No-go Conversion Witness for two Qudit Systems

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Basic aim

- To discuss about the convertibility of two bipartite quantum states.
- Usually it is done by some entanglement monotone.
- However in most of the cases entanglement monotones are not computable and also not fully informative.
- Concept of conversion witness is developed (Gour, et. al., NJP, 2015) that includes entanglement monotones.
- We provide the form of conversion witness for $O \times O$ invariant states that includes other available results of some class of symmetric states.

Entanglement: A novel resource

- This is possibly the most wonderful invention of quantum mechanics. Initially everyone thinks the correlation which is responsible for counterintuitive behavior of quantum systems is nothing but the entanglement.
- However, findings in different quantum systems show there are other candidates also. e.g., the local-indistinguishibility of a complete set of orthonormal product states in 3×3 system.
- Entanglement is used as a resource in many information processing and computational tasks, (e.g, teleportation, dense coding, quantum cryptography, etc.). Therefore, the characterization and quantification problems are of the some fundamental issues generated in the last two decades. However, there are lot of difficulties.

Entanglement

Definition

A bipartite quantum state, represented by density matrix ρ, is called *separable* if and only if it can be represented (or at least can be approximated) as a convex combination of the product of projectors on local states, i.e., if

$$\rho = \sum_{i=1}^{k} p_i \rho_i^1 \otimes \rho_i^2, \sum_i p_i = 1, p_i \geq 0$$

The states, which can not be written in the above forms, are called entangled states.

Entanglement resource theory is an example of quantum resource theories where entangled states work as resource, separable states are free and the involving parties are restricted to act locally and communicate classically (LOCC)

LOCC operations

- Physical operations are represented by completely positive maps and they have the Krause representation of the form,

$$\mathcal{E}(\rho) = \sum A_k \rho A_k^\dagger,$$

  where all A_k are linear operators, satisfy the relation $\sum A_k^\dagger A_k \leq I$.
- On composite quantum sysytems if the operators A_k have the decompostion $L_k^A \otimes L_k^B \otimes L_k^C \ldots$, we call them as separabale superoperators.
- A particular type of separable superoperators are the class of local operations alongwith classical communications (LOCC), where individual subsystems are allowed to do quantum operations locally and they can classically communicate their results to others.

PPT operations

- A larger class of operations that include separable superoperators are the class of PPT (positive under partial transposition) operations.
- Suppose C_Γ is the class of PPT operations. It is the set of completely positive maps $\mathcal{E}$ such that the partially transposed map $\mathcal{E}^\Gamma$ defined by

$$\mathcal{E}^\Gamma(\rho) = [\mathcal{E}(\rho^\Gamma)]^\Gamma,$$

  is also completely positive, where ρ^Γ is the partial transpose of the bipartite state ρ w.r.t. any one of the subsystem.

Entanglement Monotones

LOCC operations induce a partial order on the state space. Thus characterizing this partial order needs quantification of entanglement. Entanglement is basically characterized by entanglement monotones. They quantify the resourcefulness of states. e.g., entropy of entanglement, negativity, entanglement of formation, distillable entanglement, etc. The monotones are important for describing which state transformation is possible or not under LOCC.

Definition

An entanglement monotone is a real valued function f on quantum states that does not increase under LOCC operations. Mathematically,

$$\rho \longrightarrow \sigma \text{ under LOCC}$$

$$\Rightarrow f(\rho) \geq f(\sigma)$$

However most of the monotones are hard to calculate and they do not provide enough information for state transformation.

Entanglement Monotones

A family of monotones f_i is said to be complete if $\rho \longrightarrow \sigma$ under LOCC iff $f_i(\rho) \geq f_i(\sigma)$ for all $i \in \mathcal{I}$, where $\mathcal{I}$ is the index set. e.g., Trivial monotone $\{f_\tau\}_\tau$

$$f_\tau(\rho) = \begin{cases} 1, \rho \to \tau \\ 0, \rho \nrightarrow \tau \end{cases}$$

Although its value cannot be computed straightforwardly!! Hence they are not useful.

Entanglement Monotones

- For bipartite pure states a computable and complete family of monotone exists.
  Ex: Given two pure states $|\psi\rangle$ and $|\phi\rangle$ of dimension d each, LOCC conversion $|\psi\rangle \rightarrow |\phi\rangle$ is determined by the Nielsen's majorization criteria, i.e., if
  $$\sum_{i=1}^{k} \lambda_i(\psi) \leq \sum_{i=1}^{k} \lambda_i(\phi)$$
  for all $k = 1, 2, ..., d$.
  $\lambda_i' s$ are the schmidt coefficients of the states in decreasing order. A complete family of monotones thus can be defined straightforwardly as
  $$f_l(\tau) = \sum_{i=l}^{d} \lambda_i(\tau)$$
  for each $l = 2, .., d$..
- A complete family of finitely many monotones cannot exist for arbitrary mixed states.

