An analytical condition for the violation of Mermin's inequality by any three qubit state

Satyabrata Adhikari

Birla Institute of Technology, Mesra

Collaborator: Prof. Archan S. Majumdar

Quantum non-locality

- The impossiblity of reproducing the effect of quantum correlations between the outcomes of the distant measurements using local hidden variable theories is known as quantum non-locality.
- Quantum nonlocality finds applications in several information theoretic protocols such as device independent quantum key generation, quantum state estimation and communication complexity, where the amount of violation of the Bell-CHSH inequality is important.

Mermin inequality

- Like two qubit non-locality, non-locality for three qubit systems has also been studied using various approaches.
- A generalized form of the Bell-CHSH inequality was obtained for three qubits called Mermin's inequality.
- The Mermin operator is defined as

$$B_M = \hat{a}_1.\vec{\sigma} \otimes \hat{a}_2.\vec{\sigma} \otimes \hat{a}_3.\vec{\sigma} - \hat{a}_1.\vec{\sigma} \otimes \hat{b}_2.\vec{\sigma} \otimes \hat{b}_3.\vec{\sigma}$$
$$- \hat{b}_1.\vec{\sigma} \otimes \hat{a}_2.\vec{\sigma} \otimes \hat{b}_3.\vec{\sigma} - \hat{b}_1.\vec{\sigma} \otimes \hat{b}_2.\vec{\sigma} \otimes \hat{a}_3.\vec{\sigma}$$

where $\hat{a}_j$ and $\hat{b}_j$ $(j = 1, 2, 3)$ are unit vectors in R^3, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

- For any three qubit state ρ, the Mermin inequality is

$$\langle B_M \rangle_\rho \leq 2$$

$$\rho = \frac{1}{8}[I \otimes I \otimes I + \vec{l}.\vec{\sigma} \otimes I \otimes I + I \otimes \vec{m}.\vec{\sigma} \otimes I + I \otimes I \otimes \vec{n}.\vec{\sigma}$$

$$+ \vec{u}.\vec{\sigma} \otimes \vec{v}.\vec{\sigma} \otimes I + \vec{u}.\vec{\sigma} \otimes I \otimes \vec{w}.\vec{\sigma} + I \otimes \vec{v}.\vec{\sigma} \otimes \vec{w}.\vec{\sigma} +$$

$$\sum_{i,j,k=x,y,z} t_{ijk}\sigma_i \otimes \sigma_j \otimes \sigma_k$$

$$with \quad l_i = Tr(\rho(\sigma_i \otimes I \otimes I)), m_i = Tr(\rho(I \otimes \sigma_i \otimes I)),$$

$$n_i = Tr(\rho(I \otimes I \otimes \sigma_i)), (i = x, y, z)$$

$$u_i v_j = Tr(\rho(\sigma_i \otimes \sigma_j \otimes I)), u_i w_j = Tr(\rho(\sigma_i \otimes I \otimes \sigma_j)),$$

$$v_i w_j = Tr(\rho(I \otimes \sigma_i \otimes \sigma_j)), (i, j = x, y, z)$$

$$t_{ijk} = Tr(\rho(\sigma_i \otimes \sigma_j \otimes \sigma_k)), \ (i, j, k = x, y, z)$$

Mermin inequality

- Violation of the Mermin inequality has been computed for several three qubit states such as GHZ and W-states earlier.[V. Scarani, and N. Gisin, J. Phys. A 34, 6043 (2001), C. Emary, and C. W. J. Beenakker, Phys. Rev. A 69, 032317 (2004), D. P. Chi, K. Jeong, T. Kim, K. Lee, and S. Lee, Phys. Rev. A 81, 044302 (2010)]
- In order to find the maximum violation of the Mermin inequality for three qubit states one has to tackle the optimization problem numerically because there does not exist any analytical formula for even pure three qubit states.

Motivation

- Motivated by the work of Horodecki [R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A 200, 340 (1995).] in the context of two qubit systems, in the present work we perform the optimization problem involved in the Mermin inequality analytically and obtain a formula for the maximal value of the expectation of the Mermin operator in terms of the eigenvalues of symmetric matrices, that gives the maximal violation of the Mermin inequality not only for pure states but also for mixed states.

