frequency solutions and negative for negative frequency solutions

$$\begin{aligned}(u_k, u_{k'}) &\geqslant 0 \\ (u_k^*, u_{k'}^*) &\leqslant 0\end{aligned}$$

We found the culprits! Negative frequency solutions give rise to negative probabilities. However, the real problem with trying to construct a single-particle quantum field theory is the following: we know from experiment that in field theory there are multi-particle interactions.

The single-particle Hilbert space in not big enough nor appropriate to construct a quantum field theory. We need a bigger space. The right mathematical structure is given by a Fock space. The Fock space is constructed in the following way: we consider a zero-particle sector which contains the vacuum state $|0\rangle$. Then a single particle Hilbert space is constructed as proposed before. We use the vector field of solutions of the equation and impose the Lorentz invariant inner product. We now allow for multi-particle sectors which are constructed using the tensor product structure. For example, the Hilbert space for a 2-particle sector is given by $\mathcal{H}_2 = \mathcal{H} \otimes \mathcal{H}$. The Fock space therefore takes the form

$$\mathbb{C} \oplus \mathcal{H} \otimes \mathcal{H} \oplus \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \oplus \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes ...$$

It is then possible to define creation and annihilation operators $a_k^\dagger$ and a_k which, in the case of bosonic fields, satisfy the commutation relations $[a_k^\dagger, a_{k'}] = \delta_{k,k'}$. Creation operators $a_k^\dagger$ take us from the n-particle sector to the (n+1)-particle sector by creating a particle of momentum k. Annihilation operators a_k take us from the n-particle sector to the (n-1)-particle sector by annihilating a particle with momentum k. Having defined creation and annihilation operators, we can define the following operator value function being careful to treat the positive and negative solutions of the Klein-Gordon equation in a different way

$$\hat{\phi} = \int (u_k a_k + u_k^* a_k^\dagger) dk.$$

Notice that positive frequency solutions are associated with annihilation operators and negative frequency solutions with creation operators. Amazingly, this operator value function satisfies the Klein-Gordon equation $\Box\hat{\phi} = 0$. This is in fact, the right way to quantize a field. Now, we can properly define particles as the action of creation operators on the vacuum state

$$|n_1, ..., n_k'\rangle = a_1^{\dagger n} ... a_k^{\dagger n'} |0\rangle$$

and such states have positive norm.

REMARKS When there exists a global time-like Killing vector field it is meaningful to define particles. Observers flowing along timelike Killing vector fields are those who can properly describe particle states. This has important consequences to relativistic quantum information since the notion of particles (and therefore, subsystems) are indispensable to store information and thus, to define entanglement.

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Chapter 4

Technical tools in quantum field theory II

In the last lecture we learned that only observers flowing along timelike Killing vector fields can meaningfully describe particles. However, in the most general case, curved spacetimes do not admit such structures. There are in fact different kind of spacetimes

1. Spacetimes with global Killing vector fields
2. Spacetimes with no Killing vector fields
3. Spacetimes with regions which admit Killing vector fields

In this lecture we will see that in the case that the spacetime admist a timelike global Killing vector field, the vector field is not necessarily unique. A consequence of this is that the particle content of the field is observer-dependent. In the following lectures we will analyze the consequences of this for entanglement in spacetime. As an example of the observer-dependent property of the field, we will consider observers in flat-spacetime and derive the Unruh effect.

4.1 Killing observers and Bogolubov transformations

When a spacetime admits a timelike Killing vector field ∂_t it is possible to classify solutions to the Klein-Gordon equation $\{u_k, u_k^*\}$ into positive and negative frequency solutions. The field is therefore quantized as

$$\hat{\phi} = \int (u_k a_k + u_k^* a_k^\dagger) dk.$$

However, when such is the case, ∂_t is not generally unique. It is possible to find another time-like Killing vector field $\partial_{\hat{t}}$ and therefore find another basis for the solutions to the Klein-Gordon equation $\{\bar{u}_k, \bar{u}_k^*\}$ such that classification into positive and frequency solutions is possible. The field then is equivalently quantized in this basis as

$$\hat{\phi} = \int (\bar{u}_{k'}\bar{a}_{k'} + \bar{u}_{k'}^*\bar{a}_{k'}^\dagger)dk'.$$

and therefore,

$$\hat{\phi} = \int (u_k a_k + u_k^* a_k^\dagger)dk = \int (\bar{u}_{k'}\bar{a}_{k'} + \bar{u}_{k'}^*\bar{a}_{k'}^\dagger)dk'.$$

Using the inner product, it is then possible to find a transformation between them the creation and annihilation operators

$$a_k = \sum_{k'} (\alpha_{kk'}^* \bar{a}_{k'} - \beta_{kk'}^* \bar{a}_{k'}^\dagger)$$

where $\alpha_{kk'} = (u_k, \bar{u}_{k'})$ and $\beta_{kk'} = -(u_k, \bar{u}_{k'}^*)$ are called Bogolubov coefficients. Since the vacua states are defined as

$$a_k|0\rangle = \bar{a}_k|\bar{0}\rangle = 0$$

it is possible to find a transformation between the states in the two basis. We note that as long as one of the Bogoluvob coefficients $\beta_{kk'}$ is non-zero, while the un-barred state is the vacuum state, the state in the bared basis is populated with particles.

REMARK Different Killing observers observe a different particle content in the field. Therefore, particles are observer-dependent quantities. As an example we will analyze observers in flat spacetime.

4.2 Example 1: Observers in flat spacetime

Consider the scalar field $\phi(t, x)$ defined in all points of Minkowski spacetime in $1+1$ dimensions. The line element is given by $ds^2 = dt^2 - dx^2$. The Killing vector fields in this spacetime are given by the equation

$$\mathcal{L}_X \eta_{ab} = 0$$

which in components takes the form

$$X^e \partial_e \eta_{ab} + \eta_{ad}\partial_b X^d + \eta_{bd}\partial_a X^d = 0.$$

In flat spacetime there are three independent Killing vector fields corresponding to

1. Time translationss
2. Space translation

3. Observers moving on hyperbolas

To show this we use the Killing equation shown above finding that

$$\partial_t X^0 = \partial_x X^1 \Rightarrow X^\mu = \alpha \begin{pmatrix} t \\ x \end{pmatrix}$$

where α is a constant. The condition for this Killing vector field to be timelike is

$$0 \leqslant \eta_{ab} X^a X^b \Rightarrow 0 \leqslant \alpha^2 (t^2 - x^2).$$

Possible solutions correspond to the case where t and x are constants giving rise to

$$X^\mu = \alpha t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha x \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where the first summand corresponds to time translations and the second to space translations. Therefore, from time and space translations we obtain straight lines corresponding to inertial observers. Another possible solution are hyperbolas $t^2 - x^2 = const$. In what follows we will show that observers moving along these trajectories correspond to observers in uniform acceleration.

