Lectures on Entanglement

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SCHMIDT DECOMPOSITION

Theorem 1. Given $|\Psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$, there is an orthonormal basis $|\psi_1\rangle, \ldots, |\psi_m\rangle$ of $\mathcal{H}_1$ and an orthonormal basis $|\phi_1\rangle, \ldots, |\phi_n\rangle$ of $H_2\rangle$ such that

$$|\Psi\rangle = \sum_{i=1}^{m} \lambda_i |\phi_i\rangle|\psi_i\rangle \qquad \text{(if } m \leq n\text{)}$$

where $\lambda_1, \ldots, \lambda_m$ are real and non-negative.

ENTROPY OF A PROBABILITY DISTRIBUTION

Suppose a source is emitting messages, i.e. strings of symbols χ_i, where the probability of χ_i is p_i ($i = 1, \ldots, n$). In a message of length N, we expect that χ_i will occur Np_i times. Such a message has probability $p_1^{Np_1} p_2^{Np_2} \ldots p_n^{Np_n}$. Ignoring the rare untypical sequences, this probability must (since it is the same for all messages) be $1/M$ where M is the number of typical messages. Hence the number of bits required to identify such a message is

$$\log_2 M = -N \sum_i p_i \log_2 p_i$$

and the average information per symbol is

$$S(\mathbf{p}) = -\sum_i p_i \log_2 p_i$$

This is the **entropy** of the probability distribution $(p_1, \ldots, p_n)$.

GENERALISED SCHMIDT DECOMPOSITION

Given $|\Psi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, there are bases of $\mathcal{H}_1, \ldots, \mathcal{H}_n$ such that the expansion

$$|\Psi\rangle = \sum c_{i_1 \cdots i_n} |i_1\rangle \cdots |i_n\rangle$$

has the minimum number of terms, with coefficients satisfying:

1. $c_{ji\cdots i} = c_{iji\cdots i} = \cdots = c_{i\cdots ij} = 0$ if $1 \leq i < j \leq d$;
2. $c_{jd\cdots d},\ c_{djd\cdots d}, \ldots, c_{d\cdots dj}$ are real and non-negative;
3. $|c_{i\cdots i}| \geq |c_{j_1 \cdots j_n}|$ if $i \leq j_r,\ r = 1, \ldots, n$.

Three qubits

$$|\Psi\rangle = a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle + f|111\rangle$$

$$a, b, c, d \text{ real}, \qquad a \geq b \geq c \geq d \geq 0.$$

LOCAL INVARIANTS OF THREE QUBITS

$$|\Psi\rangle = \sum_{ijk} c_{ijk}|i\rangle|j\rangle|k\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

$$I_1 = c_{ijk}c^{ijk} = \langle\Psi|\Psi\rangle \qquad (c^{ijk} = c^*_{ijk})$$

$$I_2 = c_{i_1j_1k_1}c_{i_2j_2k_2}c^{i_1j_1k_2}c^{i_2j_2k_1} = \mathrm{tr}(\rho_C^2)$$

$$I_3 = c_{i_1j_1k_1}c_{i_2j_2k_2}c^{i_1j_2k_1}c^{i_2j_1k_2} = \mathrm{tr}(\rho_B^2)$$

$$I_4 = c_{i_1j_1k_1}c_{i_2j_2k_2}c^{i_2j_1k_1}c^{i_1j_2k_2} = \mathrm{tr}(\rho_A^2)$$

$$c_{i_1j_1k_1}c_{i_2j_2k_2}c_{i_3j_3k_3}c^{i_1j_2k_3}c^{i_2j_3k_1}c^{i_3j_1k_2} = \mathrm{tr}\left[(\rho_A \otimes \rho_B)\rho_{AB}\right] - \mathrm{tr}\left(\rho_A^3\right) - \mathrm{tr}\left(\rho_B^3\right)$$

(the Kempe invariant, symmetric in A,B,C)

$$I_6 = \left|\epsilon^{i_1i_2}\epsilon^{i_3i_4}\epsilon^{j_1j_2}\epsilon^{j_3j_4}\epsilon^{k_1k_3}\epsilon^{k_2k_4}c_{i_1j_1k_1}c_{i_2j_2k_3}c_{i_3j_3k_3}c_{i_4j_4k_4}\right|^2$$

$$= |\text{hyperdeterminant of } c_{ijk}|^2 \qquad \text{(the 3-tangle)}$$

FULLY ENTANGLED STATES

An n-party state is **fully entangled** if every m-party reduced state, with $m \leq n/2$, is maximally mixed.

Two qubits The Bell states

$$|\Psi_{\pm}\rangle = |00\rangle \pm |11\rangle, \qquad |\Phi_{\pm}\rangle = |01\rangle \pm |10\rangle$$

are fully entangled. Thus there is a basis of fully entangled states.

Three qubits

The GHZ state $|000\rangle + |111\rangle$ is fully entangled. It is equivalent to the tetrahedral state

$$|\Psi_{++}\rangle = |000\rangle + |011\rangle + |101\rangle + |110\rangle$$

There is a basis of fully entangled states.

BELL BASIS FOR THREE QUBITS

Three qubits

There is a basis of fully entangled states

$$
\begin{aligned}
|\Psi_{++}\rangle &= |000\rangle + |011\rangle + |101\rangle + |011\rangle \\
|\Psi_{+-}\rangle &= |000\rangle + |011\rangle - |101\rangle - |011\rangle \\
|\Psi_{-+}\rangle &= |000\rangle - |011\rangle + |101\rangle - |011\rangle \\
|\Psi_{++}\rangle &= |000\rangle - |011\rangle - |101\rangle + |011\rangle,
\end{aligned}
$$

$$
\begin{aligned}
|\Phi_{++}\rangle &= |111\rangle + |100\rangle + |010\rangle + |111\rangle \\
|\Phi_{+-}\rangle &= |111\rangle + |100\rangle - |010\rangle - |111\rangle \\
|\Phi_{-+}\rangle &= |111\rangle - |100\rangle + |010\rangle - |111\rangle \\
|\Phi_{--}\rangle &= |111\rangle - |100\rangle - |010\rangle + |111\rangle
\end{aligned}
$$

QUADRIPARTITE STATES

Four qubits

There is no fully entangled state of four qubits.
The maximally entangled state is

$$|M_4\rangle = |0011\rangle + |1100\rangle + \omega(|1010\rangle + |0101\rangle) + \omega^2(|1001\rangle + |0110\rangle)$$

where $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$

Four qudits

There is a fully entangled state of four qu*d*its for all d except $d = 2$ and (possibly) $d = 6$.

MANY-QUBIT STATES

Five qubits
There is a fully entangled 5-qubit state (Brown et al.)

$$|000\rangle|\Psi_+\rangle + |011\rangle|\Phi_+\rangle + |101\rangle|\Psi_-\rangle + |110\rangle|\Phi_-\rangle$$

Six qubits
There is a fully entangled 6-qubit state (Borras et al.)

$$\begin{aligned} &|000\rangle|\Psi_{++}\rangle + |011\rangle|\Psi_{+-}\rangle + |101\rangle|\Psi_{-+}\rangle + |110\rangle|\Psi_{--}\rangle \\ +&|111\rangle|\Phi_{++}\rangle + |100\rangle|\Phi_{+-}\rangle + |010\rangle|\Phi_{-+}\rangle + |001\rangle|\Phi_{--}\rangle \end{aligned}$$

Seven qubits

Open Question Is there a fully entangled 7-qubit state?

Eight qubits

There is no fully entangled n-qubit state for $n \geq 8$ (Scott).