Maximum expectation value of the Mermin operator in terms of eigenvalues

- The expectation value of the Mermin operator with respect to the state ρ is given by

$$\langle B_M \rangle_\rho = Tr(B_M \rho) = (\widehat{a}_1, \widehat{a}_3{}^T \vec{T} \widehat{a}_2) - (\widehat{a}_1, \widehat{b}_3{}^T \vec{T} \widehat{b}_2) - (\widehat{b}_1, \widehat{b}_3{}^T \vec{T} \widehat{a}_2) - (\widehat{b}_1, \widehat{a}_3{}^T \vec{T} \widehat{b}_2)$$

$where\ \ \widehat{a}_s = (a_{sx}, a_{sy}, a_{sz}), \widehat{b}_s = (b_{sx}, b_{sy}, b_{sz}), (s = 1, 2, 3)$

$(\vec{x}, \vec{y})$ denote the inner product of two vectors $\vec{x}$ and $\vec{y}$ and is defined as

$(\vec{x}, \vec{y}) = \|\vec{x}\|\|\vec{y}\| \cos\theta, \theta$ being the angle between $\vec{x}$ and $\vec{y}$.

$$\vec{T} = (T_x, T_y, T_z)$$

$$where,\ T_x = \begin{pmatrix} t_{xxx} & t_{xyx} & t_{xzx} \\ t_{xxy} & t_{xyy} & t_{xzy} \\ t_{xxz} & t_{xyz} & t_{xzz} \end{pmatrix}, T_y = \begin{pmatrix} t_{yxx} & t_{yyx} & t_{yzx} \\ t_{yxy} & t_{yyy} & t_{yzy} \\ t_{yxz} & t_{yyz} & t_{yzz} \end{pmatrix}, T_z = \begin{pmatrix} t_{zxx} & t_{zyx} & t_{zzx} \\ t_{zxy} & t_{zyy} & t_{zzy} \\ t_{zxz} & t_{zyz} & t_{zzz} \end{pmatrix}$$

Result-1

If the symmetric matrices $T_x{}^T T_x, T_y{}^T T_y$, and $T_z{}^T T_z$ has unique largest eigenvalue $\lambda_x^{\max}, \lambda_y^{\max}, \lambda_z^{\max}$ respectively then Mermin's inequality is violated if

$$\left\langle B_M^{\max} \right\rangle_\rho = \max\left\{2\sqrt{\lambda_x^{\max}}, 2\sqrt{\lambda_y^{\max}}, 2\sqrt{\lambda_z^{\max}}\right\} > 2$$

Proof : In the expression for $\left\langle B_M \right\rangle_\rho$, we first simplify the vectors $\widehat{a}_3^T \vec{T} \widehat{a}_2$, $\widehat{b}_3^T \vec{T} \widehat{b}_2$, $\widehat{b}_3^T \vec{T} \widehat{a}_2$, $\widehat{a}_3^T \vec{T} \widehat{b}_2$. We choose the vectors $\widehat{a}_2, \widehat{a}_3, \widehat{b}_2, \widehat{b}_3$ in such a way that they maximize the quantity $\left\langle B_M \right\rangle_\rho$ over all operators B_M.

$Case-I:$ In this case we will show $\left\langle B_M^{(1)} \right\rangle_\rho = 2\sqrt{\lambda_x^{\max}}$.

(i) The vector $\widehat{a}_3^T \vec{T} \widehat{a}_2$ can be simplified as

$\widehat{a}_3^T \vec{T} \widehat{a}_2 = ((\widehat{a}_3, T_x \widehat{a}_2), (\widehat{a}_3, T_y \widehat{a}_2), (\widehat{a}_3, T_z \widehat{a}_2)) = (\left\|\widehat{a}_3^{\max}\right\| \left\|T_x \widehat{a}_2\right\|, 0, 0)$

where $\widehat{a}_3^{\max}$ is the unit vector along $T_x \widehat{a}_2$ and perpendicular to $T_y \widehat{a}_2$ and $T_z \widehat{a}_2$. Since $\widehat{a}_3^{\max}$ is the unit vector so $\left\|\widehat{a}_3^{\max}\right\| = 1$.

Again $\left\|T_x \widehat{a}_2\right\|^2 = (T_x \widehat{a}_2, T_x \widehat{a}_2) = (\widehat{a}_2, T_x^T T_x \widehat{a}_2)$. If $\lambda_x^{\max}$ is the largest eigenvalue of the symmetric matrix $T_x^T T_x$ and $\widehat{a}_2^{\max}$ is the corresponding unit vector then $\left\|T_x \widehat{a}_2\right\|^2 = (\widehat{a}_2, T_x^T T_x \widehat{a}_2)$
$= (\widehat{a}_2, \lambda_x^{\max} \widehat{a}_2^{\max})$. If $\widehat{a}_2$ is the unit vector along $\widehat{a}_2^{\max}$ then $\left\|T_x \widehat{a}_2\right\|^2 = \lambda_x^{\max}$. Thus $\widehat{a}_3^T \vec{T} \widehat{a}_2 = (\sqrt{\lambda_x^{\max}}, 0, 0)$.

