Aspects of classification of quantum nets

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Wigner Distributions on $\mathbb{R}$

Consider a quantum system whose classical configuration space $Q = \mathbb{R}$ is the real line $\mathbb{R}$. Let $\{\ |q\rangle \mid q \in \mathbb{R}\}$ denone the coordinate basis in the corresponding Hilbert space $\mathcal{H}$:

$$\langle q|q'\rangle \;=\; \delta\left(q-q'\right), \int_{-\infty}^{\infty} dq|q\rangle\langle q| = \mathbb{I};$$

Given the coordinate basis, one defines a 'momentum basis' $\{\ |p\rangle \mid p \in \mathbb{R}\}$ related to it by Fourier transformation:

$$|p\rangle \;=\; \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dq\ e^{iqp/\hbar}|q > ;$$

$$\langle p|p'\rangle \;=\; \delta\left(p-p'\right), \int_{-\infty}^{\infty} dp|p\rangle\langle p| = \mathbb{I};$$

We may arrange the values of q and p in the usual Cartesian fashion and call it the 'classical phase space' associated with the quantum system. (Note that this classical phase space is not always the same as T^*Q).
In 1932, Wigner introduced a quantum analogue of the classical phase space distribution which associates with any quantum state $\widehat{\rho}$ a function $W_{\widehat{\rho}}(q,p)$ as follows:

$$\begin{aligned} \widehat{\rho} \mapsto W_{\widehat{\rho}}(q,p) &= \mathrm{Tr}\left\{\widehat{\rho}\,\widehat{W}(q,p)\right\} \ ; \\ \widehat{W}(q,p) &= \frac{1}{(2\pi\hbar)} \int_{-\infty}^{\infty} dq' \ |q + \frac{1}{2}q'\rangle\langle q - \frac{1}{2}q'| \ e^{i\,pq'/\hbar}, \end{aligned}$$

The operators $\widehat{W}(q,p)$ will be referred to as phase point operators

The Wigner distribution defined above has the following properties

1. Reality : $W_{\widehat{\rho}}(q,p) = W_{\widehat{\rho}}(q,p)^*$.
2. Marginals property :Average of the Wigner distribution along a line in phase space yields a probability density
3. Traciality: $\mathrm{Tr}\,\{\widehat{\rho}'\widehat{\rho}\} = \frac{1}{(2\pi\hbar)}\int\limits_{-\infty}^{\infty} dq \int\limits_{-\infty}^{\infty} dp\; W_{\widehat{\rho}'}(q,p)\; W_{\widehat{\rho}}(q,p)$.
4. $W_{\widehat{\rho}}(q,p)$ not necessarily positive for all $\widehat{\rho}$. For pure states $|\psi >\in \mathcal{H}$ the Wigner distribution is positive if and only if the state is a Gaussian state. (The Wigner distribution for such states is itself a Gaussian).

Correspondingly for the phase point operators

1. Hermiticity : $\widehat{W}(q,p) = \widehat{W}^{\dagger}(q,p)$.
2. Marginals property :Average of the phase point operator along a line in phase space yields a Projector

Wigner distributions have played an important role in semi classical approximations, classical optics etc. Of late they came into prominence in Quantum Information Theory largely due to the work of R. Simon and also Duan et. al in the context of continuous variable entanglement where necessary and sufficient conditions for entanglement in two mode Gaussian pure states were derived.

Two important operations at the level of the ‘classical phase space’

1. Translations: $(q, p) \mapsto (q', p') = (q + q_0, p + p_0)$. These transform an (isotropic) line to another line parallel to it
2. Symplectic transformations $SL(2, \mathbb{R})$:

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix};$$

$$a, b, c, d \in \mathbb{R}; ad - bc = 1$$

   These change the orientation of the isotropic lines,

At the quantum level, these operations are implemented by the operators $D(q, p)$, the displacement operators, and by unitary operators $\{U(S) \mid S \in SL(2, \mathbb{R})\}$ respectively.