In relativity acceleration is uniform if, at each instant, the acceleration in an inertial frame travelling with the same velocity as the particle has the same value. We therefore consider two observers: the first with coordinates (t, x) and the second, who is inertial, with coordinates $(\tilde{t}, \tilde{x})$ moving with constant velocity v with respect to the first . Using the Lorentz transformations

$$\tilde{t} = \gamma \left(t - \frac{vx}{c^2} \right), \quad \tilde{x} = \gamma (x - vt)$$

where $\gamma = (1 - \frac{v^2}{c^2})^{-1}$ is the Lorentz factor, we aim at finding the transformation between the accelerations in the two reference frames

$$a = \ddot{x} = \frac{d^2 x}{dt^2}, \quad \tilde{a} = \ddot{\tilde{x}} = \frac{d^2 \tilde{x}}{d\tilde{t}^2}.$$

In our notation dot represents differentiation with respect to the time coordinate. Since $d\tilde{x}$ and $d\tilde{t}$ are given by

$$d\tilde{x} = \gamma(dx - vdt), \quad c^2 d\tilde{t} = \gamma(c^2 dt - vdx)$$

we obtain the following transformation between the velocities

$$\frac{\dot{\tilde{x}}}{c^2} = \frac{1}{c^2}\frac{d\tilde{x}}{d\tilde{t}} = \frac{\gamma(dx - vdt)}{\gamma(c^2 dt - vdx)} = \frac{(dx - vdt)}{(c^2 dt - vdx)} = \frac{(\dot{x} - v)}{(c^2 - v\dot{x})}.$$

Denoting $\dot{\tilde{x}} \equiv \tilde{u}$ and $\dot{x} \equiv u$ and differentiating yields

$$\begin{aligned} c^{-2} d\tilde{u} &= d[u - v](c^2 - vu)^{-1} + (u - v) d[c^2 - vu]^{-1} \\ &= du (c^2 - vu)^{-1} + v(u - v)(c^2 - vu)^{-2} du \\ &= (c^2 - vu)^{-2} [c^2 - vu + v(u - v)] du \\ &= (c^2 - vu)^{-2} (c^2 - v^2) du. \end{aligned}$$

Hence

$$\begin{aligned}\frac{d\tilde{u}}{d\tilde{t}} &= c^4\frac{(c^2-vu)^{-2}(c^2-v^2)}{\gamma(c^2-vu)}\frac{du}{dt}\\ \tilde{a} &= c^3(c^2-vu)^{-3}(c^2-vu)^{\frac{3}{2}}a\end{aligned}$$

which yields the transformation between accelerations we were looking for

$$\tilde{a}=\left(1-\frac{uv}{c^2}\right)^{-3}\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}a.$$

The inverse transformation yields,

$$a=\tilde{a}\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}\left(1+\frac{\tilde{u}v}{c^2}\right)^{-3}.$$

To consider an observer moving in uniform acceleration we set $\tilde{a}=\alpha$ where α is a constant. We also must consider that the inertial observer moves at each time with the same velocity as the first observer. Therefore, $\tilde{u}=0$ and $v=u$ which results in

$$\frac{d^2x}{dt^2}=\alpha\left(1-\frac{1}{c^2}\left(\frac{dx}{dt}\right)\right)^{\frac{3}{2}}.$$

To find the trajectory of the observer in uniform acceleration we integrate the above equation

$$\frac{du}{(1-\frac{u^2}{c^2})^{\frac{3}{2}}} = \alpha dt$$

and assuming the particle starts from rest at time $t=t_0$, we find

$$\frac{u}{(1-\frac{u^2}{c^2})^{\frac{1}{2}}} = \alpha(t-t_0)$$

and integrating once more we obtain,

$$(x-x_0)=\frac{c}{\alpha}[c^2+\alpha^2(t-t_0)^2]^{\frac{1}{2}}-\frac{c^2}{\alpha}$$

or equivalently,

$$\frac{(x-x_0+c^2/\alpha)^2}{(c^2/\alpha)}-\frac{(ct-ct_0)^2}{(c^2/\alpha)}=1.$$

Setting $x_0-(c^2/\alpha)=t_0=0$, we obtain $x^2-c^2t^2=c^2/\alpha$. Therefore, we have found that an observer in uniform acceleration moves along hyperbolas. In the next lecture we will investigate how the field looks like from the perspective of an observer in uniform acceleration given that the field is in the vacuum from the inertial perspective.

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Chapter 5

The Unruh effect

In this lecture we continue with our example of observers in flat spacetime. We have shown that in flat spacetime there are two kinds of observers who can meaningfully descibe particles: inertial observers who's trajectories correspond to straight lines and observers in uniform acceleration who follow hyperbolas. We have learned that, in principle, the particle content of the field can be different when described from the perspective of different observers. Therefore, we will quantize the field from the perspective of inertial observers and then describe it from the perspective of observers in uniform acceleration.

Minkowski coordinates (t, x) are a convenient choice of coordinates for inertial observers. We saw in our last lecture that their trajectories follow straight lines. In this coordinates the Klein-Gordon equation takes the following form

$$(\partial_t^2 - \partial_x^2)\phi = 0.$$

The solutions to this equation are plane waves

$$\begin{aligned} u_k &= \frac{1}{\sqrt{2\pi\omega}} e^{i(kx-\omega t)} \\ u_k^* &= \frac{1}{\sqrt{2\pi\omega}} e^{-i(kx-\omega t)} \end{aligned}$$

with $\omega = |k|$ and $-\infty < k < \infty$. The spacetime admits global timelike Killing vector fields. In this case, the timelike Killing vector field corresponds to ∂_t. With respect to this field, we can classify the above solutions in positive and negative frequency solutions. Since

$$\begin{aligned} i\partial_t u_k &= -i\omega u_k \\ i\partial_t u_k^* &= i\omega u_k^* \end{aligned}$$

we identify u_k as positive and u_k^* as negative frequency solutions. Since these solutions form a complete set of orthonormal functions in Minkowski spacetime the quantized field can be expressed as

$$\hat{\phi} = \int (u_k a_k + u_k^* a_k^\dagger) dk$$

were the creation and annihilation operators follow the appropriate commutation relations $[a_k^\dagger, a_{k'}] = \delta_{k,k'}$. The vacuum state from this perspective is therefore defined as $a_k |0\rangle^{\mathcal{M}} = 0$. The state can be written as $|0\rangle^{\mathcal{M}} = \prod_k |0_k\rangle^{\mathcal{M}}$ where $|0_k\rangle^{\mathcal{M}}$ is the vacuum state of mode k.

Now we want to describe the field from the perspective of an observer moving with uniform acceleration. The trajectory of such observer follows a hyperbola. We will parametrize the trajectory in the following way

$$x = \frac{e^{a\chi}}{a}\cosh(a\eta)$$
$$t = \frac{e^{a\chi}}{a}\sinh(a\eta)$$

such that $x^2 - t^2 = \frac{e^{a\chi}}{a}$. Here a is an arbitrary reference acceleration. The observer's proper acceleration is given by $ae^{-a\chi}$. This suggests that appropriate coordinates for accelerated observers are (η, χ) which are know as Rindler coordinates. The temporal coordinates $\eta = cte$ correspond to straight lines going through the origin and the spacial coordinates $\chi = cte$ are hyperbolas. Note that in the limit that $\chi \to -\infty$ we have $x^2 = t^2$ which corresponds to lines at 45 degrees. The transformation is defined in the region $|x| \geq t$ which defines a wedge known as Rindler wedge I. When $\eta \to \infty$ then

$$\frac{x}{t} = \tanh(a\eta) \to 1 \Rightarrow x = t$$

Observers uniformly accelerated asymptotically approach the speed of light and are constrained to move in wedge I. Since the transformation does not cover the whole Minkowski space, we must define a second region called Rindler wedge II by considering the coordinate transformation

$$x = \frac{-e^{a\chi}}{a}\cosh(a\eta)$$
$$t = \frac{-e^{a\chi}}{a}\sinh(a\eta)$$

which differ from the first set of transformations by a sign in both coordinates. Rindler regions I and II are causally disconnected and the lines at 45 degrees define the Rindler horizon. No information can flow between these regions.