(ii) The vector $\widehat{a}_3^T \vec{T} \widehat{b}_2$ can be simplified as

$$\widehat{a}_3^T \vec{T} \widehat{b}_2 = ((\widehat{a}_3, T_x \widehat{b}_2), (\widehat{a}_3, T_y \widehat{b}_2), (\widehat{a}_3, T_z \widehat{b}_2)) = (0, -\left\|\widehat{a}_3^{\min}\right\| \left\|T_y \widehat{b}_2\right\|, 0)$$

where $\widehat{a}_3^{\min}$ is the unit vector antiparallel to $T_y \widehat{b}_2$ and perpendicular to $T_x \widehat{b}_2$ and $T_z \widehat{b}_2$. Since $\widehat{a}_3^{\min}$ is the unit vector, $\left\|\widehat{a}_3^{\min}\right\| = 1$. Again $\left\|T_y \widehat{b}_2\right\|^2 = (T_y \widehat{b}_2, T_y \widehat{b}_2) = (\widehat{b}_2, T_y^T T_y \widehat{b}_2)$. If $\lambda_y^{\max}$ is the largest eigenvalue of the symmetric matrix $T_y^T T_y$ and $\widehat{b}_2^{\max}$ is the corresponding unit vector then $\left\|T_y \widehat{b}_2\right\|^2 =$ $(\widehat{b}_2, T_y^T T_y \widehat{b}_2) = (\widehat{b}_2, \lambda_y^{\max} \widehat{b}_2^{\max})$. If $\widehat{b}_2$ is the unit vector along $\widehat{b}_2^{\max}$ then $\left\|T_y \widehat{b}_2\right\|^2 = \lambda_y^{\max}$. Thus $\widehat{a}_3^T \vec{T} \widehat{b}_2 = (0, -\sqrt{\lambda_y^{\max}}, 0)$.

(iii) The vector $\widehat{b}_3^T \vec{T} \widehat{b}_2$ can be simplified as

$$\widehat{b}_3^T \vec{T} \widehat{b}_2 = ((\widehat{b}_3, T_x \widehat{b}_2), (\widehat{b}_3, T_y \widehat{b}_2), (\widehat{b}_3, T_z \widehat{b}_2)) = (0, -\left\|\widehat{b}_3^{\min}\right\| \left\|T_y \widehat{b}_2\right\|, 0)$$

$$= (0, -\sqrt{\lambda_y^{\max}}, 0)$$

where $\widehat{b}_3^{\min}$ is the unit vector antiparallel to $T_y \widehat{b}_2$ and perpendicular to $T_x \widehat{b}_2$ and $T_z \widehat{b}_2$. Since $\widehat{b}_3^{\min}$ is the unit vector, $\left\|\widehat{b}_3^{\min}\right\| = 1$.

(iv) The vector $\widehat{b}_3^T \vec{T} \widehat{a}_2$ can be simplified as

$$\widehat{b}_3^T \vec{T} \widehat{a}_2 = ((\widehat{b}_3, T_x \widehat{a}_2), (\widehat{b}_3, T_y \widehat{a}_2), (\widehat{b}_3, T_z \widehat{a}_2)) = (\left\|\widehat{b}_3^{\max}\right\| \left\|T_x \widehat{a}_2\right\|, 0, 0)$$

$$= (\sqrt{\lambda_x^{\max}}, 0, 0)$$

where $\widehat{b}_3^{\max}$ is the unit vector along $T_x \widehat{a}_2$ and perpendicular to $T_y \widehat{a}_2$ and $T_z \widehat{a}_2$. Since $\widehat{b}_3^{\max}$ is the unit vector, $\left\|\widehat{b}_3^{\max}\right\| = 1$.

We have

$$\begin{aligned}
\left\langle B_M^{(1)} \right\rangle_\rho &= \max_{\widehat{a}_1, \widehat{b}_1} [(\widehat{a}_1, (\sqrt{\lambda_x^{\max}}, 0, 0)) + (\widehat{a}_1, (0, \sqrt{\lambda_y^{\max}}, 0)) \\
&\quad - (\widehat{b}_1, (\sqrt{\lambda_x^{\max}}, 0, 0)) + (\widehat{b}_1, (0, \sqrt{\lambda_y^{\max}}, 0))] \\
&= \left\| \widehat{a}_1^{\max} \right\| \left\| (\sqrt{\lambda_x^{\max}}, 0, 0) \right\| + \left\| \widehat{b}_1^{\max} \right\| \left\| (\sqrt{\lambda_x^{\max}}, 0, 0) \right\| \\
&= 2\sqrt{\lambda_x^{\max}}
\end{aligned}$$

where $\widehat{a}_1^{\max}$ is the unit vector parallel to $(\sqrt{\lambda_x^{\max}}, 0, 0)$ and perpendicular to $(0, \sqrt{\lambda_y^{\max}}, 0)$; $\widehat{b}_1^{\max}$ is the unit vector antiparallel to $(\sqrt{\lambda_x^{\max}}, 0, 0)$ and perpendicular to $(0, \sqrt{\lambda_y^{\max}}, 0)$