Dirac inspired 'square root' approach to Wigner distributions

Consider Tr $\left\{\widehat{A}\widehat{B}\right\}$. It is easy to show that

$$\begin{aligned}\text{Tr}\left\{\widehat{A}\widehat{B}\right\} &= \int\int dqdp\langle q|\widehat{A}|p\rangle\langle p|\widehat{B}|q\rangle \\ &= 2\pi\hbar\int\int dqdpA_l\left(q,p\right)B_r\left(q,p\right) \\ &= 2\pi\hbar\int\int dqdpA_r\left(q,p\right)B_l\left(q,p\right).\end{aligned}$$

where $A_l\left(q,p\right) = \langle q|\widehat{A}|p\rangle\langle p|q\rangle$ and $A_r\left(q,p\right) = \langle p|\widehat{A}|q\rangle\langle q|p\rangle$. Note that the RHS lacks the manifest symmetry of the LHS under interchange of $\widehat{A}$ and $\widehat{B}$. This symmetry can be restored by at the expense of introducing a kernel

$$\text{Tr}\left\{\widehat{A}\widehat{B}\right\} = \int\int\int\int dqdpdq'dp'A_l\left(q,p\right)K_l\left(q,p;q',p'\right)B_l\left(q',p'\right),$$

where

$$\begin{aligned} K_l(q,p';q',p) &= (2\pi\hbar)^2 \langle q|p'\rangle\langle p'|q'\rangle\langle q'|p\rangle\langle p|q\rangle \\ &= \exp\{i(q-q')(p'-p)/\hbar\} \end{aligned}$$

The kernel $K_l(q,p;q',p')$ is explicitly symmetric under: $(q,p) \longleftrightarrow (q',p')$, so we have a classical phase-space expression for $\mathrm{Tr}\{\widehat{A}\widehat{B}\}$ manifestly symmetric in $\widehat{A}$ and $\widehat{B}$. (Here we have chosen to work with symbols carrying subscripts ℓ, One could, if one wishes, carry out a similar analysis with symbols carrying subscripts r). A natural question to ask if this kernel can in some sense be 'transformed away' while maintaining manifest symmetry in $\widehat{A}$ and $\widehat{B}$. This can be done if we can express it as the 'square' or the convolution of some more elementary kernel, say in the form:

$$K_l(q,p;q',p') = \int\int dq''dp''\xi(q'',p'';q,p)\,\xi(q'',p'';q',p').$$

We would then have

$$\mathrm{Tr}\left\{\widehat{A}\widehat{B}\right\} = \frac{1}{2\pi\hbar}\int\int dq dp A\left(q,p\right) B\left(q,p\right),$$

where $A\left(q,p\right)$ arises from $A_l\left(q,p\right)$ via:

$$A\left(q,p\right) = \sqrt{2\pi\hbar}\int\int dq' dp' \xi\left(q,p;q',p'\right) A_l\left(q',p'\right).$$

What properties do we demand of $\xi\left(q,p;q',p'\right)$ other than? Naturally, the same as those of $K_l\left(q,p;q',p'\right)$ viz, Symmetry, essential Unitarity, translational invariance, marginals property. For K_ℓ given above, its square root ξ satisfying the above properties is easily found to be

$$\xi\left(q,p;q',p'\right) = \sqrt{\frac{2}{\pi\hbar}}\,\mathrm{exp}\left\{2i\left(q-q'\right)\left(p-p'\right)/\hbar\right\},$$

and it is easy to check that the symbol $A(q,p)$ defined above is indeed the Wigner symbol of $\widehat{A}$.

Wigner Distribution for finite state quantum systems: $Q = \mathbb{Z}_N$

Consider a quantum system described by a (complex) Hilbert space of dimension N. We designate $q \in \mathbb{Z}_N$ as the 'coordinates' and $\{|q\rangle \mid q \in \mathbb{Z}_N\}$ as the 'coordinate basis". ($\mathbb{Z}_N = \{0, 1, \cdots, N-1\}$ denotes the ring of integers with addition and multiplication modulo N)

$$\langle q|q'\rangle = \delta_{q,q'}, \quad \sum_{q=0}^{N-1} |q\rangle\langle q| = \mathbb{I};$$