In Lecture 2 we found that metric in Rindler coordinates takes the form

$$ds^2 = e^{2a\chi}(d\eta^2 - d\chi^2)$$

The factor $e^{2a\chi}$ is known as the conformal factor. A part from this factor, the metric has the same form as metric in Rindler coordinates. Therefore, the Klein-Gordon equation takes the form

$$(\partial_\eta^2 - \partial_\chi^2)\phi = 0$$

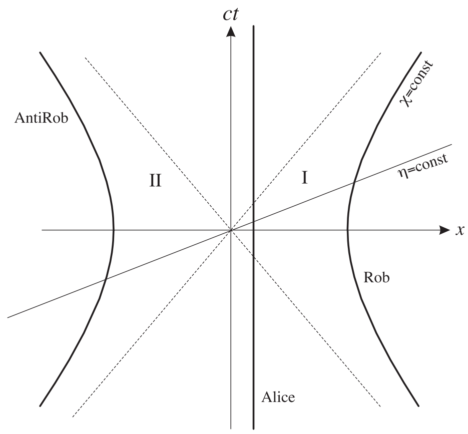

Figure 5.1: Rindler space-time diagram: lines of constant position χ = const. are hyperbolae and all curves of constant η are straight lines that pass through the origin. An uniformly accelerated observer Rob travels along a hyperbola constrained to either region I or region II.

The solutions are again plane waves

$$\begin{aligned}\bar{u}_k^I &= \frac{1}{2\pi\omega} e^{i(k\chi - \omega\eta)} \\ \bar{u}_k^{*I} &= \frac{1}{2\pi\omega} e^{-i(k\chi - \omega\eta)}\end{aligned}$$

with $\omega = |k|$ and $-\infty < k < \infty$. These solutions only have support in region I and therefore, are not a complete set of solutions. The solutions $\bar{u}_k^I$ and $\bar{u}_k^{*I}$ are identified as positive and negative frequency solutions, respectively, with respect to the timelike Killing vector field ∂_η. The transformation which defines Rindler region II also gives rise to the spacetime $ds^2 = e^{2a\chi}(d\eta^2 - d\chi^2)$. However, in this case as t grows η becomes small. The timelike Killing vector field in this case is $-\partial_\eta$ and the solutions are

$$\begin{aligned}u_k^{II} &= \frac{1}{\sqrt{2\pi\omega}} e^{i(k\chi + \omega\eta)} \\ u_k^{*II} &= \frac{1}{\sqrt{2\pi\omega}} e^{-i(k\chi + \omega\eta)}\end{aligned}$$

with support in region II. The solutions of region I together with the solutions in region II form a complete set of orthonormal solutions. Therefore we can quantize the field in this basis. We thus, obtain that the field in Rindler coordinates is given by

$$\hat{\phi} = \int (u_k^I a_k^I + u_k^{II} a_k^{II} + h.c.) dk$$

The vacuum state in the Rindler basis is $|0\rangle_R = |0\rangle_I \otimes |0\rangle_{II}$ where $a_k^I |0\rangle^I = 0$ and $a_k^{II} |0\rangle^{II} = 0$.

We would now like to express the Minkowski vacuum $|0\rangle^{\mathcal{M}}$ in terms of Rindler states. For this we make use of the inner product and find the Bogoliubov coefficents, obtaining

$$a_k = \int ((u_k, u_{k'}^I) a_k^I + (u_k, u_{k'}^{*I}) a_k^{\dagger I} + (u_k, u_{k'}^{II}) a_k^{II} + (u_k, u_{k'}^{*II}) a_k^{\dagger II}) dk$$

where, for example, $(u_k, u_k^I) = \int (u_k \partial_t u_{k'}^{I\dagger} - \partial_t u_k u_{k'}^{I\dagger}) dx$. This calculation is quiet involved. The Minkowski creation and annihilation operators result in a infinite sum of Rindler operators. However, by introducing an alternative basis for the inertial observers, the Unruh basis, the problem is significantly simplified. The Unruh solutions correspond to

$$u_k^U = \cosh r u_k^I + \sinh r u_k^{II*},$$

where $\operatorname{sech}^2(r) = 1 - e^{\frac{2\pi\omega}{a}}$. An similar procedure to the one followed above must be carried out between Unruh solutions and Minkowski solutions. We then find that the Unruh annihilation operators result in an intergal of Minkowski annihilation operators of the form

$$A_k = \int C_{k'} a_{k'} dk'.$$

Therefore, the Unruh vacuum and the Minkowski vacuum coincide, i.e. $A_k |0\rangle^{\mathcal{M}} = 0$. The transformation between Unruh and Rindler operators yields

$$A_k = \cosh(r) a_k^I - \sinh(r) a_k^{II\dagger}$$

This transformation is much simpler since it involves a single Unruh and Rindler frequency. Considering the Ansatz

$$|0_k\rangle^{\mathcal{M}} = \sum_n A_n |n_k\rangle^I |n_k\rangle^{II}$$

we can find the Minkowski vacuum in terms of Rindler states by solving the equation

$$0 = A_k |0_k\rangle^{\mathcal{M}} = (\cosh(r) a_k^I - \sinh(r) a_k^{II\dagger}) \sum_n A_n |n_k\rangle^I |n_k\rangle^{II}$$

this yields

$$\begin{aligned} 0 &= \sum A_n \cosh(r)\sqrt{n}\, |(n-1)_k\rangle^I |n_k\rangle^{II} - \sum A_n \sinh(r)\sqrt{n+1}\, |n_k\rangle^I |(n+1)_k\rangle^{II} \\ &= \sum A_{n+1} \cosh(r)\sqrt{n+1}\, |n_k\rangle^I |(n+1)_k\rangle^{II} - \sum A_n \sinh(r)\sqrt{n+1}\, |n_k\rangle^I |(n+1)_k\rangle^{II} \end{aligned}$$

We obtain the following recurrence relation

$$A_{n+1} = \frac{\sinh(r)}{\cosh(r)} A_n \Rightarrow A_n = \tanh^n(r) A_0$$

and therefore,

$$|0_k\rangle^{\mathcal{M}} = \sum A_0 \tanh^n(r) \, |n_k\rangle^I \, |n_k\rangle^{II}$$

From the normalization condition for the vacuum state we obtain

$$|A_0|^2 \sum_n \tanh^{2n}(r) = 1.$$

Since

$$\begin{aligned} |A_0|^2 \sum_n \tanh^{2n}(r) &= |A_0|^2 \frac{1}{1-\tanh^2(r)} \\ &= |A_0|^2 \cosh^2(r) \end{aligned}$$

we finally find

$$|0_k\rangle^{\mathcal{M}} = \frac{1}{\cosh(r)} \sum_n \tanh^n(r) \, |n_k\rangle^I \, |n_k\rangle^{II}$$

The state in the Rindler basis corresponds to a two mode squeezed state. Here $\tanh r = e^{-\frac{2\pi\omega}{a}}$. Since the accelerated observer is constrained to move in region I we must trace over the states in region II. We will carry out this calculation in the next lecture. The result will be that in the Rindler wedge I the state is a thermal state. This is the well known Unruh effect: while inertial observers describe the state of the field to be the vacuum, observers in uniform acceleration observe a thermal state.

REMARK Therefore, the particle content of a field is observer dependent.

