Stronger Error-Disturbance Relations

Namrata Shukla

Post Doctoral Fellow
Harish-chandra Research Institute Allahabad

arXiv:1503.05085

February 18, 2016

Stronger E-D Relations

Namrata Shukla

Uncertainty Relations

Error-Disturbance Relations

New Error-Disturbance Relations

Comparison with Examples

Conclusions

Overview

Uncertainty

- Uncertainty principle encapsulates the impossibility of simultaneous measurement of two incompatible physical observables.

Uncertainty

- Uncertainty principle encapsulates the impossibility of simultaneous measurement of two incompatible physical observables.

- The Robertson version of the uncertainty relation

  $$\Delta A^2 \Delta B^2 \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle|^2,$$

  where $\Delta A^2 = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$.

Uncertainty

- Uncertainty principle encapsulates the impossibility of simultaneous measurement of two incompatible physical observables.

- The Robertson version of the uncertainty relation

  $\Delta A^2 \Delta B^2 \geq \frac{1}{2}|\langle\psi|[A, B]|\psi\rangle|^2,$

  where $\Delta A^2 = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$.

- These uncertainty relations may happen to be trivial even if the observables are incompatible on the state of system. Stronger uncertainty relations by Maccone and Pati capture the concept of incompatible observables.

- These uncertainty relations differ from the original uncertainty principle in a crucial way as these are relations concerning intrinsic quantum fluctuations for observables of a quantum system.

- These uncertainty relations differ from the original uncertainty principle in a crucial way as these are relations concerning intrinsic quantum fluctuations for observables of a quantum system.

- In a realistic measurement scenario, we must couple the system to a probe through an interaction and read the result from the measuring apparatus.

- These uncertainty relations differ from the original uncertainty principle in a crucial way as these are relations concerning intrinsic quantum fluctuations for observables of a quantum system.

- In a realistic measurement scenario, we must couple the system to a probe through an interaction and read the result from the measuring apparatus.

- Arthurs and Kelly derived an expression akin to the Robertson uncertainty relation for error ϵ_A in measurement of observable A and corresponding disturbance η_B on observable B

$$\epsilon_A \eta_B \geq \frac{1}{2}|\langle\psi|[A, B]|\psi\rangle|$$

Measurement Process

- System and apparatus are initially non-entangled and in states $|\psi\rangle_s$ and $|\phi\rangle_p$.

Measurement Process

- System and apparatus are initially non-entangled and in states $|\psi\rangle_s$ and $|\phi\rangle_p$.
- The Physical observables of the system to be measured $A_{in} = A \otimes \mathbb{I},\ B_{in} = B \otimes \mathbb{I}$.

Measurement Process

- System and apparatus are initially non-entangled and in states $|\psi\rangle_s$ and $|\phi\rangle_p$.
- The Physical observables of the system to be measured $A_{in} = A \otimes \mathbb{I}, \ B_{in} = B \otimes \mathbb{I}$.
- Fix an operator M (after measurement) to read off and estimate of the value of A.
  $M_{in} = \mathbb{I} \otimes M$

Measurement Process

- System and apparatus are initially non-entangled and in states $|\psi\rangle_s$ and $|\phi\rangle_p$.
- The Physical observables of the system to be measured $A_{in} = A \otimes \mathbb{I}, \ B_{in} = B \otimes \mathbb{I}$.
- Fix an operator M (after measurement) to read off and estimate of the value of A.
  $M_{in} = \mathbb{I} \otimes M$
- $A_{out} = U^{\dagger} \left(A \otimes \mathbb{I}\right) U, \ B_{out} = U^{\dagger} \left(B \otimes \mathbb{I}\right) U,$
  $M_{out} = U^{\dagger} \left(\mathbb{I} \otimes M\right) U$