$Case-II$: In this case we choose the vectors in such a way that the maximized expectation value of the Mermin operator is given by $\left\langle B_M^{(2)} \right\rangle_\rho = 2\sqrt{\lambda_y^{\max}}$.

(i) The vector $\widehat{a}_3^T \vec{T} \widehat{a}_2$ can be simplified as

$$\widehat{a}_3^T \vec{T} \widehat{a}_2 = ((\widehat{a}_3, T_x \widehat{a}_2), (\widehat{a}_3, T_y \widehat{a}_2), (\widehat{a}_3, T_z \widehat{a}_2)) = (0, \left\| \widehat{a}_3^{\max} \right\| \left\| T_y \widehat{a}_2 \right\|, 0)$$

where $\widehat{a}_3^{\max}$ is the unit vector along $T_y \widehat{a}_2$ and perpendicular to $T_x \widehat{a}_2$ and $T_z \widehat{a}_2$. Repeating the steps of case-I, we find

$$\widehat{a}_3^T \vec{T} \widehat{a}_2 = (0, \sqrt{\lambda_y^{\max}}, 0).$$

Similarly we obtain

(ii) $\widehat{a}_3^T \vec{T} \widehat{b}_2 = (-\sqrt{\lambda_x^{\max}}, 0, 0)$; (iii) $\widehat{b}_3^T \vec{T} \widehat{b}_2 = (-\sqrt{\lambda_x^{\max}}, 0, 0)$;

(iv) $\widehat{b}_3^T \vec{T} \widehat{a}_2 = (0, \sqrt{\lambda_y^{\max}}, 0)$.

Therefore, we have

$$
\begin{aligned}
\left\langle B_M^{(2)} \right\rangle_\rho &= \max_{\widehat{a}_1, \widehat{b}_1} [(\widehat{a}_1, (0, \sqrt{\lambda_y^{\max}}, 0)) + (\widehat{a}_1, (\sqrt{\lambda_x^{\max}}, 0, 0)) \\
&\qquad - (\widehat{b}_1, (0, \sqrt{\lambda_y^{\max}}, 0)) + (\widehat{b}_1, (\sqrt{\lambda_x^{\max}}, 0, 0))] \\
&= \left\| \widehat{a}_1^{\max} \right\| \left\| (0, \sqrt{\lambda_y^{\max}}, 0) \right\| + \left\| \widehat{b}_1^{\max} \right\| \left\| (0, \sqrt{\lambda_y^{\max}}, 0) \right\| \\
&= 2\sqrt{\lambda_y^{\max}}
\end{aligned}
$$

where $\widehat{a}_1^{\max}$ is the unit vector parallel to $(0, \sqrt{\lambda_y^{\max}}, 0)$ and perpendicular to $(\sqrt{\lambda_x^{\max}}, 0, 0)$; $\widehat{b}_1^{\max}$ is the unit vector antiparallel to $(0, \sqrt{\lambda_y^{\max}}, 0)$ and perpendicular to $(\sqrt{\lambda_x^{\max}}, 0, 0)$.

$Case - III$: Proceeding in a similar fashion as in Case-I and Case-II, we obtain

$$\left\langle B_M^{(3)} \right\rangle_\rho = \max_{\widehat{a}_1, \widehat{b}_1} [(\widehat{a}_1, (0,0,\sqrt{\lambda_z^{\max}})) + (\widehat{a}_1, (\sqrt{\lambda_x^{\max}}, 0, 0))$$

$$- (\widehat{b}_1, (0,0,\sqrt{\lambda_z^{\max}})) + (\widehat{b}_1, (\sqrt{\lambda_x^{\max}}, 0, 0))]$$

$$= \left\|\widehat{a}_1^{\max}\right\| \left\|(0,0,\sqrt{\lambda_z^{\max}})\right\| + \left\|\widehat{b}_1^{\max}\right\| \left\|(0,0,\sqrt{\lambda_z^{\max}})\right\|$$

$$= 2\sqrt{\lambda_z^{\max}}$$

where $\widehat{a}_1^{\max}$ is the unit vector parallel to $(0,0,\sqrt{\lambda_z^{\max}})$ and perpendicular to $(\sqrt{\lambda_x^{\max}}, 0, 0)$; $\widehat{b}_1^{\max}$ is the unit vector antiparallel to $(0,0,\sqrt{\lambda_z^{\max}})$ and perpendicular to $(\sqrt{\lambda_x^{\max}}, 0, 0)$.