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Chapter 6

Entanglement in flat spacetime

In the last lecture we learned that while an inertial observer detects the field to be in the vacuum state, the state in the Rindler basis corresponds to a two mode squeezed state. In this lecture we will show that from the perspective of observers in uniform acceleration constrained to move in Rindler region I, the field is in a thermal state with temperature. We will analyze the effects of this on entanglement. For this, we will consider an entangled state between two modes of the field and show that while for inertial observers the state is maximally entangled, the entanglement degrades when considering observers in uniform acceleration. We found that the inertial vacuum state corresponds to a two mode squeezed state in the Rindler basis

$$|0_k\rangle^{\mathcal{M}} = \frac{1}{\cosh(r)} \sum_n \tanh^n(r) \, |n_k\rangle^I \, |n_k\rangle^{II}$$

The density matrix of this state is given by $\rho_0 = |0_k\rangle \langle 0_k|^{\mathcal{M}}$. Given that we are working with a single mode we will, for now, drop the frequency index k. Since an observer constrained to move in region I has no access to information in region II, he must trace over the states in region II. Therefore, the state in region I corresponds to the following reduced density matrix

$$\begin{aligned}
\rho_I &= Tr_{II}\left(\frac{1}{\cosh^2(r)} \sum_{n,m} \tanh^{n+m}(r) \, |n\rangle^I \, |n\rangle^{II} \, \langle m|^I \, \langle m|^{II}\right) \\
&= \frac{1}{\cosh^2(r)} \sum_{n,m} \tanh^{n+m}(r) \, \langle m|n\rangle^I \, |n\rangle^{II} \, \langle m|^{II} \\
&= \frac{1}{\cosh^2(r)} \sum_{n,m} \tanh^{n+m}(r) \delta_{mn} \, |n\rangle^{II} \, \langle m|^{II} \\
&= \frac{1}{\cosh^2(r)} \sum_{n} \tanh^{2n}(r) \, |n\rangle^{II} \, \langle n|^{II} .
\end{aligned}$$

Since $\tanh r = e^{-\frac{2\pi\omega}{a}}$ and $\mathrm{sech}^2(r) = 1 - e^{\frac{2\pi\omega}{a}}$, the density matrix corresponds to a canonical thermal state

$$\rho_I = (e^{-\frac{2\pi\omega}{a}} - 1)\sum_n (e^{-\frac{2\pi\omega}{a}})^n \left|n\right\rangle^{II} \left\langle n\right|^{II}.$$

with temperature $T_U = \frac{a}{2\pi k_B}$ (where k_B is the Boltzman constant) proportional to the observer's acceleration. The temperature is known as the Unruh temperature.

We now analyze the effects of this on the entanglement between two field modes of the field. In the canonical scenario considered in the study of entanglement in non-inertial frames the field, from the inertial perspective, is considered to be in a state where all modes are in the vacuum state except for two of them which are in a two-mode entangled state. For example, the Bell state

$$\left|\Psi\right\rangle^{\mathcal{U}} = \frac{1}{\sqrt{2}}\left(\left|0_k\right\rangle^{\mathcal{U}}\left|0_{k'}\right\rangle^{\mathcal{U}} + \left|1_k\right\rangle^{\mathcal{U}}\left|1_{k'}\right\rangle^{\mathcal{U}}\right), \tag{6.1}$$

where $\mathcal{U}$ labels Unruh states and k, k' are two Unruh frequencies. Two inertial observers, Alice and Bob, each carrying a monocromatic detector sensitive to frequencies k and k' respectively, would find maximal correlations in their measurements since the Bell state is maximally entangled. It is then interesting to investigate to what degree the state is entangled when described by observers in uniform acceleration. In the simplest scenario, Alice is again considered to be inertial and an uniformly accelerated observer Rob is introduced, who carries a monocromatic detector sensitive to mode k'. To study this situation, the states corresponding to Rob must be transformed into the appropriate basis, in this case, the Rindler basis. Note that, from the inertial perspective, we employ the Unruh basis, which is an alternative basis for inertial observers, since the transformation into the Rindler basis is very simple. We have already calculated the transformation for the vacuum state, and with that in hand, we can calculate the single particle state $\left|1_{k'}\right\rangle^{\mathcal{U}} = A^{\dagger}_{k'}\left|0_{k'}\right\rangle^{\mathcal{U}}$. This calculation will be left as an exercise. The resulting state must be

$$A^{\dagger}_{k'}\left|0_{k'}\right\rangle^{\mathcal{U}} = \left|1_{k'}\right\rangle^{\mathcal{U}} \quad = \quad \frac{1}{\cosh(r)}\sum_n \tanh^n(r)\sqrt{n+1}\left|(n+1)_{k'}\right\rangle_I \left|n_{k'}\right\rangle_{II}.$$

Thus, the maximally entangled state from the perspective of inertial Alice and accelerated Rob is

$$\begin{aligned}\left|\Psi\right\rangle^{\mathcal{U}} &= \frac{1}{\sqrt{2}}\left|0_k\right\rangle \otimes \frac{1}{\cosh(r)}\sum_n \tanh^n(r)\left|n_{k'}\right\rangle^I \left|n_{k'}\right\rangle^{II} \\ &+ \frac{1}{\sqrt{2}}\left|1_k\right\rangle \otimes \frac{1}{\cosh(r)}\sum_n \tanh^n(r)\sqrt{n+1}\left|(n+1)_{k'}\right\rangle^I \left|n_{k'}\right\rangle^{II}.\end{aligned}$$

Since Rob is causally disconnected from region II we must take the trace over region II. The density matrix for the Alice-Rob subsystem is

$$\rho_{AR} \quad = \quad \frac{1}{\cosh^2(r)}\sum_{n=0}^{\infty}\tanh^{2n}(r)\rho_n$$

where

$$\begin{aligned}\rho_n &= |0_k, n_{k'}\rangle\langle 0_k, n_{k'}| + \frac{\sqrt{n+1}}{\cosh(r)}\left(|0_k, n_{k'}\rangle\langle 1_k, (n+1)_{k'}| + |1_k, (n+1)_{k'}\rangle\langle 0_k, n_{k'}|\right)\\ &+ \frac{n+1}{\cosh^2(r)}|1_k, (n+1)_{k'}\rangle\langle 1_k, (n+1)_{k'}|\end{aligned}$$

We calculate the entanglement between the modes detected by Alice and Rob using the logarithmic negativity

$$E_N(\rho_{AR}) = \log_2 \|\rho_{AR}^{PT}\|$$

where $||\rho_{AR}^{PT}||$ is the trace norm of the partial transpose of the density matrix for Alice and Rob. Recall that the trace norm is equivalent to summing over the negative eigenvalues of the matrix. Therefore, we must find the eigenvalues of the partial transpose of the Alice-Rob density matrix. The partial transpose is found by exchanging the states corresponding to Alice

$$\rho_{AR}^{PT} = \frac{1}{\cosh^2(r)}\sum_{n=0}^{\infty}\tanh^{2n}(r)\rho_n^{PT}$$

with

$$\begin{aligned}\rho_n^{PT} &= |0_k, n_{k'}\rangle\langle 0_k, n_{k'}| + \frac{\sqrt{n+1}}{\cosh(r)}\left(|1_k, n_{k'}\rangle\langle 0_k, (n+1)_{k'}| + |0_k, (n+1)_{k'}\rangle\langle 1_k, n_{k'}|\right)\\ &+ \frac{n+1}{\cosh^2(r)}|1_k, (n+1)_{k'}\rangle\langle 1_k, (n+1)_{k'}|\end{aligned}$$