The 'momentum basis' $\{|p\rangle \mid p \in \mathbb{Z}_N\}$ is then obtained by a discrete Fourier transform

$$|p\rangle = \frac{1}{\sqrt{N}} \sum_{q \in \mathbb{Z}_N} \omega^{qp} |q >; \quad p \in \mathbb{Z}_N, \omega = e^{2\pi i/N}$$

$$\langle p|p'\rangle = \delta_{p,p'}, \sum_{p\in\mathbb{Z}_N} |p\rangle\langle p| = \mathbb{I};$$

We can think of the phase space, $\Gamma_0 = \mathbb{Z}_N \times \mathbb{Z}_N$, a discrete $N \times N$ lattice. The analogues of the operations on the phase space for the continuum case now are

1. Translations: $(q, p) \mapsto (q', p') = (q + q_0, p + p_0)$. These transform a line to another line parallel to it
2. Symplectic transformations $SL(2, \mathbb{Z}_N)$: These take an isotropic line to another.

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix};$$

$$a, b, c, d \in \mathbb{Z}_N; ad - bc = 1 \text{mod } N$$

The kernel $K_l\left(q,p';q',p\right)$ is now given by

$$\begin{aligned} K_l\left(q,p';q',p\right) &= N^2\langle q|p'\rangle\langle p'|q'\rangle\langle q'|p\rangle\langle p|q\rangle \\ &= \omega^{(q-q')(p'-p)}. \end{aligned}$$

Choosing $\sigma=(q,p)$ to label rows and columns, we have here a $N^2\times N^2$ matrix $K_l\left(\sigma;,\sigma'\right)$. Setting up the Wigner distributions amounts to finding the square root $\xi\left(\sigma;,\sigma'\right)$ of this matrix (satisfying Symmetry, essential Unitarity, translational invariance, marginals property). This can explicitly be done:

$$\xi(\sigma;\sigma')=\frac{1}{N^{3/2}}\sum_{\sigma''\in\Gamma_0}\tau^{q''p''}S(\sigma'')\omega^{\langle\sigma,\sigma''\rangle-\langle\sigma',\sigma''\rangle}$$

where $\tau = -e^{i\pi/N}$; $\tau^2 = \omega$ and $S(\sigma)$ take values $\pm$. For the phase point operators we then have

$$\sigma \in \Gamma_0 : \ \widehat{W}(\sigma) = \frac{1}{N} \sum_{\sigma' \in \Gamma_0} \omega^{\langle \sigma, \sigma' \rangle} S(\sigma') D(\sigma')$$

where $D(\sigma)$ denote the unitary displacement operators obeying

$$\begin{aligned} D^\dagger(\sigma) &= D(-\sigma) \\ D(\sigma)D(\sigma') &= \tau^{\langle \sigma, \sigma' \rangle} D(\sigma + \sigma') \\ D(\sigma + N\sigma_0) &= D(\sigma) \text{ if } N \text{ odd} \\ &= (-1)^{\langle \sigma, \sigma_0 \rangle} D(\sigma) \text{ if } N \text{ even} \end{aligned}$$

Hermiticity of $\widehat{W}(\sigma)$ requires $S(\sigma)$ to obey

$$S(\sigma) = S([N - \sigma]) \text{ if } N \text{ odd}$$

and for N even

$$\begin{aligned} S(\sigma) &= S([N - \sigma]) \text{ if } q \text{ or } p = 0 \\ &= (-1)^{(q+p)} S([N - \sigma]) \text{ otherwise} \end{aligned}$$

Trace orthogonality of $D(\sigma)$ leads to

$$\mathsf{Tr}(\widehat{W}(\sigma')\widehat{W}(\sigma)) = N\delta_{\sigma',\sigma}$$

Also

$$\mathsf{Tr}(\widehat{W}(\sigma)) = 1,$$

Requiring standard $q - p$ marginals conditions

$$S(q,0) = S(0,p) = 1$$

Isotropic line conditions

The conditions so far on $S(\sigma)$ reduce the number of undetermined signs to about half. To generate more conditions, we consider more marginals conditions, based on isotropic lines.