Entanglement Monotones

- Concurrence is a computable monotone for two-qubit pure states but fails to detect complete convertibility in two-qubit mixed states.
- Entanglement of formation is hardly calculable in higher dimensional mixed states.
- Negativity is the only known computable monotone for arbitrary states.

Since, LOCC$\subset$ PPT, PPT monotones are also LOCC monotone and PPT monotones have good mathematical structure.

Conversion Witness

Definition

Let W be a real valued function on a pair of bipartite quantum states. If $W(\rho,\sigma) \geq 0 \Rightarrow \rho \longrightarrow \sigma$ under LOCC then W is said to be a go witness. If $W(\rho,\sigma) < 0 \Rightarrow \rho \nrightarrow \sigma$, then W is said to be a no-go witness. W is said to be a complete witness if it is both a go and no-go witness.

The main goal of no-go witness is to determine whether a pair (ρ,σ) is non-convertible or not. Let us consider the pair of states which are LOCC non-convertible

$$N = \{(\rho,\sigma) : \rho \nrightarrow \sigma\}$$

Also, consider the set where the convertibility is detected by a no-go witness W, i.e.,

$$D_W = \{(\rho,\sigma) : W(\rho,\sigma) < 0\}$$

Clearly,

$$D_W \subseteq N$$

Conversion Witness

Now consider two no-go witnesses W_1 and W_2. Their sets of detected pairs must be subsets of N, i.e., $D_{W_1} \subseteq N$ and $D_{W_2} \subseteq N$. Suppose that $D_{W_1} \subseteq D_{W_2}$. Then all of the pairs detected by W_2 are already detected by W_1, but W_1 might detect more pairs than W_2 does. In this case, we might as well use only W_1, since W_1 tells us all of the information given by W_2, and also more! So we write $W_1 \succeq W_2$, and this is indeed a partial order.

Partial Order

Explicitly, given two no-go witness W_1, W_2, we say

$$W_1 \succeq W_2 \,\text{if}\, W_2(\rho,\sigma) < 0 \Rightarrow W_1(\rho,\sigma) < 0.$$

The relation is a partial order relation.

Conversion Witness

Few Notes:

1. Any complete witness is stronger than any go witness or no-go witness
2. Given two witnesses, they may be incomparable
3. Given a monotone f, a no-go witness can be defined as $W_f(\rho, \sigma) = f(\rho) - f(\sigma)$. Hence entanglement monotones are special types of conversion witness.
4. Similar partial order can be defined for go witness as well.

Conversion Witness

An operator inequality

For positive semi-definite operators $A, B \in H_{n,+}$ we have $(A - B)_+ \leq A$ and $(A - B)_- \leq B$.

we will use use the result in the subsequent construction method.

Computable no-go Conversion Witness: Construction

Let C_Γ denotes the set of all PPT operations, i.e., if $\mathcal{E} \in \mathcal{C}_\Gamma$ then $\mathcal{E}$ is CPTP map such that its partial transposed map is defined by $\mathcal{E}^\Gamma(\rho) := \left[\mathcal{E}(\rho^\Gamma)\right]^\Gamma$. Negativity ($:=Tr[\rho^{\Gamma_-}]$) is a PPT monotone, i.e.,

$$
\begin{aligned}
&\rho \longrightarrow \sigma \quad \text{under PPT} \\
\Rightarrow (\mathcal{E}(\rho))^{\Gamma_-} &= [\mathcal{E}^\Gamma(\rho^\Gamma)]_- \\
&= [\mathcal{E}^\Gamma(\rho^{\Gamma_+} - \rho^{\Gamma_-})]_- \\
&= [\mathcal{E}^\Gamma(\rho^{\Gamma_+}) - \mathcal{E}^\Gamma(\rho^{\Gamma_-})]_- \\
&\leq \mathcal{E}^\Gamma(\rho^{\Gamma_-}) \\
&\Rightarrow N(\rho) \geq N(\sigma) \, \text{(taking trace on both sides)}
\end{aligned}
$$

We have used the symbol $\rho^{\Gamma_\pm} = (\rho^\Gamma)_\pm$.

Computable no-go Conversion Witness: Construction

Support function of a subset C is defined as

$$h_C(\rho) := \sup_{\gamma \in C} Tr[\gamma\rho]$$

Consider the sets

$$\mathcal{N}_c = \{\gamma : N(\gamma) \leq c\} \text{ with } \; c \geq 0$$

$$\mathcal{C}(\rho) = \{\mathcal{E}(\rho) : \mathcal{E} \; \text{ is CPTP map}\}$$