This matrix is infinite dimensional, however it has a block diagonal form which allows us to diagonalize the matrix block by block. Considering the density matrix for the $(n, n+1)$ sector, each block takes the form

$$\rho_n^{PT} = \frac{1}{\cosh^2(r)}\begin{pmatrix} \tanh^{2n}(r) & 0 & 0 & 0\\ 0 & \frac{n}{\cosh(r)}\tanh^{2(n-1)}(r) & \frac{\sqrt{n+1}}{\cosh(r)}\tanh^{2n}(r) & 0\\ 0 & \frac{\sqrt{n+1}}{\cosh(r)}\tanh^{2n}(r) & \tanh^{2(n+1)}(r) & 0\\ 0 & 0 & 0 & \frac{n+1}{\cosh^2(r)}\tanh^{2n}(r)\end{pmatrix}$$

were we have used the basis $\{|0_k, n_{k'}\rangle, |0_k, (n+1)_{k'}\rangle, |1_k, n_{k'}\rangle, |1_k, (n+1)_{k'}\rangle\}$. We are looking for the negative eigenvalues of this matrix. We can see that the eigenvalues corresponding to the first and last diagonal entries of the matrix are always positive. Therefore, we must simply diagonalize the 2x2 matrix

$$\frac{1}{\cosh^2(r)}\begin{pmatrix} \frac{n}{\cosh(r)}\tanh^{2(n-1)}(r) & \frac{\sqrt{n+1}}{\cosh(r)}\tanh^{2n}(r)\\ \frac{\sqrt{n+1}}{\cosh(r)}\tanh^{2n}(r) & \tanh^{2(n+1)}(r). \end{pmatrix}$$

The eigenvalues of the density matrix are

$$\lambda_\pm = \frac{\tanh^n(r)}{4\cosh^2(r)}\left(\frac{n}{\sinh^2(r)} + \tanh^2(r) \pm Z_n\right)$$

with

$$Z_n = \left(\frac{n}{\sinh^2(r)} + \tanh^2(r)\right)^2 + \frac{4}{\cosh^2(r)}.$$

Since one of the eigenvalues is always negative for finite r, the state is always entangled. To calculate the logarithmic negativity we sum over all negative eigenvalues and find $\mathcal{N}(\rho_{AR}) = \log_2((1/2\cosh^2(r)) + \Sigma)$ where

$$\Sigma = \sum_n \frac{\tanh^{2n}(r)}{2\cosh^2(r)} \sqrt{\left(\frac{n}{\sinh^2(r)} + \tanh^2(r)\right)^2 + \frac{4}{\cosh^2(r)}}$$

We conclude that entanglement is degraded when one of the observers moves in uniform acceleration. This means that entanglement is observer dependent. In the flat case, one can conclude that this is an effect of Rob's acceleration. Rob must be in a spaceship to be accelerated and energy must be supplied into the system. One can argue that, in the flat case, inertial observers play a special role and that therefore, a well defined notion of entanglement corresponds to the entanglement described from the inertial perspective. However, in curved spacetime different inertial observers describe a different particle content in the field which results in different degrees of entanglement in the field. In that case, there is no well-defined notion of entanglement.

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Chapter 7

Particle creation in an expanding universe

In this lecture we will consider an example of a curved spacetime which does not admit a global timelike Killing vector field. However, the spacetime has two assymthoticaly flat regions in which timelike Killing vector fields can be found and therefore positive and negative solutions to the Klein-Gordon equations can be distinguished. We will consider a $(1+1)$-dim expanding Robertson-Walker universe which is asymptotically flat in the future and past infinity. Since in the past infinity spacetime is flat, particles states can be defined. In this region, we will consider the field to be in the vacuum state. We will then analyze the state from the perspective of observers in the future infinity. We will show that in the future infinity there has been particle creation.

The spacetime of a Robertson-Walker Universe in $(1+1)$-dim is given by

$$ds^2 = dt^2 - a^2(t)d\chi^2$$

where the spatial sections of the space time are expanding (or contracting) uniformly according to the function $a^2(t)$. Considering the infintesimal coordinate transformation

$$d\eta = \frac{dt}{a(t)}$$

the metric is written as $ds^2 = a^2(t)(d\eta^2 - d\chi^2)$. Defining $a^2(t) = c^2(\eta)$ we obtain the metric

$$ds^2 = c^2(\eta)(d\eta^2 - d\chi^2)$$

We now suppose that $c(\eta) = 1 + \epsilon(1 + \tanh(\sigma\eta))$ where ϵ and σ are constants. This describes a toy model of a universe undergoing a period of smooth expansion. The parameter ϵ is known as the expansion volume and σ is the expansion rate. In the limit $\eta \to -\infty$ the metric is $ds^2 = (d\eta^2 - d\chi^2)$ and in the limit $\eta \to -\infty$ then $ds^2 = (1+2\epsilon)(d\eta^2 - d\chi^2)$. Therefore, the metric is flat in these regions and the vector field ∂_η has Killing properties in these regions.

We now consider the massive Klein-Gordon equation $(\Box + m)\phi = 0$ in the Robertson-Walker spacetime described above. Here m is the mass of the field. Since the d'Alambertian in a curved spacetime is defined by

$$\Box\phi := \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)$$

in this spacetime the metric tensor g_{ab} has components

$$g_{ab} = c(\eta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

therefore $g = det(g_{ab}) = -c^2(\eta)$ and the contravariant metric is given by

$$g^{ab} = \frac{1}{c(\eta)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence

$$\begin{aligned}
\Box\phi &= \frac{1}{c(\eta)}\partial_\mu(c(\eta)g^{\mu\nu}\partial_\nu\phi) \\
&= \frac{1}{c(\eta)}[\partial_0 c(\eta)g^{00}\partial_0 + \partial_1 c(\eta)g^{11}\partial_1]\phi \\
&= \frac{1}{c(\eta)}[\partial_\eta c(\eta)\frac{1}{c(\eta)}\partial_\eta + \partial_\chi c(\eta)\frac{1}{c(\eta)}\partial_\chi]\phi \\
&= \frac{1}{c(\eta)}[\partial_\eta^2 - \partial_\chi^2]\phi
\end{aligned}$$

Therefore, the Klein-Gordon equation takes the form

$$((\partial_\eta^2 - \partial_\chi^2) + c(\eta)m)\phi = 0$$

Exploiting the resulting spacial translational invariance we separate the solutions into

$$u_k = \frac{1}{\sqrt{2\pi\omega}}e^{ik\chi}\xi_k(\eta).$$

The equation then becomes

$$(\frac{1}{\sqrt{2\pi\omega}}e^{ik\chi}\partial_\eta^2\xi_k(\eta) + \frac{k^2}{\sqrt{2\pi\omega}}e^{ik\chi}\xi_k(\eta) + c(\eta)m\frac{1}{\sqrt{2\pi\omega}}e^{ik\chi}\xi_k(\eta)) = 0$$

and therefore,

$$\partial_\eta^2\xi_k(\eta) + (k^2 + c(\eta)m^2)\xi_k(\eta) = 0.$$

This equation can be solved for the whole spacetime in terms of two hypergeometric functions. We find two types of solutions