An isotropic line λ is a maximal set of N distinct points in Γ_0, a subset of Γ_0, including $\sigma = (0,0)$ and obeying:

$$\sigma', \sigma \in \lambda \Rightarrow \langle \sigma', \sigma \rangle = 0 \text{ mod } N.$$

Each point $\sigma \in \Gamma_0$ belongs to at least one isotropic line. (Further details later) We now require that the operator P_λ obtained by averaging $\widehat{W}$'s over an isotropic line:

$$\begin{aligned} P_\lambda &= \frac{1}{N} \sum_{\sigma \in \lambda} \widehat{W}(\sigma) = \frac{1}{N^2} \sum_{\sigma \in \lambda} \sum_{\sigma' \in \Gamma_0} \omega^{\langle \sigma, \sigma' \rangle} S(\sigma') D(\sigma') \\ &= \frac{1}{N} \sum_{\sigma \in \lambda} S(\sigma) D(\sigma) \end{aligned}$$

be a rank one projector

$$P_\lambda^2 = P_\lambda, \mathrm{Tr}(P_\lambda) = 1$$

This yields

$$S(\sigma)S(\sigma') \quad = \quad \tau^{\langle\sigma',\sigma\rangle}\epsilon(\sigma',\sigma)S([\sigma+\sigma']),$$

for all $\sigma', \sigma \in \lambda$. Here

$$\epsilon(\sigma',\sigma) = \epsilon(\sigma,\sigma') = (-1)^{((q'+q)[p'+p]-[q'+q](p'+p))/N}$$

N odd case

In the case when N is odd, the isotropic line conditions have a remarkably simple solution

$$S(\sigma) = 1$$

and results by using the fact that every σ on a given isotropic line can be uniquely written as $2\sigma'$ where σ' also lies on λ. (Group theoreticaly : every element of a group of odd order,abelian or non abelian, can be uniquely written as the square of another group element). Thus in the odd case, marginals conditions fix all the signs leading to a unique Wigner distribution. No detailed properties of the isotropic lines or how they go into each other under $SL(2, \mathbb{Z}_N)$ are ever required. The Wigner distribution thus obtained has all the properties of the Wigner distribution in the continuum. In particular,

$$\widehat{W}(0,0) = \text{Parity Operator}$$

N even : special case $N = 4$

Here we have seven lines, six generated by $(1,0),(1,1),(1,2),(1,3),(0,1),(2,1)$ and one generated by $(2,0)$ and $(0,2)$. Applying the isotropic line conditions to these lines one finds $S(2,2)$ must both be $+1$ as well as -1 implying that it is impossible to satisfy marginals property on all the isotropic lines. Given this circumstance, can we at least consistently satisfy marginals property on a subset if not on all the isotropic lines? This forces us to examine detailed structure isotropic lines in the even case, in particular, how they divide themselves up in to orbits under $SL(2,\mathbb{Z}_N)$.

Isotropic Lines and orbits under $SL(2, Z_N)$

[Albouy J. Phys. 42 072001 (2009)] Towards handling the case of general N (essentially even N) we may note the following. Any N can be uniquely written as product of (increasing) primes in the form :

$$N = N_1 N_2 \cdots N_k = \prod_{j=1}^{k} N_j$$

$$N_j = p_j^{n_j}, p_j = \text{j}^{\text{th}} \text{ prime : } p_1 = 2,\ p_2 = 3,\ p_3 = 5, \cdots ,$$

$$p_j = \text{odd } j \geq 2; \text{ and } n_j = 0 \text{ or } 1 \text{ or } 2 \cdots$$