We have for PPT operation

$$C_\Gamma(\rho) \subset \mathcal{N}_{N(\rho)}$$

and

$$h_{\mathcal{C}_\Gamma}(\rho) \leq h_{\mathcal{N}_c}(\rho)$$

Computable no-go Conversion Witness: Construction

Trace, generally, reduce the information. Instead, we proceed as follows

$$(\mathcal{E}(\rho))^{\Gamma_-} \leq \mathcal{E}^{\Gamma}(\rho^{\Gamma_-})$$
$$\Leftrightarrow Tr[\tau(\mathcal{E}(\rho))^{\Gamma_-}] \leq Tr[\tau\mathcal{E}^{\Gamma}(\rho^{\Gamma_-})]$$

where τ is any quantum state.
Since, $\mathcal{E}^{\Gamma}$ is also a CPTP map,

$$Tr[\tau\mathcal{E}^{\Gamma}(\rho^{\Gamma_-})] \leq h_{C_{\Gamma}(\rho^{\Gamma_-})}(\tau)$$

No-go conversion witness for PPT operation is defined as

$$\widehat{W}_{\tau}(\rho,\sigma) := h_{C_{\Gamma}(\rho^{\Gamma_-})}(\tau) - Tr[\tau\sigma^{\Gamma_-}]$$

Computable no-go Conversion Witness: construction

Using normalization $\tilde{\rho} = \frac{\rho^{\Gamma_-}}{N(\rho)}$ for non-PPT states, the witness reads

$$\widehat{W}_\tau(\rho, \sigma) := N(\rho) h_{C_\Gamma(\tilde{\rho})}(\tau) - Tr[\tau \sigma^{\Gamma_-}]$$

whenever $\tau = \frac{1}{n} I$,

$$\widehat{W}_{\frac{I}{n}}(\rho, \sigma) = \frac{1}{n}(N(\rho) - N(\sigma)) = \frac{1}{n} W_N(\rho, \sigma)$$

.

If we define,

$$\widehat{W}(\rho, \sigma) := n \min_\tau \widehat{W}_\tau(\rho, \sigma)$$

then we will at once see

$$\widehat{W} \succeq \widehat{W}_\tau$$

Computable no-go Conversion Witness: construction

Replacing $C_\Gamma(\tilde{\rho})$ by $\mathcal{N}_{N(\tilde{\rho})}$ we still have

$$Tr[\tau\sigma^{\Gamma_-}] \leq N(\rho) h_{\mathcal{N}_{N(\tilde{\rho})}}(\tau)$$

Hence we have the computable PPT conversion witness

$$W_\tau(\rho, \sigma) := N(\rho) h_{\mathcal{N}_{N(\tilde{\rho})}}(\tau) - Tr[\tau\sigma^{\Gamma_-}]$$

Clearly, $\widehat{W}_\tau(\rho, \sigma) \leq W_\tau(\rho, \sigma)$, i.e., the new witness is weaker but we will see it is better computable. We define stronger(than W_τ) witness

$$W(\rho, \sigma) := n \min_\tau W_\tau(\rho, \sigma)$$

Whenever $\tau = \frac{I}{n}$, we get $W_\tau(\rho, \sigma) = \frac{1}{n}(N(\rho) - N(\sigma)) = W_N(\rho, \sigma)$. Thus hierarchy $\widehat{W} \succeq W \succeq W_N$ holds. However the support function is hard to evaluate for arbitrary τ. We will show that whenever τ is orthogonally invariant state it is enough to optimize over orthogonally invariant states to evaluate the support function.

Orthogonally Invariant Class of states

Any $\mathcal{O} \otimes \mathcal{O}$ invariant state from a $n \otimes n$ system can be taken as

$$\rho_{oo} = a\,\mathbb{I}_n + b\,\mathbb{F} + c\,\hat{\mathbb{F}}$$

with $n(na + b + c) = 1$ (trace condition) and proper positivity constraints. $\mathbb{I}$ is the identity operator, $\mathbb{F} = d|\Phi\rangle\langle\Phi|^{\Gamma}$ is the flip operator and $\hat{\mathbb{F}}$ is the projection on maximally entangled state.

The operators satisfy the algebra,

$$\mathbb{F}^2 = \mathbb{I}$$
$$\mathbb{F}\hat{\mathbb{F}} = \hat{\mathbb{F}}\mathbb{F} = \hat{\mathbb{F}}$$
$$\hat{\mathbb{F}}^2 = n\,\hat{\mathbb{F}}$$

n is the dimension of each subsystem. The state can also be written in terms of orthogonal projectors

$$\rho_{oo} = \frac{\hat{f}}{d}U + \frac{1-\hat{f}}{d(d-1)}V + \frac{1-\hat{f}}{d(d+2)(d-1)}W$$

with $U = \frac{\hat{\mathbb{F}}}{d}$, $V = \frac{I-\mathbb{F}}{2}$, $W = \frac{I+\mathbb{F}}{2} - \frac{\hat{\mathbb{F}}}{d}$