$$\begin{aligned}
u_k^{(1)}(\eta,\chi) &= \frac{1}{\sqrt{4\pi\omega_{in}}}e^{ik\chi - i\omega_+\eta\left(\frac{i\omega_-}{\sigma}\right)\ln 2\cosh(\sigma\eta)}{}_2F_2(\alpha,\beta,\gamma_1,\delta) \\
u_k^{(2)}(\eta,\chi) &= \frac{1}{\sqrt{4\pi\omega_{out}}}e^{ik\chi - i\omega_+\eta\left(\frac{i\omega_-}{\sigma}\right)\ln 2\cosh(\sigma\eta)}{}_2F_1(\alpha,\beta,\gamma_2,\delta)
\end{aligned}$$

where the constants above are defined as

$$\alpha \quad = \quad 1 + \frac{i\omega_-}{\rho}, \quad \beta = \frac{i\omega_-}{\rho}, \quad \gamma_1 = 1 - \frac{i\omega_{in}}{\rho}, \quad \gamma_2 = 1 + \frac{i\omega_{out}}{\rho} \quad \delta = \frac{1}{2}(1 - \tanh(\rho\eta))$$

and the frequencies

$$\begin{aligned}
\omega_{in} &= [k^2 + m^2]^{\frac{1}{2}} \\
\omega_{out} &= [k^2 + m^2(1 + 2\epsilon)]^{\frac{1}{2}} \\
\omega_\pm &= \frac{1}{2}(\omega_{out} \pm \omega_{in}).
\end{aligned}$$

We note that in the limit $\eta \to -\infty$ the first solution becomes

$$u_k^{(1)} \to \frac{1}{\sqrt{4\pi\omega_{in}}} e^{ik\chi - i\omega_{in}\eta}$$

And in the case $\eta \to +\infty$ the second solution is

$$u_k^{(2)} \to \frac{1}{\sqrt{4\pi\omega_{out}}} e^{ik\chi - i\omega_{out}\eta}$$

One can see that in these limits the asymptotic solutions $u_k^{(1)}$ and $u_k^{(2)}$ to the Klein-Gordon equation are plane waves which can then be associated with positive mode solutions. The negative mode solutions correspond to u_k^{1*} and u_k^{2*}. Since u_k^1 is associated with a plan wave at $\eta \to -\infty$ (past infinity) we call these solutions in-waves $u_k^{(1)} \equiv u_k^{(in)}$. The solutions $u_k^{(2)} \equiv u_k^{(out)}$ which are associated with plane waves at $\eta \to +\infty$ (future infinity) will be called out-waves.

Using the linear transformation properties of hypergeometric functions we can write $u_k^{(in)}$ in terms of $u_k^{(out)}$. This is easier than caluclating the Bogolubov coefficients using the inner product. We then obtain

$$u_k^{(in)}(\eta, \chi) = \alpha_k u_k^{(out)}(\eta, \chi) + \beta_k u_k^{(out*)}(\eta, \chi)$$

where

$$\begin{aligned}
\alpha_k &= \left(\frac{\omega_{out}}{\omega_{in}}\right)^{\frac{1}{2}} \frac{\Gamma\left(1 - \frac{i\omega_{in}}{\sigma}\right)\Gamma\left(-\frac{i\omega_{out}}{\sigma}\right)}{\Gamma\left(-\frac{i\omega_+}{\sigma}\right)\Gamma\left(1 - \frac{i\omega_+}{\sigma}\right)} \\
\beta_k &= \left(\frac{\omega_{out}}{\omega_{in}}\right)^{\frac{1}{2}} \frac{\Gamma\left(1 - \frac{i\omega_{in}}{\sigma}\right)\Gamma\left(-\frac{i\omega_{out}}{\sigma}\right)}{\Gamma\left(-\frac{i\omega_-}{\sigma}\right)\Gamma\left(1 - \frac{i\omega_-}{\sigma}\right)}
\end{aligned}$$

Here Γ are Gamma functions. From the above expressions we can read off the Bogolubov coefficients $\alpha_{kk'} = \alpha_k \delta_{kk'}$ and $\beta_{kk'} = \beta_k \delta_{-kk'}$. Therefore the transformation between annihilation operators yields

$$a_k^{in} = \alpha_k^* a_k^{out} - \beta_k^* a_{-k}^{out\dagger}.$$

We now consider that the state of the field in the past infinity is the vacuum state (no entanglement)

$$|0\rangle^{in} = \bigotimes_{k=-\infty}^{\infty} |0\rangle_k^{\;in}$$

and use the expression for the in-mode annihilation operator to calculate the state in the future infinity. Since the transformation between in and out annihilation operators only mixes modes of frequency k and $-k$, we consider the following Ansatz for the state in the future infinity

$$|0\rangle^{in} \quad = \quad \sum_n A_n \, |n\rangle_k^{out} \, |n\rangle_{-k}^{out}$$

Given that the vacuum state is defined as $a_k^{in} \, |0\rangle^{in} = 0$ we obtain the equation

$$\begin{aligned} a_k^{in} \, |0\rangle^{in} \quad &= \quad \alpha_k^* \sum_n A_n \sqrt{n} \, |n-1\rangle_k^{out} \, |n\rangle_{-k}^{out} - \beta_k^* \sum_n A_n \sqrt{n} \, |n\rangle_k^{out} \, |n-1\rangle_{-k}^{out} \\ &= \quad \alpha_k^* \sum_n A_{n+1} \sqrt{n+1} \, |n\rangle_k^{out} \, |n+1\rangle_{-k}^{out} - \beta_k^* \sum_n A_{n+1} \sqrt{n+1} \, |n+1\rangle_k^{out} \, |n\rangle_{-k}^{out} . \end{aligned}$$

We then obtain the following recurrence relation

$$A_{n+1} = \frac{\beta_k^*}{\alpha_k^*} A_n \implies A_n = \left(\frac{\beta_k^*}{\alpha_k^*} \right)^n A_0 .$$

The vacuum state from the perspective of observers in the past infinity becomes

$$|0\rangle^{in} = A_0 \sum_n \left(\frac{\beta_k^*}{\alpha_k^*} \right)^n |n\rangle_k^{out} \, |n\rangle_{-k}^{out}$$

for observers in the future infinity. Employing the normalization condition $\langle 0|0\rangle^{in} = 1$ and defing $\gamma = \left| \frac{\beta_k}{\alpha_k} \right|^2$ we obtain

$$|A_0|^2 = 1 - \gamma$$

and therefore,

$$|0\rangle^{in} = \sqrt{1-\gamma} \sum_n \gamma^n \, |n\rangle_k^{out} \, |n\rangle_{-k}^{out} .$$

In terms of the explicit Bobolubov coefficients

$$\gamma = \frac{\sinh^2(\pi\omega_-/\sigma)}{\sinh^2(\pi\omega_+/\sigma)} .$$

REMARKS The vacuum state from the perspective of observers in the remote past has particles in the remote future. Due to the expansion of the universe there has been particle creation.

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Chapter 8

Entanglement in curved spacetime

In the last lecture we considered an example of curved spacetime where particles can be defined in two asymptotically flat regions regions. We found that while in the remote past the field was in the vacuum state, in the remote future particles have been created. We recall however, that in the interim region, where the universe is undergoing expansion, no sensible notion of particles exist. In this lecture we will continue with this example and show that in the future infinity entanglement has been created between field modes. Interestingly, it is possible to learn about the expansion parameters of the Universe from the entanglement generated. As a last example, we will consider the entanglement between two field modes in the spacetime of an eternal black hole. We will see that while two observers falling into the black hole describe the state of the field to be in a maximally entangled state, the entanglement in the state becomes degraded from the perspective of an inertial observer, Alice, who falls into a black hole and an non-inertial observer, Rob, escaping the black hole.