If $n_1 = 0$: then N is odd , previous results are in hand. Something new arise only if $n_1 \geq 1$.
We may thus treat first the case:

$$N = p^n = \text{power of a single prime,}$$

Isotropic Lines and orbits under $SL(2, \mathbb{Z}_N)$ for $N = p^n$

1. The number of isotropic lines is $(p^{n+1} - 1)/(p - 1)$
2. The number of isotropic lines passing through a point $\sigma \in Z_N \times Z_N$ is given by $(p^{t+1} - 1)/(p - 1)$, where t is the p-valuation of σ. [By p–valuation one means the following ; Every element a of Z_N can be uniquely written as $a_0 p^0 + a_1 p^1 + \cdots a_{n-1} p^{n-1}$ with $a_i \in \{0, 1, \cdots, p_j - 1\}$. The p–valuation of a is then the smallest i for which a_i is non zero. The p– valuation of 0 is taken to be n_j. For $\sigma = (q, p)$, p–valuation is defined to be the minimum of the p-valuations of q and p]. Thus for $(0, 0)$, $t = p$ and for a σ such that either q or p is a unit., we have $t = 1$, i.e only one line passes through it.

As to the action of the group $SL(2, Z_N)$ on the isotropic lines one finds that

1. The isotropic lines, $(p^{n+1} - 1)/(p - 1)$ in number, divide themselves into Int$[n/2] + 1$ orbits under $SL(2, Z_N)$. The orbits denoted by $O_k(p^n), k = 0, 1, \cdots,$ Int$[n/2]$ contain $p^{n-2k-1}(p + 1)$ isotropic lines if $k < Int[n/2]$ and a singleton if $k = Int[n/2]$. In, particular the largest orbit, corresponding to $k = 0$, contains $p^{n-1}(p + 1)$ lines.
2. Only the largest orbit has the property that it covers all points in $Z_N \times Z_N$. The $p^{n-1}(p + 1)$ isotropic lines in this orbit are all generated by single generators of order N which may be taken to be $(1, \alpha)$, $\mu \in \{0, 1, \cdots, N - 1\}$ and $(\alpha, 1), \alpha \in \{$non units in $Z_N\}$.

Isotropic Lines and orbits in the general case

Turning now to the case of a general N we note that ring Z_N can be factored as

$$Z_N = Z_{N_1} \times Z_{N_2} \times \cdots \times Z_{N_k}$$

The explicit correspondence between elements of Z_N and those of the rings Z_{N_j} is provided by the chinese remainder theorem which tells us that an element $q \in Z_N$ can be uniquely decomposed as

$$q = \sum_{j=1}^{k} q_j \cdot \nu_j \cdot \mu_j$$

where $q_j = [q \bmod N_j] \in Z_{N_j}$, $\nu_j = N/N_j$ and μ_j denotes the (multiplicative) inverse of ν_j in Z_{N_j}. Thus each element $q \in Z_N$ can uniquely represented as an array

$$q \longleftrightarrow \{q_1, q_2, \cdots, q_k\}, \quad q_i \in Z_{N_i}$$

In particular the elements 0 and 1 are represented by

$$0 \longleftrightarrow \{0, 0, \cdots, 0\}; \quad 1 \longleftrightarrow \{1, 1, \cdots, 1\}$$

Further, this correspondence has the nice property that

$$q + q' \quad \longleftrightarrow \quad \{q_1 + q_1', q_2 + q_2', \cdots, q_k + q_k'\}, \quad q_i \in Z_{N_i}$$

$$qq' \quad \longleftrightarrow \quad \{q_1 q_1', q_2 q_2', \cdots, q_k q_k'\}, \quad q_i \in Z_{N_i}$$

In view of this we have the following results:

1. A point $\sigma \in Z_N \times Z_N$ can be represented as

$$\sigma \longleftrightarrow \{\sigma_1, \sigma_2, \cdots, \sigma_k\}, \quad \sigma_i \in Z_{N_i} \times Z_{N_i}$$

2. The symplectic product of $\langle \sigma, \sigma' \rangle$ vanishes if and only if each of the components $\langle \sigma_i, \sigma_i' \rangle$ vanish.
3. The group $SL(2, Z_N)$ also factorises as

$$SL(2, Z_N) = SL(2, Z_{N_1}) \times SL(2, Z_{N_2}) \times SL(2, Z_{N_k})$$

Isotropic lines in $N = 2^n$ case

From the above it is evident that the isotropic lines in $\sigma \in Z_N \times Z_N$ and $SL(2, Z_N)$ action are completely determined by those in each of the factors $Z_{N_j} \times Z_{N_j}$. Further, having disposed off the odd case we need to examine only the $N = 2^n$ case. Specialising the earlier results to this we have