Support function for $O \otimes O$ states

In terms of the parameters $f = \mathrm{tr}[\mathbb{F}\rho_{oo}]$ and $\hat{f} = \mathrm{tr}[\hat{\mathbb{F}}\rho_{oo}]$, negativity of the above state can be written as

$$N(\rho_{oo}) = \frac{1}{2}\left(\left[\frac{|f|}{d} + \frac{|1-\hat{f}|}{2} + \frac{d + d\hat{f} - 2f}{2d}\right] - 1\right)$$

Consider the regions

$$R_0 = \{(f, \hat{f}) : f \geq 0, \hat{f} \leq 1\} \cap \mathcal{O}$$

$$R_1 = \{(f, \hat{f}) : f \geq 0, \hat{f} \geq 1\} \cap \mathcal{O}$$

$$R_2 = \{(f, \hat{f}) : f \leq 0, \hat{f} \geq 1\} \cap \mathcal{O}$$

$$R_3 = \{(f, \hat{f}) : f \leq 0, \hat{f} \leq 1\} \cap \mathcal{O}$$

Support function for $O \otimes O$ states

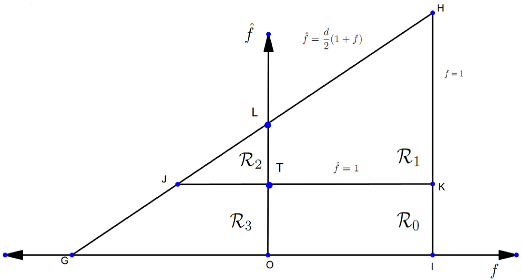

Figure: The set of all orthogonally invariant states are represented through the figure in the parametric space of $f, \hat{f}$. The triangular region HGI denotes the set of feasible states. The whole region is divided into four sub-regions depending on the value of negativity.

Support function for $O \otimes O$ states

Negative part of ρ_{oo}^{T} is given by

$$(\rho_{oo}^{f,\hat{f}})^{\Gamma_-} = \begin{cases} \frac{\hat{f}-1}{d(d-1)}V, & (f,\hat{f}) \in \mathcal{R}_1 \ , \\ -\frac{f}{d}U - \frac{1-\hat{f}}{d(d-1)}V, & (f,\hat{f}) \in \mathcal{R}_2 \ , \\ -\frac{f}{d}U, & (f,\hat{f}) \in \mathcal{R}_3 \ . \end{cases}$$

Maximum value of negativity over all orthogonally invariant states can be obtained as,

$$\max_{\rho \in O\otimes O} N(\rho) = \begin{cases} 0, & \rho \in \mathcal{R}_0, \\ \frac{d-1}{2}, & \rho \in \mathcal{R}_1 \ , \\ \frac{d-2}{4}, & \rho \in \mathcal{R}_2 \ , \\ \frac{1}{d}, & \rho \in \mathcal{R}_3 \ . \end{cases}$$

Support function for $O \otimes O$ states

To calculate the support function we fix $c_1 \geq 0$. Hence, for any orthogonally invariant state $\rho_{oo}^{s,\hat{s}}$,

$$\begin{aligned} h_{\mathcal{N}_{c_1}}(\rho_{oo}^{s,\hat{s}}) &= \sup_{\gamma \in O \otimes O} \mathrm{Tr}[\gamma \rho_{oo}^{s,\hat{s}}], \\ &= \mathrm{Tr}[\gamma^* \rho_{oo}^{s,\hat{s}}], \\ &= \mathrm{Tr}[\mathcal{T}_{o \otimes o}(\gamma^*) \rho_{oo}^{s,\hat{s}}], \\ &= \mathrm{Tr}[\gamma^* \mathcal{T}_{o \otimes o}(\rho_{oo}^{s,\hat{s}})], \\ &= \max_{\substack{\rho_{oo}^{f,\hat{f}} \in O \otimes O, \\ \mathcal{N}(\rho_{oo}^{f,\hat{f}}) \leq c_1}} \mathrm{Tr}[\rho_{oo}^{f,\hat{f}} \rho_{oo}^{s,\hat{s}}]. \end{aligned}$$

Support function for $O \otimes O$ states

- The third line follows from the invariance of orthogonal states under twirling operation.
- Fourth line follow from the self-adjoint property of the twirling operation.
- Finally, the state $\mathcal{T}_{o\otimes o}(\rho_{oo}^{s,\hat{s}})$ is the same orthogonal invariant state and γ^* is any orthogonally invariant state and it can be taken as $\rho_{oo}^{f,\hat{f}}$.
- Negativity of the state $\rho_{oo}^{f,\hat{f}}$ can not be greater than initial negativity c_1 because orthogonal operations cannot increase negativity. Hence, for orthogonally invariant states, it is sufficient to perform the optimization over all orthogonally invariant state with negativity less than or equal to c_1.