8.1 Entanglement in an expanding universe

We found that the state corresponding to the vacuum state in the past infinity corresponds to the following two mode state in the future infinity

$$|0\rangle^{in} = \sqrt{1-\gamma}\sum_n \gamma^n \, |n\rangle_k^{out} \, |n\rangle_{-k}^{out} \, .$$

Since the state is pure, we can employ the Von-Neuman entropy to quantify the entanglement generated the field modes k and $-k$. In order to do this we need to compute the reduced density matrix for one of the modes. The densitity matrix for the state is

$$\begin{aligned}\rho_0 &= |0\rangle^{in}\langle 0|^{in} \\ &= (1-\gamma)\sum_{n,m}\gamma^{(n+m)}|n\rangle_k^{out}|n\rangle_{-k}^{out}\langle m|_k^{out}\langle m|_{-k}^{out}.\end{aligned}$$

The reduced density matrix for mode k is obtained by tracing over mode $-k$

$$\begin{aligned}\rho_k &= tr_{-k}[\rho_0] \\ &= (1-\gamma)\sum_{n,m}\gamma^{(n+m)}\delta_{nm}|n\rangle_k^{out}\langle m|_k^{out} \\ &= (1-\gamma)\sum_{n,m}\gamma^{2n}|n\rangle_k^{out}\langle n|_k^{out}\end{aligned}$$

Since the reduced denstiy matrix is already in diagonal form with eigenvalues $\lambda_n = (1-\gamma)\gamma^{2n}$, it is straight forward to compute the Von-Neumann entropy

$$\begin{aligned}S(\rho_k) &= -Tr\left(\rho_k\log_2\rho_k\right) \\ &= -(1-\gamma)\sum_n\gamma^{2n}\log_2((1-\gamma)\gamma^{2n}) \\ &= (\gamma-1)\sum_n(2n\gamma^{2n}\log_2\gamma+\gamma^{2n}\log_2(1-\gamma)) \\ &= 2(\gamma-1)\log_2\gamma\sum_n n\gamma^{2n}+(\gamma-1)\log_2(1-\gamma)\sum_n\gamma^{2n} \\ &= (\gamma-1)\log_2\gamma\sum_n 2n\gamma^{2n}+(\gamma-1)\log_2(1-\gamma)\frac{1}{1-\gamma} \\ &= (\gamma-1)\log_2\gamma\frac{\partial}{\partial\gamma}(1-\gamma)^{-1}-\log_2(1-\gamma)\end{aligned}$$

which yields

$$S(\rho_k) = \log_2\left[\frac{\gamma^{\frac{\gamma}{\gamma-1}}}{1-\gamma}\right]$$

Entanglement has been created in the remote future due to the expansion of the universe. The entanglement depends on the cosmological constants since the coefficient γ depends on the expansion rate σ, the expansion volume ϵ and the frequency of the modes involved through

$$\gamma = \frac{\sinh^2(\pi\omega_-/\sigma)}{\sinh^2(\pi\omega_+/\sigma)}$$

with

$$\begin{aligned}\omega_\pm &= \frac{1}{2}(\omega_{out}\pm\omega_{in}) \\ \omega_{in} &= [k^2+m^2]^{\frac{1}{2}} \\ \omega_{out} &= [k^2+m^2(1+2\epsilon)]^{\frac{1}{2}}.\end{aligned}$$

In the case of light particles, the equations can be inverted and we can show that we can estimate the expansion parameters from the entanglement.

REMARK Entanglement has been created between field modes due to the expansion of the universe. It is possible to learn from the past history of the universe from the entanglement in the modes.

8.2 Alice falls into a black hole

The Schwarzchild spacetime of an eternal black hole describes the geometry of a spherical non-rotating mass m. Considering only the radial component, the metric is

$$ds^2 = (1 - (2m/R))dT^2 - (1 - (2m/R))^{-1}dR^2.$$

The spacetime is curved and admits no global timelike Killing vector fields. However, close to the horizon of the black hole defined by $R = 2m$, the space time is approximately flat. To see this, we consider the following coordinate change $R - 2m = x^2/8m$, such that

$$1 - (2m/R) = x^2/8mR = \frac{(x^2/8m)}{(x^2/8m + 2m)} = \frac{(Ax)^2}{(1 + (Ax)^2)} \approx (Ax)^2$$

when $x \approx 0$ with $A = 1/4m$. This means that $dR^2 = (Ax^2)dx^2$. Therefore, very close to the horizon $R \approx 2m$ the Schwarzschild spacetime can be approximated by Rindler space

$$ds^2 = -(Ax)^2dT^2 + dx^2.$$

where the acceleration parameter $a = A^{-1}$. This means that, very close to the horizon of the black hole, we can consider Alice being inertial and falling into the black hole while Rob escapes the fall by being accelerated. If Alice claims that the state of the field is the vacuum state, then Rob detects a thermal state since the state has the form

$$|0_k\rangle^{\mathcal{M}} = \frac{1}{\cosh(r)} \sum_n \tanh^n(r) |n_k\rangle^{in} |n_k\rangle^{out}.$$

were r is a function of the mass of the black hole. Here *in* and *out* denote the modes inside and outside the black hole. $|0_k\rangle^{\mathcal{M}}$ is the state detected by Alice. If we then consider that the field, from Alice's perspective is a maximally entangled state of two modes, Rob will detect less entanglement between the modes due to the Hawking effect.

8.3 Final remarks

The phenomenon of entanglement has been extensively studied in non-relativistic settings. Much of the interest on this quantum property has stemmed from its relevance in quantum information theory. However, relatively little is known about relativistic effects on entanglement despite the fact that many of the systems used in the implementation of quantum information involve relativistic systems such as photons. The vast majority of investigations on entanglement assume that the world is flat and non-relativistic. Understanding entanglement in spacetime is ultimately necessary because the world is fundamentally relativistic. Moreover, entanglement plays a prominent role in black hole thermodynamics and in the information loss problem.

The question of understanding entanglement in non-inertial frames has been central to the development of the emerging field of relativistic quantum information. The main aim of this field is to incorporate relativistic effects to improve quantum information tasks (such as quantum teleportation) and to understand how such protocols would take place in curved space-times. In most quantum information protocols entanglement plays a prominent role. Therefore, it is of a great interest to understand how it can be degraded or created by the presence of horizons or spacetime dynamics.

In this series of lectures we showed that studies on relativistic entanglement show that conceptually important qualitative differences to a non-relativistic treatment arise. For instance, entanglement was found to be an observer-dependent property that is degraded from the perspective of accelerated observers moving in flat spacetime. These results suggest that entanglement in curved spacetime might not be an invariant concept.