1. There are $2^{n+1} - 1$ isotropic lines in $Z_{2^n} \times Z_{2^n}$
2. These can be divided into two categories, (a) those in the largest orbit, 3.2^{n-1} in number and generated by single generators. (b) and the rest which involve two generators of orders 2^r and 2^s where $r + s = n$.
3. The isotropic lines in the category a cover all the phase points in $Z_{2^n} \times Z_{2^n}$. Further, only the even points appear on more than one line.
4. The isotropic lines in the category (b) are $2^{n-1} - 1$ in number. These cover all the even phase points in $Z_{2^n} \times Z_{2^n}$ only.

It turns out that it is impossible to satisfy isotropic line conditions on all the isotropic lines and that they can be consistently implemented orbit by orbit. It is therefore sensible to demand marginals property restricting oneself to the largest orbit as it is the only one whose lines cover all phase points. These conditions relate or fix the signs for points on the isotropic lines in the largest orbit. The number of unfixed signs equals $3.2^{n-1}-2$. Thus, for $N=2$ and 4 respectively, the unfixed signs are shown below:

$$\begin{array}{cc} 1 & S(1.1) \\ 1 & 1 \end{array}$$

$$\begin{array}{cccc} 1 & S(1,3) & -S(1,2)1 & S(1,1 \\ 1 & S(2,1) & 1 & -S(2,1) \\ 1 & S(1,1) & S(1,2) & S(1,3) \\ 1 & 1 & 1 & 1 \end{array}$$

and we have $2, 2^4$ different choices for Wigner distributions

Spectra of phase point operators

We now examine the dependence of the eigenvalues of the phase point operators as a function of the signs that remain unfixed. For this purpose it is sufficient to look at the eigenvalues of $\widehat{W}(0,0)$.

For $N = 2$ there is only one unfixed sign $S(1,1)$ and the spectrum of $\widehat{W}(0,0)$ is the same for $S(1,1) = \pm 1$

For $N = 4$ one finds that there are three distinct spectra for $\widehat{W}(0,0)$ depending on the values of the four free signs $S(1,1) \equiv a, S(1,2) \equiv b, S(1,3) \equiv c, S(2,1) \equiv d$. They are

- $((1+\sqrt{6})/2, (1-\sqrt{6})/2, -1/2, 1/2)$.
- $((1+2\sqrt{2})/2, -1/2, (1-\sqrt{2})/2, (1-\sqrt{2})/2)$
- $((1+\sqrt{2})/2, (1+\sqrt{2})/2, (1-2\sqrt{2})/2, -1/2)$

For$N = 8$, one has 4 and $N = 16$, one has 15 distinct spectra. Thus, although the number of different Wigner distributions base on choices for the signs for $N = 2, 2^2, 2^3, 2^4$ is $2, 2^4, 2^{10}, 2^{22}$, those which have distinct spectra are only $1, 3, 4, 15$ in number. (It seems that the number of distinct spectra for $N = 2^n$ equals 2^{n-1} if n even and $2^n - 1$ if n odd) This motivates us to examine the action of the Clifford group on the families of Wigner distributions (quantum nets) obtained so far.

Clifford Group

Here by Clifford group we mean the set of all unitaries that take the generalised Pauli group – the set $\{e^{i\xi}D(\sigma)\}$ into itself. The structure of this group has been examined in great detail by Appleby in connection with SICPOVMS. The set of unitaries that are relevant here consists of matrices:

$$V_F = \frac{1}{\sqrt{N}} \sum_{r,s=1}^{N-1} \tau^{\beta^{-1}(\alpha s^2 - 2rs + \delta r^2)} |r >< s|$$

labelled by

$$F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, Z_{2N})$$

and act on the displacement operators as follows:

$$V_F D(\sigma) V_F^\dagger = D(F\sigma)$$

and thus take the displacement operators to displacement operators upto a sign.