Support function for $O \otimes O$ states

The support function from (24) can be simplified further in our case as,

$$
\begin{aligned}
h_{\mathcal{N}_{c_1}}(\rho_{oo}^{s,\hat{s}}) &= \max_{\substack{\rho_{oo}^{f,\hat{f}} \in O\otimes O, \\ \mathcal{N}(\rho_{oo}^{f,\hat{f}}) \leq c_1}} \mathrm{Tr}[\rho_{oo}^{f,\hat{f}} \rho_{oo}^{s,\hat{s}}], \\
&= \max_{\substack{\rho_{oo}^{f,\hat{f}} \in O\otimes O, \\ \mathcal{N}(\rho_{oo}^{f,\hat{f}}) \leq c_1}} (Af + B\hat{f} + E).
\end{aligned}
$$

Clearly, the optimizing function is a linear one: $Af + B\hat{f} + E$ where A, B, E are function of $s, \hat{s}$. With $d \geq 4$, this linear optimization over f, $\hat{f}$ can be performed depending upon the sign of A and B and the range of c_1. We will use the no-go PPT conversion witness for normalized states. Hence, from now on, it is necessary to take for non-PPT states,

$$
c_1 = \min\{\frac{d-1}{2}, \frac{\mathrm{Tr}(\rho^{\Gamma_-\Gamma_-})}{\mathrm{Tr}(\rho^{\Gamma_-})}\}
$$

No-go Conversion witness $d \geq 4$

Corresponding to an orthogonally invariant state τ we have the following conversion witness,

Table: Conversion witness at the extreme points of the orthogonally invariant set of states for different values of c_1.

$(s, \hat{s})$	$c_1 \geq \frac{d-1}{2}$	$\frac{d-2}{4} \leq c_1 \leq \frac{d-1}{2}$	$\frac{1}{d} \leq c_1 \leq \frac{d-2}{4}$	
Extreme points	$h_{\mathcal{N}_{c_1}}(\rho_{oo}^{s,\hat{s}})$	$h_{\mathcal{N}_{c_1}}(\rho_{oo}^{s,\hat{s}})$	$h_{\mathcal{N}_{c_1}}(\rho_{oo}^{s,\hat{s}})$	$\mathrm{Tr}[\rho_{oo}^{s,\hat{s}} \sigma^{\Gamma}-]$
A1 $(1, 0)$	$\frac{2}{(d-1)(d+2)}$	$\frac{2}{(d-1)(d+2)}$	$\frac{2}{(d-1)(d+2)}$	$\frac{2}{(d-1)(d+2)} \mathrm{Tr}[W \sigma^{\Gamma}-]$
A2 $(1, \frac{2}{1+d})$	$\frac{2}{d(d+1)}$	$\frac{2}{d(d+1)}$	$\frac{2}{d(d+1)}$	$\frac{2}{d(d+1)} \mathrm{Tr}[(U+W)\sigma^{\Gamma}-]$
A3 $(1, d)$	1	$\frac{2c_1+1}{d}$	$\frac{d(2c_1+1)-2}{d^2-4}$	$\mathrm{Tr}[U \sigma^{\Gamma}-]$
A4 $(-1, 0)$	$\frac{2}{d(d-1)}$	$\frac{2}{d(d-1)}$	$\frac{2}{d(d-1)}$	$\frac{2}{d(d-1)} \mathrm{Tr}[V \sigma^{\Gamma}-]$
A5 $(\frac{1}{d}, \frac{1}{d})$,	$\frac{1}{d^2}$	$\frac{1}{d^2}$	$\frac{1}{d^2}$	$\frac{1}{d^2} N(\sigma)$
A6 $(\frac{1}{1+d}, 0)$	$\frac{1}{d^2-1}$	$\frac{1}{d^2-1}$	$\frac{1}{d^2-1}$	$\frac{1}{d^2-1} \mathrm{Tr}[(V+W)\sigma^{\Gamma}-]$

The witness at $A5$ is nothing but the negativity witness. Hence, our witness is also a betterment of Negativity.

No-go Conversion witness $d \geq 4$

Table: Conversion witness at the extreme points of the orthogonally invariant set of states for $c_1 \leq \frac{1}{d}$.