Measurement Process

- System and apparatus are initially non-entangled and in states $|\psi\rangle_s$ and $|\phi\rangle_p$.
- The Physical observables of the system to be measured $A_{in} = A \otimes \mathbb{I}, \;\; B_{in} = B \otimes \mathbb{I}$.
- Fix an operator M (after measurement) to read off and estimate of the value of A.
  $M_{in} = \mathbb{I} \otimes M$
- $A_{out} = U^{\dagger} \left(A \otimes \mathbb{I} \right) U, \;\; B_{out} = U^{\dagger} \left(B \otimes \mathbb{I} \right) U,$
  $M_{out} = U^{\dagger} \left(\mathbb{I} \otimes M \right) U$
- Noise $\epsilon_A = \sqrt{\langle \Psi | \left(M_{out} - A_{in} \right)^2 | \Psi \rangle},$
  Disturbance $\eta_B = \sqrt{\langle \Psi | \left(B_{out} - B_{in} \right)^2 | \Psi \rangle},$
  where $|\Psi\rangle = |\psi\rangle_s \otimes |\phi\rangle_p$.

Error-Disturbance Relations

- Ozawa proved, for error-disturbance and intrinsic quantum fluctuations for two incompatible observables

$$\epsilon_A \eta_B + \epsilon_A \Delta B + \Delta A \eta_B \geq |\mathcal{C}_{AB}|,$$

where $|\mathcal{C}_{AB}| = \frac{1}{2}|\langle\psi|[A, B]|\psi\rangle|$.

M. Ozawa, Phys. Rev. A **67**, 042105 (2003)

Error-Disturbance Relations

- Ozawa proved, for error-disturbance and intrinsic quantum fluctuations for two incompatible observables

  $$\epsilon_A \eta_B + \epsilon_A \Delta B + \Delta A \eta_B \geq |\mathcal{C}_{AB}|,$$

  where $|\mathcal{C}_{AB}| = \frac{1}{2}|\langle\psi|[A, B]|\psi\rangle|$.

- Hall's error-disturbance relation

  $$\epsilon_A \eta_B + \epsilon_A \Delta B_{out} + \Delta M_{out} \eta_B \geq |\mathcal{C}_{AB}|$$

M. Ozawa, Phys. Rev. A **67**, 042105 (2003)

Error-Disturbance Relations

- Ozawa proved, for error-disturbance and intrinsic quantum fluctuations for two incompatible observables

$$\epsilon_A \eta_B + \epsilon_A \Delta B + \Delta A \eta_B \geq |\mathcal{C}_{AB}|,$$

where $|\mathcal{C}_{AB}| = \frac{1}{2} |\langle\psi|[A, B]|\psi\rangle|$.

- Hall's error-disturbance relation

$$\epsilon_A \eta_B + \epsilon_A \Delta B_{out} + \Delta M_{out} \eta_B \geq |\mathcal{C}_{AB}|$$

- Weston *et al.* formulated a new error-disturbance relation

$$\epsilon_A (\Delta B + \Delta B_{out}) + \eta_B (\Delta A + \Delta M_{out}) \geq 2|\mathcal{C}_{AB}|$$

M. Ozawa, Phys. Rev. A **67**, 042105 (2003)

- Branciard's error-disturbance relation

$$\epsilon_A^2 \Delta B^2 + \eta_B^2 \Delta A^2 + 2\epsilon_A \eta_B \sqrt{\Delta A^2 \Delta B^2 - \mathcal{C}_{AB}^2} \geq \mathcal{C}_{AB}^2$$

C. Branciard, Proc. Natl. Acad. Sci. **110**, 6742 (2013)

- Branciard's error-disturbance relation

$$\epsilon_A^2 \Delta B^2 + \eta_B^2 \Delta A^2 + 2\epsilon_A \eta_B \sqrt{\Delta A^2 \Delta B^2 - \mathcal{C}_{AB}^2} \geq \mathcal{C}_{AB}^2$$

For dichotomic set of observables A, B with eigenvalues ± 1 and the states with $\langle A \rangle = \langle B \rangle = 0$,

$$\epsilon_A^2 \left(1 - \frac{\epsilon_A^2}{4}\right) + \eta_B^2 \left(1 - \frac{\eta_B^2}{4}\right) + 2\sqrt{1 - \mathcal{C}_{AB}^2}\, \epsilon_A \eta_B \sqrt{1 - \frac{\epsilon_A^2}{4}} \sqrt{1 - \frac{\eta_B^2}{4}} \geq \mathcal{C}_{AB}^2$$