As a result they take a phase point operators $\widehat{W}(\sigma, \mathrm{s})$ to $\widehat{W}(\sigma, \mathrm{s}')$. We have analysed the orbits of $\widehat{W}(\sigma, \mathrm{s})$ under the conjugate action of $\{V_F\}$ and found that they permit us to classify the Wigner distributions into $1, 3, 4$ families for $N = 2, 4, 8$ consistent with that obtained by spectral considerations.

Wigner distributions on $\mathbb{F}_{p^n}$

Consider a quantum system described by a (complex) Hilbert space of dimension $N = p^n$. We designate $q \in \mathbb{F}_N$ as the 'coordinates' and $\{|q\rangle \mid q \in \mathbb{F}_N\}$ as the 'coordinate basis". ($\mathbb{F}_N$ denotes the finite field of order $N = p^n$)

$$\langle q|q'\rangle \quad = \quad \delta_{q,q'}, \; \sum_{q\in\mathbb{F}_N} |q\rangle\langle q| = \mathbb{I};$$

The 'momentum basis' $\{|p\rangle \mid p \in \mathbb{F}_N\}$ is then obtained by a discrete Fourier transform

$$|p\rangle \quad = \quad \frac{1}{\sqrt{N}} \sum_{q\in\mathbb{F}_N} \omega^{\mathrm{tr}[qp]} |q>; \quad p \in \mathbb{F}_N, \omega = e^{2\pi i/p}$$

$$\langle p|p'\rangle \;=\; \delta_{p,p'}, \; \sum_{p\in\mathbb{F}_N} |p\rangle\langle p| = \mathbb{I};$$

We can think of the phase space, $\Gamma_0 = \mathbb{F}_N \times \mathbb{F}_N$, a discrete $N \times N$ lattice. The analogues of the operations on the phase space for the continuum case now are

1. Translations: $(q,p) \mapsto (q',p') = (q+q_0, p+p_0)$. These transform a line to another line parallel to it
2. Symplectic transformations $SL(2,\mathbb{F}_N)$: These take an isotropic line to another.

$$\begin{pmatrix} q \\ p \end{pmatrix} \;\mapsto\; \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix};$$

$$a,b,c,d \in \mathbb{F}_N; ad - bc = 1$$

The kernel $K_l(q,p';q',p)$ is now given by

$$\begin{aligned} K_l(q,p';q',p) &= N^2\langle q|p'\rangle\langle p'|q'\rangle\langle q'|p\rangle\langle p|q\rangle \\ &= \omega^{\mathrm{tr}[(q-q')(p'-p)]}. \end{aligned}$$

For the phase point operators we then have

$$\sigma \in \Gamma_0 : \ \widehat{W}(\sigma) = \frac{1}{N}\sum_{\sigma'\in\Gamma_0}\omega^{\langle\sigma,\sigma'\rangle}S(\sigma')D(\sigma')$$

and hence

$$P_\lambda = \sum_{\sigma\in\lambda} S(\sigma)D(\sigma)$$

Requiring P_λ to be rank one projectors gives

$$S(\sigma)S(\sigma')\tau^{-\mathrm{tr}[pq]-\mathrm{tr}[p'q']+\mathrm{tr}[(p+p')(q+q')]\omega^{(qp')}} = S(\sigma+\sigma')$$

The structure of the isotropic lines here is rather simple –there are exactly $N+1$ isotropic lines which constitute a single orbit under $SL(2,\mathbb{F}_N)$.
When p is odd, the marginals condition have a simple solution—$S(q,p)=(-1)^{\mathrm{tr}[qp]}$
When p=2, one recovers the results of Wootters et al obtained using a geometric approach.

Conclusions

- We have a developed a unified approach to setting up Wigner distributions which works for $Q = \mathbb{R},\ \mathbb{Z}_N,\ \mathbb{F}_{p^n}$
- In the ring case there is no need to double the number of coordinates for $N = even$.
- For $N = 2^n$ many definitions possible, all consistent with the restricted marginals property
- Number of spectrally distinct Wigner distributions is much smaller than that that suggested by the residual freedom in the choice of signs
- In the finite field case one recovers the results of Wootters et al purely algebraically square root approach.