$(s, \hat{s})$	$c_1 \leq \frac{1}{d}$	
Extreme points	$h_{\mathcal{N}_{c_1}}(\rho_{oo}^{s,\hat{s}})$	$\mathrm{Tr}[\rho_{oo}^{s,\hat{s}}\sigma^{\Gamma}-]$
A1	$\frac{2}{(d-1)(d+2)}$	$\frac{2}{(d-1)(d+2)}\mathrm{Tr}[W\sigma^{\Gamma}-]$
A2	$\frac{2}{d(d+1)}$	$\frac{2}{d(d+1)}\mathrm{Tr}[(U+W)\sigma^{\Gamma}-]$
A3	$\frac{d(1-c_1 d)}{2}$ when $0 \leq c_1 \leq \frac{d-2}{d^2-4}$;	$\mathrm{Tr}[U\sigma^{\Gamma}-]$
	$\frac{2c_1+1}{d}$ when $\frac{d-2}{d^2-4} \leq c_1 \leq \frac{1}{d}$	
A4	$\frac{c_1 d+1}{d(d-1)}$ when $\frac{d-2}{d^2} \leq c_1 \leq \frac{1}{d}$;	$\frac{2}{d(d-1)}\mathrm{Tr}[V\sigma^{\Gamma}-]$
	$\frac{-2d(2c_1+1)+2d^2-4}{d(d-1)(d^2-4)}$ when $0 \leq c_1 \leq \frac{d-2}{d^2}$	
A5	$\frac{1}{d^2}$	$\frac{1}{d^2}N(\sigma)$
A6	$\frac{1}{d^2-1}$	$\frac{1}{d^2-1}\mathrm{Tr}[(V+W)\sigma^{\Gamma}-]$

We can now perform optimization over all orthogonally invariant states to get,

$$W_{oo}(\rho,\sigma) = d^2 \min_{\tau \in o \otimes o} W_{\tau}(\rho,\sigma). \tag{1}$$

However the optimization reduces to finding the minimum at the above extreme points.

No-go Conversion witness $d = 3$

Table: Conversion witnesses corresponding to $\frac{1}{4} \leq c_1 \leq \frac{1}{3}$ and $c_1 \leq \frac{1}{4}$ at the extreme points of 3×3 orthogonally invariant state space

$(s, \hat{s})$ Extreme points	$\frac{1}{4} \leq c_1 \leq \frac{1}{3}$ $W_\tau(\rho, \sigma)$	$c_1 \leq \frac{1}{4}$ $W_\tau(\rho, \sigma)$
A1 $(1, 0)$	$\frac{1}{5}(N(\rho) - \mathrm{Tr}[W\sigma^{\Gamma}-])$	$\frac{1}{5}(N(\rho) - \mathrm{Tr}[W\sigma^{\Gamma}-])$
A2 $(1, \frac{1}{2})$	$\frac{1}{5}(N(\rho) - \mathrm{Tr}[(U+W)\sigma^{\Gamma}-])$	$\frac{1}{6}(N(\rho) - \mathrm{Tr}[(U+W)\sigma^{\Gamma}-])$
A3 $(1, 3)$	$\frac{2c_1+1}{3}N(\rho) - \mathrm{Tr}[U\sigma^{\Gamma}-]$	$\max\{\frac{2c_1+1}{3}, \frac{1-3c_1}{2}\}N(\rho) - \mathrm{Tr}[U\sigma^{\Gamma}-]$
A4 $(-1, 0)$	$\frac{1}{3}(N(\rho) - \mathrm{Tr}[V\sigma^{\Gamma}-])$	$\frac{1}{3}N(\rho) - \frac{1}{3}\mathrm{Tr}[V\sigma^{\Gamma}-]$
A5 $(\frac{1}{3}, \frac{1}{3})$	$\frac{1}{9}(N(\rho) - \mathrm{Tr}[(U+V+W)\sigma^{\Gamma}-])$	$\frac{1}{9}(N(\rho) - N(\sigma))$
A6 $(\frac{1}{4}, 0)$	$\frac{1}{8}(N(\rho) - \mathrm{Tr}[(V+W)\sigma^{\Gamma}-])$	$\frac{1}{8}(N(\rho) - \mathrm{Tr}[(V+W)\sigma^{\Gamma}-])$

Other cases can be obtained from the general case by just substituting $d = 3$.

No-go Conversion witness > 0: Example 1

Example 1: Let us consider a Werner state

$$\rho_w = \frac{d\alpha - 1}{d(d^2-1)}\mathbb{F} + \frac{d-\alpha}{d(d^2-1)}\mathbb{I}, -1 \leq \alpha \leq 1$$

and another Isotropic state

$$\rho_{iso} = \beta d\hat{\mathbb{F}} + \frac{1-\beta}{d^2-1}(\mathbb{I} - \hat{\mathbb{F}}/d), 0 \leq \beta \leq 1$$

The states are non-PPT in the region $-1 \leq \alpha \leq 0$ and $\frac{1}{d} \leq \beta \leq 1$ respectively. Hence, whenever $\beta = \frac{d-2\alpha}{d^2}$, both the states have same negativity $N(\rho_w) = N(\rho_{iso}) = -\frac{\alpha}{d}$. Thus, it is impossible to judge by negativity alone whether one state can be converted to other by using PPT operations.