C. Branciard, Proc. Natl. Acad. Sci. **110**, 6742 (2013)

- Branciard's error-disturbance relation

$$\epsilon_A^2 \Delta B^2 + \eta_B^2 \Delta A^2 + 2\epsilon_A \eta_B \sqrt{\Delta A^2 \Delta B^2 - \mathcal{C}_{AB}^2} \geq \mathcal{C}_{AB}^2$$

For dichotomic set of observables A, B with eigenvalues ± 1 and the states with $\langle A \rangle = \langle B \rangle = 0$,

$$\epsilon_A^2 \left(1 - \frac{\epsilon_A^2}{4}\right) + \eta_B^2 \left(1 - \frac{\eta_B^2}{4}\right)$$
$$+ 2\sqrt{1 - \mathcal{C}_{AB}^2}\,\epsilon_A \eta_B \sqrt{1 - \frac{\epsilon_A^2}{4}}\sqrt{1 - \frac{\eta_B^2}{4}} \geq \mathcal{C}_{AB}^2$$

This gives strictly stronger bound than Branciard or Ozawa Bound and signifies maximum Bell non-locality among all well known error-disturbance relations.

C. Branciard, Proc. Natl. Acad. Sci. **110**, 6742 (2013)

New Error-Disturbance Relations

Theorem

For the noise operator $N_A = M_{out} - A_{in}$ and corresponding disturbance operator $D_B = B_{out} - B_{in}$, if the system and the probe are in joint state $|\Psi\rangle = |\psi\rangle_s \otimes |\phi\rangle_p$, the following inequality holds:

$$\begin{aligned} \epsilon_A^2 + \eta_B^2 \quad &\geq \quad \pm i\langle\psi|[A,B]|\psi\rangle \mp i\langle\Psi|[M_{out}, B_{in}]|\Psi\rangle \\ &\mp i\langle\Psi|[A_{in}, B_{out}]|\Psi\rangle + |\langle\Psi|N_A \pm iD_B|\Psi^{\perp}\rangle|^2, \end{aligned}$$

where the sign is chosen such that $\pm i\langle\psi|[A,B]|\psi\rangle$ is positive and $|\Psi^{\perp}\rangle$ is orthogonal to $|\Psi\rangle$.

L. Maccone and A. K. Pati, Phys. Rev. Lett. **113**, 260401 (2014)

Example:Qubit System

- System, probe, observables and estimator

$$|\psi\rangle_s = \alpha|0\rangle + \beta|1\rangle = u|0\rangle, \ |\phi\rangle_p = |1\rangle$$

Example:Qubit System

- System, probe, observables and estimator

$$|\psi\rangle_s = \alpha|0\rangle + \beta|1\rangle = u|0\rangle, \; |\phi\rangle_p = |1\rangle$$

- $A_{in} = \sigma'_x \otimes \mathbb{I}, \; B_{in} = \sigma'_y \otimes \mathbb{I}, \; M_{in} = \mathbb{I} \otimes \sigma_x$

Example:Qubit System

- System, probe, observables and estimator

$$|\psi\rangle_s = \alpha|0\rangle + \beta|1\rangle = u|0\rangle, \ |\phi\rangle_p = |1\rangle$$

- $A_{in} = \sigma'_x \otimes \mathbb{I}, \ B_{in} = \sigma'_y \otimes \mathbb{I}, \ M_{in} = \mathbb{I} \otimes \sigma_x$

$$u = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha \end{pmatrix}, \qquad \sigma'_x = u\sigma_x u^\dagger, \ \sigma'_y = u\sigma_y u^\dagger$$

where $\alpha = \cos\theta, \ \beta = \sin\theta \ e^{i\phi}$.