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Chapter 9

Problems

Entanglement of Dirac Fields in non-inertial frames

In a parallel analysis to the bosonic case, we consider a Dirac field ϕ satisfying the equation $\{i\gamma^\mu(\partial_\mu - \Gamma_\mu) + m\}\phi = 0$ where γ^μ are the Dirac-Pauli matrices and Γ_μ are spinorial affine connections. The field expansion in terms of the Minkowski solutions of the Dirac equation is

$$\phi = N_{\mathrm{M}} \int \left(c_{k,\mathrm{M}}\, u^{+}_{k,\mathrm{M}} + d^{\dagger}_{k,\mathrm{M}}\; u^{-}_{k,\mathrm{M}} \right) dk, \tag{9.1}$$

Where N_{M} is a normalisation constant and the label $\pm$ denotes respectively positive and negative energy solutions (particles/antiparticles) with respect to the Minkowskian Killing vector field ∂_t. The label k is a multilabel including energy and spin $k = \{E_\omega, s\}$ where s is the component of the spin on the quantisation direction. c_k and d_k are the particle/antiparticle operators that satisfy the usual anticommutation rule

$$\{c_{k,\mathrm{M}}, c^{\dagger}_{k',\mathrm{M}}\} = \{d_{k,\mathrm{M}}, d^{\dagger}_{k',\mathrm{M}}\} = \delta_{kk'}, \tag{9.2}$$

and all other anticommutators vanishing.

The Dirac field operator in terms of Rindler modes is given by

$$\phi = N_{\mathrm{R}} \int \left(c_{j,\mathrm{I}} u^{+}_{j,\mathrm{I}} + d^{\dagger}_{j,\mathrm{I}} u^{-}_{j,\mathrm{I}} + c_{j,\mathrm{II}} u^{+}_{j,\mathrm{II}} + d^{\dagger}_{j,\mathrm{II}} u^{-}_{j,\mathrm{II}} \right) dj, \tag{9.3}$$

Where N_{R} is, again, a normalisation constant. $c_{j,\Sigma}, d_{j,\Sigma}$ with $\Sigma = \mathrm{I}, \mathrm{II}$ represent Rindler particle/antiparticle operators. The usual anticommutation rules again apply. Note that operators in different regions $\Sigma = \mathrm{I}, \mathrm{II}$ do not commute but anticommute. $j = \{E_\Omega, s'\}$ is again a multi-label including all the degrees of freedom. Here $u^{\pm}_{k,\mathrm{I}}$ and $u^{\pm}_{k,\mathrm{II}}$ are the positive/negative frequency solutions of the Dirac equation in Rindler coordinates with respect to the Rindler timelike Killing vector field in region I and II, respectively. The modes $u^{\pm}_{k,\mathrm{I}}$, $u^{\pm}_{k,\mathrm{II}}$ do not have support outside the right, left Rindler wedge. The annihilation operators

$c_{k,\mathrm{M}}, d_{k,\mathrm{M}}$ define the Minkowski vacuum $|0\rangle_{\mathrm{M}}$ which must satisfy

$$c_{k,\mathrm{M}}\,|0\rangle_{\mathrm{M}} = d_{k,\mathrm{M}}\,|0\rangle_{\mathrm{M}} = 0, \qquad \forall k. \tag{9.4}$$

In the same fashion $c_{j,\Sigma}, d_{j,\Sigma}$, define the Rindler vacua in regions $\Sigma = \mathrm{I}, \mathrm{II}$

$$c_{j,\mathrm{R}}\,|0\rangle_{\Sigma} = d_{j,\mathrm{R}}\,|0\rangle_{\Sigma} = 0, \qquad \forall j,\ \Sigma = \mathrm{I}, \mathrm{II}. \tag{9.5}$$

As in the bosonic case, we will work with Unruh modes for the inertial observers, where the transformation between Unruh and Rindler operators is given by and the operators

$$C_k \quad \equiv \quad \left(\cos r_k\, c_{k,\mathrm{I}} - \sin r_k\, d^{\dagger}_{k,\mathrm{II}}\right).$$

It can be shown that for a massless Dirac field the Unruh operators have the same form as Eq. (9.6) however in this case $\tan r_k = e^{-\pi\Omega_a/a}$. In this case, to find the Minkowski vacuum in the Rindler basis we consider the following ansatz

$$|0\rangle_{\mathrm{M}} = \bigotimes_{\Omega} |0_\Omega\rangle_{\mathrm{M}}\,, \tag{9.6}$$

were

$$|0_\Omega\rangle_M \quad = \quad \sum_{n,s}\left(F_{n,\Omega,s}\,|n_{\Omega,s}\rangle^{+}_{\mathrm{I}}\,|n_{\Omega,-s}\rangle^{-}_{\mathrm{II}}\right) \tag{9.7}$$

where the label $\pm$ denotes particle/antiparticle modes and s labels the spin. The minus signs on the spin label in region II show explicitly that spin, as all the magnitudes which change under time reversal, is opposite in region I with respect to region II.

Due to the anticommutation relations we must introduce the following sign conventions

$$\begin{aligned}
|1_\Omega\rangle^{+}_{\mathrm{I}}|1_\Omega\rangle^{-}_{\mathrm{II}} &= d^{\dagger}_{\Omega,IV}c^{\dagger}_{\Omega,\mathrm{I}}|0_\Omega\rangle^{+}_{\mathrm{I}}|0_\Omega\rangle^{-}_{\mathrm{II}}\,,\\
&= -c^{\dagger}_{\Omega,\mathrm{I}}d^{\dagger}_{\Omega,\mathrm{II}}|0_\Omega\rangle^{+}_{\mathrm{I}}|0_\Omega\rangle^{-}_{\mathrm{II}}\,,\\
|1_\Omega\rangle^{-}_{\mathrm{I}}|1_\Omega\rangle^{+}_{\mathrm{II}} &= c^{\dagger}_{\Omega,IV}d^{\dagger}_{\Omega,\mathrm{I}}|0_\Omega\rangle^{-}_{\mathrm{I}}|0_\Omega\rangle^{+}_{\mathrm{II}}\,,\\
&= -d^{\dagger}_{\Omega,\mathrm{I}}c^{\dagger}_{\Omega,\mathrm{II}}|0_\Omega\rangle^{-}_{\mathrm{I}}|0_\Omega\rangle^{+}_{\mathrm{II}}\,.
\end{aligned} \tag{9.8}$$

We will now consider the simplest case that preserves the fundamental Dirac characteristics which corresponds to Grassman scalars. In this case the Pauli exclusion principle limits the sums to $n = 0, 1$ and there is no spin. For convenience, is it suitable to introduce the following notation,

$$|nn'n''n'''\rangle_\Omega \equiv |n_\Omega\rangle^{+}_{\mathrm{I}}\,|n'_\Omega\rangle^{-}_{\mathrm{II}}\,|n''_\Omega\rangle^{-}_{\mathrm{I}}\,|n'''_\Omega\rangle^{+}_{\mathrm{II}}\,. \tag{9.9}$$

- Considering Grassman scalars, obtain the form of the coefficients $F_{n,\Omega}$ for the vacuum by imposing that the Minkowski vacuum is annihilated by the particle annihilator C_k for all frequencies and values for the spin third component, ie. $C_\Omega\,|0_\Omega\rangle_{\mathrm{R}} = 0$.

- Obtain the Unruh one particle state by applying the creation operator to the vacuum state $|1_j\rangle_{\rm U} = C^\dagger_\Omega |0_j\rangle_{\rm M}$.
- Consider the following fermionic maximally entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0_\omega\rangle_{\rm M}|0_\Omega\rangle_{\rm U} + |1_\omega\rangle^+_{\rm M}|1_\Omega\rangle^+_{\rm U}\right), \tag{9.10}$$

  which is the fermionic analog to the bosonic maximally entangled state studied in our lectures. Compute Alice-Rob partial density matrix by tracing over region II .
- Obtain the partial transpose of density matrix and find its negative eigenvalues. Note that the matrix is block diagonal and only two blocks contribute.
- Compute the negativity.
- Compare results with the bosonic case.

Chapter 10

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