No-go Conversion witness > 0: Example 1

For the conversion $\rho_W \longrightarrow \rho_{iso}$ under PPT, we note that $(\rho_w^{\Gamma_-})^{\Gamma_-} = \frac{\alpha}{d^2}\mathbb{F}_-$, where $\mathbb{F}_-$ is the negative part of the operator $\mathbb{F}$. Hence, $c_1 = \frac{1}{d}$. The extreme point A4 from Table 2 gives

$$W_{A4}(\rho_w, \rho_{iso}) = \frac{2}{d(d-1)} N(\rho_w) - \mathrm{Tr}(V\rho_{iso}^{\Gamma_-})$$

$$= \frac{2}{d(d-1)} N(\rho_w) - \frac{d\beta - 1}{d(d-1)} \mathrm{Tr}(V\mathbb{F}_-)$$

$$= -\frac{\alpha}{d^2(d-1)} \geq 0$$

Hence, the conversion witness cannot detect the above conversion too.

No-go Conversion witness=0: Example 2

Example 2: Next, we consider transformation between two orthogonally invariant states with same amount of negativity. For this purpose, let us consider two states $\rho_{oo}^{f_1,\hat{f}}$ and $\rho_{oo}^{f_2,\hat{f}} \in \mathcal{R}_1$. Both the states have negativity $N(\rho_{oo}^{f_1,\hat{f}}) = N(\rho_{oo}^{f_2,\hat{f}}) = \frac{\hat{f}-1}{2}$. We note the following relations,

$$\begin{aligned} U^{\Gamma} &= \frac{1}{d}(U - V + W) \\ W^{\Gamma} &= \frac{1+d}{2} - \frac{1}{d}U + \frac{d+2}{2d}V + \frac{d-2}{2d}W \\ V^{\Gamma} &= \frac{1-d}{2}U + \frac{1}{2}V + \frac{1}{2}W \end{aligned}$$

Using these relations it is straightforward to obtain $c_1 = \frac{1}{d}$. We can now easily obtain conversion witness corresponding to the extreme point A4 and it gives

$$W_{A4}(\rho_{oo}^{f_1,\hat{f}}, \rho_{oo}^{f_2,\hat{f}}) = \frac{\hat{f}-1}{d(d-1)} - (V(\rho_{oo}^{f,\hat{f}_2})^{\Gamma_-}) = 0$$

Hence the the conversion $\rho_{oo}^{f_1,\hat{f}} \longrightarrow \rho_{oo}^{f_2,\hat{f}}$ can not be determined in this case too. Similar conclusion follows if we assume the initial states belong to the region $\mathcal{R}_3$.

No-go Conversion witness < 0: Example 3

Example 3: Now we consider the conversion between the pure state $\rho = |x\rangle\langle x|$ and orthogonally invariant state $\sigma = \rho_{oo}^{f,\hat{f}}$. The pure state can be written as $\sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle$. It can be shown that the witness corresponding to negativity become,

$$W_N(\rho, \sigma) = \begin{cases} \frac{1}{d}(d\sqrt{\lambda_0\lambda_1} + f), & \sigma \in \mathcal{R}_1, \\ \sqrt{\lambda_0\lambda_1} - \frac{\hat{f}-1}{2}, & \sigma \in \mathcal{R}_2 \ , \\ \sqrt{\lambda_0\lambda_1} + \frac{d+2f-d\hat{f}}{4}, & \sigma \in \mathcal{R}_3 \ . \end{cases}$$

Conversion witness corresponding to the transformation gives

$$W(\rho, \sigma) = \frac{2N(\rho)}{d} - \mathrm{Tr}(U\sigma^{\Gamma_-})$$

It is to be noted that $c_1 = 0.5$.

No-go Conversion witness < 0: Example 3

- Whenever $\sigma \in \mathcal{R}_1$, we get $W(\rho, \sigma) > 0$. Hence, the conversion cannot be determined from the conversion witness.
- However, if $\sigma \in \mathcal{R}_2$, there is a region inside $\mathcal{R}_2$ determined by the relation $f + \hat{f} - 1 < 0$ where $W_N(\rho, \sigma) > 0$, $W(\rho, \sigma) < 0$ and the range of pure state parameter needs to satisfy the relation $\frac{\hat{f}-1}{2} < \sqrt{\lambda_0 \lambda_1} < -\frac{f}{2}$. Hence, the conversion witness proves non-convertibility between such pure states to those orthogonally invariant states.
- Similar conclusion holds whenever $\sigma \in \mathcal{R}_3$. In this case the pure state which satisfies $\frac{\hat{f}-1}{2} < \sqrt{\lambda_0 \lambda_1} < -\frac{f}{2}$ can not be converted to any $\sigma \in \mathcal{R}_3$.