Example:Qubit System

- System, probe, observables and estimator

  $$|\psi\rangle_s = \alpha|0\rangle + \beta|1\rangle = u|0\rangle, \; |\phi\rangle_p = |1\rangle$$

- $A_{in} = \sigma'_x \otimes \mathbb{I}, \; B_{in} = \sigma'_y \otimes \mathbb{I}, \; M_{in} = \mathbb{I} \otimes \sigma_x$

  $$u = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha \end{pmatrix}, \qquad \sigma'_x = u\sigma_x u^\dagger, \; \sigma'_y = u\sigma_y u^\dagger$$

  where $\alpha = \cos\theta, \; \beta = \sin\theta \; e^{i\phi}$.

- Coupling through CNOT interaction

  $$U = P_0 \otimes \mathbb{I} + P_1 \otimes \sigma_x,$$

  $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$.

Branciard's EDR

$$\epsilon_A^2 + \eta_B^2 \geq 1$$

Branciard's EDR

$$\epsilon_A^2 + \eta_B^2 \geq 1$$

New EDR

$$\epsilon_A^2 + \eta_B^2 \geq 2 - 8\cos^2\theta\sin^2\theta\sin^2\phi + 16\cos^4\theta\sin^4\theta\sin^2\phi$$

Branciard's EDR

$$\epsilon_A^2 + \eta_B^2 \geq 1$$

New EDR

$$\begin{aligned} \epsilon_A^2 + \eta_B^2 \quad &\geq \quad 2 - 8\cos^2\theta \sin^2\theta \sin^2\phi \\ &\quad +16\cos^4\theta \sin^4\theta \sin^2\phi \end{aligned}$$

For $\phi = \pi/2$

$$\epsilon_A^2 + \eta_B^2 \quad \geq \quad 1 + \cos^4(2\theta)$$

- $|\Psi^{\perp}\rangle$ can be generated by projecting any state $|r\rangle$ to the orthogonal subspace of $|\Psi\rangle$.

- $|\Psi^{\perp}\rangle$ can be generated by projecting any state $|r\rangle$ to the orthogonal subspace of $|\Psi\rangle$.

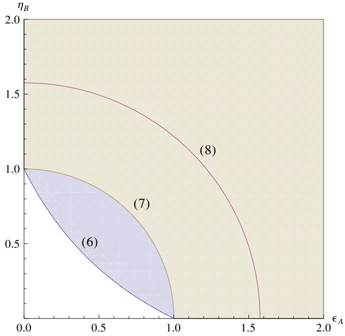

Figure : Error-disturbance relations for the fixed values of observables and state such that $|\mathcal{C}_{AB}| = 1$.

Example:Qubit System

Define the following,

- $$L_{Ozawa} = \epsilon_A \eta_B + \epsilon_A \Delta B + \Delta A \eta_B$$
- $$L_{Branciard} = \sqrt{\epsilon_A^2 \Delta B^2 + \eta_B^2 \Delta A^2 + 2\epsilon_A \eta_B \sqrt{\Delta A^2 \Delta B^2 - \mathcal{C}_{AB}^2}}$$
- $$L_{New}^{(1)} = \frac{1}{2}\Big[\epsilon_A^2 + \eta_B^2 \pm i\langle\Psi|[M_{out}, B_{in}]|\Psi\rangle \pm i\langle\Psi|[A_{in}, B_{out}]|\Psi\rangle - |\langle\Psi|N_A \pm iD_B|\Psi^{\perp}\rangle|^2\Big]$$

- $|\psi\rangle_s = cos\theta|0\rangle + sin\theta|1\rangle$, $|\phi\rangle_p = |1\rangle$

- $|\psi\rangle_s = cos\theta|0\rangle + sin\theta|1\rangle$, $|\phi\rangle_p = |1\rangle$
- $A_{in} = \sigma_x \otimes \mathbb{I}$, $B_{in} = \sigma_y \otimes \mathbb{I}$, $M_{in} = \mathbb{I} \otimes \sigma_x$

- $|\psi\rangle_s = cos\theta|0\rangle + sin\theta|1\rangle$, $|\phi\rangle_p = |1\rangle$
- $A_{in} = \sigma_x \otimes \mathbb{I}, \ B_{in} = \sigma_y \otimes \mathbb{I}, \ M_{in} = \mathbb{I} \otimes \sigma_x$
- $U = P_0 \otimes \mathbb{I} + P_1 \otimes \sigma_x$