Some other Computable no-go Conversion Witness

Instead of orthogonally invariant state the optimization can be performed over Werner or isotropic class of states

- The support function is evaluated for Werner and Isotropic class of states

$$W_{wer}(\rho,\sigma) = \min\{W_N(\rho,\sigma), \frac{2}{d(d-1)} W'_{wer}(\rho,\sigma)\}$$

$$W_{iso}(\rho,\sigma) = \min\{W_N(\rho,\sigma), W'_{iso}(\rho,\sigma)\}$$

where

$$W'_{wer}(\rho,\sigma) = \frac{dN(\tilde{\rho})+1}{2} N(\rho) - Tr[F_{-}\sigma^{\Gamma_{-}}]$$

$$W'_{iso}(\rho,\sigma) = \frac{2N(\tilde{\rho})+1}{2} N(\rho) - \langle\Phi|\sigma^{\Gamma_{-}}|\Phi\rangle$$

the sub-witness W'_{wer}, W'_{iso} and W_N are incomparable, however $W_{iso}, W_{wer} \succeq W_N$. The new witness $W_\Gamma = \min\{W_N, W'_{wer}, W'_{iso}\}$ is an computable witness better than negativity.

Some other Computable no-go Conversion Witness

- W_{wer} is improvement to negativity if $N(\tilde{\rho}) < 1/d$ and W_{iso} is an improvement if $N(\tilde{\rho}) < d - 1/2$.
- Any pure entangled state with negativity less than 1/3 cannot be converted to an entangled Werner state in same dimension and same negativity by PPT operation. Take pure state

$$\sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle$$

with $\lambda_0 = \frac{d+\sqrt{d^2-4}}{2d}$, $\lambda_1 = \frac{d-\sqrt{d^2-4}}{2d}$. Then, $N(|x\rangle\langle x|) = 1/d$. Take another Werner state σ with $\alpha = -1$. $N(\sigma) = 1$. However

$$W'_{iso} = 2/d^2 - 1/d < 0.$$

Hence the said pure state can not be converted to the said Werner state by PPT operations.

Further no-go PPT conversion witness

The set $C_\Gamma(\tilde{\rho})$ is hard to characterize hence we have considered the larger set $\mathcal{N}_{\mathcal{N}(\tilde{\rho})}$. So, can we obtain any other set in between them to construct the orbit?

Consider the set of Hermitian operators

$$\mathcal{N}_1(\rho) = \{\gamma \in H_{n,+} | Tr(\gamma) = N(\rho), N(\gamma) \leq N(\rho^{\Gamma_-})\}$$

Let,

$$\gamma \in C_\Gamma(\tilde{\rho}), \Rightarrow \gamma = \mathcal{E}(\tilde{\rho})$$

Thus

$$N(\gamma) = N(\mathcal{E}(\tilde{\rho})) \leq N(\tilde{\rho}) = N(\rho^{\Gamma_-})$$

Also,

$$Tr(\gamma) = Tr(\mathcal{E}(\tilde{\rho})) = Tr(\rho^{\Gamma_-}) = N(\rho)$$

Thus we have $C_\Gamma(\tilde{\rho}) \subseteq \mathcal{N}_1(\rho)$. On the other hand the definition of $\mathcal{N}_1(\rho)$ justifies that $\mathcal{N}_1(\rho) \subseteq \mathcal{N}_{\mathcal{N}(\tilde{\rho})}$ Hence,

$$C_\Gamma(\tilde{\rho}) \subseteq \mathcal{N}_1(\rho) \subseteq \mathcal{N}_{\mathcal{N}(\tilde{\rho})}$$

Further no-go PPT conversion witness

The above set can be refined more. Particularly, the relation $N(\gamma) \leq N(\rho^{\Gamma_-})$ is modified by stronger relation and we can define a new set

$$\mathcal{N}_2(\rho) = \left\{ \gamma \in H_{n,+} \middle| {}^{Tr(\gamma)=N(\rho),}_{\substack{\exists \eta \in H_{n,+} \text{ with } Tr(\eta)=N(\rho^{\Gamma_-}), \\ N(\eta) \leq N(\rho^{\Gamma_- \Gamma_-}), \gamma^{\Gamma_-} \leq \eta}} \right\}$$

It can be shown,

$$C_\Gamma(\tilde{\rho}) \subseteq \mathcal{N}_2(\rho) \subseteq \mathcal{N}_1(\rho) \subseteq \mathcal{N}_{\mathcal{N}(\tilde{\rho})}$$

This process can be continued to get the orbit $\mathcal{N}_k(\rho)$ with

$$C_\Gamma(\tilde{\rho}) \subseteq \ldots \subseteq \mathcal{N}_2(\rho) \subseteq \mathcal{N}_1(\rho) \subseteq \mathcal{N}_{\mathcal{N}(\tilde{\rho})}$$

Conclusion

- Conversion witness is a generalization of monotones and can determine convertibility when monotones fail. However they are not perfect.
- We have obtained the explicit form of conversion witness for some symmetric class of states and they provide betterment over negativity. However our witness is not more efficient than the witness as provided by Gour et.al. although incomparable.
- We have shown two states, having same negativity, but can not be converted to other using conversion witness.
- Conversion witness can be studied in other resource theories (such as coherence+LICC restriction), possibly beyond quantum theory too.

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Thank You