- $|\psi\rangle_s = cos\theta|0\rangle + sin\theta|1\rangle$, $|\phi\rangle_p = |1\rangle$
- $A_{in} = \sigma_x \otimes \mathbb{I}, \ B_{in} = \sigma_y \otimes \mathbb{I}, \ M_{in} = \mathbb{I} \otimes \sigma_x$
- $U = P_0 \otimes \mathbb{I} + P_1 \otimes \sigma_x$

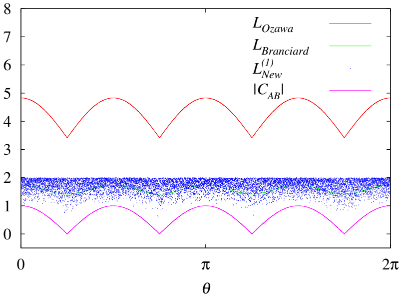

Figure : 25% of states, show tighter bound for qubit initial states.

Example:Qutrit System

- $|\psi\rangle_s = sin\theta cos\phi|0\rangle + sin\theta sin\phi|1\rangle + cos\theta|2\rangle$, $|\phi\rangle_p = |1\rangle$

Example:Qutrit System

- $|\psi\rangle_s = sin\theta cos\phi|0\rangle + sin\theta sin\phi|1\rangle + cos\theta|2\rangle$, $|\phi\rangle_p = |1\rangle$
- $A_{in} = S_x \otimes \mathbb{I}$
  $B_{in} = S_y \otimes \mathbb{I}$
  $M_{in} = \mathbb{I} \otimes S_x$

  S_x, S_y, S_z being spin matrices for spin 1.

Example:Qutrit System

- $|\psi\rangle_s = sin\theta cos\phi|0\rangle + sin\theta sin\phi|1\rangle + cos\theta|2\rangle$, $|\phi\rangle_p = |1\rangle$
- $A_{in} = S_x \otimes \mathbb{I}$
  $B_{in} = S_y \otimes \mathbb{I}$
  $M_{in} = \mathbb{I} \otimes S_x$

  S_x, S_y, S_z being spin matrices for spin 1.
- Qutrit CNOT

$$U = \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \end{pmatrix}$$

Example:Qutrit System

- $|\psi\rangle_s = sin\theta cos\phi|0\rangle + sin\theta sin\phi|1\rangle + cos\theta|2\rangle$, $|\phi\rangle_p = |1\rangle$

- $A_{in} = S_x \otimes \mathbb{I}$
  $B_{in} = S_y \otimes \mathbb{I}$
  $M_{in} = \mathbb{I} \otimes S_x$

  S_x, S_y, S_z being spin matrices for spin 1.

- Qutrit CNOT

$$U = \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \end{pmatrix}$$

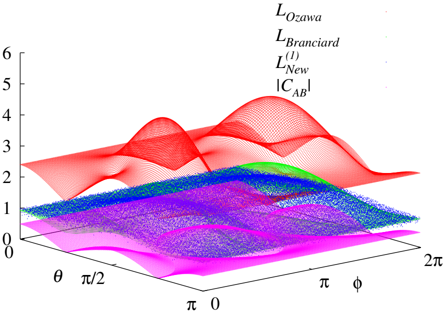

Figure : 47.1% of states, show tighter bound for qutrit initial states.

Theorem

For Noise operator N_A and corresponding Disturbance operator D_B defined as, $N_A = M_{out} - A_{in}$ and $D_B = B_{out} - B_{in}$, if the system and the probe are in joint state $|\Psi\rangle = |\psi\rangle_s \otimes |\phi\rangle_p$, the following inequality holds:

$$\epsilon_A\eta_B + \eta_B\Delta A + \epsilon_A\Delta B - \frac{1}{2}\frac{|\langle\Psi|N_A\Delta D_B \pm iD_B\Delta N_A|\Psi^\perp\rangle|^2}{\epsilon_A\eta_B}$$
$$-\frac{1}{2}\frac{|\langle\Psi|A\Delta D_B \pm iD_B\Delta A|\Psi^\perp\rangle|^2}{\Delta A\eta_B} - \frac{1}{2}\frac{|\langle\Psi|N_A\Delta B \pm iB\Delta N_A|\Psi^\perp\rangle|^2}{\epsilon_A\Delta B} \geq |\mathcal{C}_{AB}|$$

where $\mathcal{C}_{AB}$ is defined previously and the sign is chosen such that $\pm i\langle\psi|[A,B]|\psi\rangle$ is positive.

Qubit System

- $|\psi\rangle_s = cos\theta|0\rangle + sin\theta|1\rangle$, $|\phi\rangle_p = |1\rangle$
- $A_{in} = \lambda\ (\sigma_x \otimes \mathbb{I}),\ \ B_{in} = \sigma_y \otimes \mathbb{I},\ \ M_{in} = \mathbb{I} \otimes \sigma_x$ with $\lambda = 0.01$.
- $U = P_0 \otimes \mathbb{I} + P_1 \otimes \sigma_x$

Qubit System

- $|\psi\rangle_s = cos\theta|0\rangle + sin\theta|1\rangle$, $|\phi\rangle_p = |1\rangle$
- $A_{in} = \lambda\ (\sigma_x \otimes \mathbb{I}),\ \ B_{in} = \sigma_y \otimes \mathbb{I},\ \ M_{in} = \mathbb{I} \otimes \sigma_x$ with $\lambda = 0.01$.
- $U = P_0 \otimes \mathbb{I} + P_1 \otimes \sigma_x$

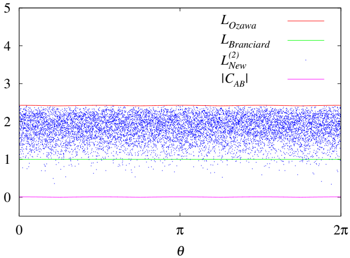

Figure : 1.5% of states, show tighter bound for qubit initial states.

Qubit System

- $|\psi\rangle_s = cos\theta|0\rangle + sin\theta|1\rangle$, $|\phi\rangle_p = |1\rangle$
- $A_{in} = \lambda\ (\sigma_x \otimes \mathbb{I}),\ \ B_{in} = \sigma_y \otimes \mathbb{I},\ \ M_{in} = \mathbb{I} \otimes \sigma_x$ with $\lambda = 0.01$.
- $U = P_0 \otimes \mathbb{I} + P_1 \otimes \sigma_x$

$$L^{(2)}_{New} = \epsilon_A\eta_B + \eta_B\Delta A + \epsilon_A\Delta B - \frac{1}{2}\frac{|\langle\Psi|N_A\Delta D_B \pm iD_B\Delta N_A|\Psi^\perp\rangle|^2}{\epsilon_A\eta_B} - \frac{1}{2}\frac{|\langle\Psi|A\Delta D_B \pm iD_B\Delta A|\Psi^\perp\rangle|^2}{\Delta A\eta_B} - \frac{1}{2}\frac{|\langle\Psi|N_A\Delta B \pm iB\Delta N_A|\Psi^\perp\rangle|^2}{\epsilon_A\Delta B}.$$

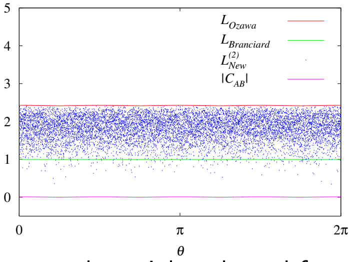

Figure : 1.5% of states, show tighter bound for qubit initial states.

Conclusions

- Formulation of a new inherent fluctuation free error-disturbance relation.
- Achieved better bounds than Branciard for some initial states and specific measurement settings.
- Modification of Ozawa's error-disturbance relation for product of variances.
- Better bounds than Ozawa and Branciard for some given states.

Collaborators

Chiranjib

Arun K Pati

Thank you