Thus finally, the maximum expectation value of the Mermin operator with respect to the state ρ is given by

$$\left\langle B_M^{\max} \right\rangle_\rho = \max\left\{ \left\langle B_M^{(1)} \right\rangle_\rho, \left\langle B_M^{(2)} \right\rangle_\rho, \left\langle B_M^{(3)} \right\rangle_\rho \right\}$$

$$= \max\left\{ 2\sqrt{\lambda_x^{\max}}, 2\sqrt{\lambda_y^{\max}}, 2\sqrt{\lambda_z^{\max}} \right\}$$

The Mermin inequality is violated if

$$\left\langle B_M^{\max} \right\rangle_\rho > 2 \Rightarrow \max\left\{ 2\sqrt{\lambda_x^{\max}}, 2\sqrt{\lambda_y^{\max}}, 2\sqrt{\lambda_z^{\max}} \right\} > 2$$

Result-2

If the symmetric matrices $T_x^T T_x, T_y^T T_y$, and $T_z^T T_z$ have two equal largest eigenvalue $\lambda_x^{\max}, \lambda_y^{\max}, \lambda_z^{\max}$ respectively then Mermin's inequality is violated if

$$\langle B_M^{\max} \rangle_\rho = \max\left\{4\sqrt{\lambda_x^{\max}}, 4\sqrt{\lambda_y^{\max}}, 4\sqrt{\lambda_z^{\max}}\right\} > 2$$

Proof : In the expression for $\langle B_M \rangle_\rho$, we first simplify the vectors $\hat{a}_3^T \vec{T} \hat{a}_2$, $\hat{b}_3^T \vec{T} \hat{b}_2$, $\hat{b}_3^T \vec{T} \hat{a}_2$, $\hat{a}_3^T \vec{T} \hat{b}_2$. We choose the vectors $\hat{a}_2, \hat{a}_3, \hat{b}_2, \hat{b}_3$ in such a way that they maximize the quantity $\langle B_M \rangle_\rho$ over all operators B_M.

$Case - I$: We consider the symmetric matrix $T_x^T T_x$ which has two equal largest eigenvalues $\lambda_x^{\max}$.

In this case, we will show $\left\langle B_M^{(4)} \right\rangle_\rho = 4\sqrt{\lambda_x^{\max}}$.

(i) The vector $\widehat{a}_3^T \vec{T} \widehat{a}_2$ can be simplified as

$\widehat{a}_3^T \vec{T} \widehat{a}_2 = ((\widehat{a}_3, T_x \widehat{a}_2), (\widehat{a}_3, T_y \widehat{a}_2), (\widehat{a}_3, T_z \widehat{a}_2)) = (\left\|\widehat{a}_3^{\max}\right\| \left\|T_x \widehat{a}_2\right\|, 0, 0)$

where $\widehat{a}_3^{\max}$ is the unit vector along $T_x \widehat{a}_2$ and perpendicular to $T_y \widehat{a}_2$ and $T_z \widehat{a}_2$. Since $\widehat{a}_3^{\max}$ is the unit vector so $\left\|\widehat{a}_3^{\max}\right\| = 1$.

Again $\left\|T_x \widehat{a}_2\right\|^2 = (T_x \widehat{a}_2, T_x \widehat{a}_2) = (\widehat{a}_2, T_x^T T_x \widehat{a}_2)$. If $\lambda_x^{\max}$ is the largest eigenvalue of the symmetric matrix $T_x^T T_x$ and $\widehat{a}_2^{\max}$ is the corresponding unit vector then $\left\|T_x \widehat{a}_2\right\|^2 = (\widehat{a}_2, T_x^T T_x \widehat{a}_2)$ $= (\widehat{a}_2, \lambda_x^{\max} \widehat{a}_2^{\max})$. If $\widehat{a}_2$ is the unit vector along $\widehat{a}_2^{\max}$ then $\left\|T_x \widehat{a}_2\right\|^2 = \lambda_x^{\max}$. Thus $\widehat{a}_3^T \vec{T} \widehat{a}_2 = (\sqrt{\lambda_x^{\max}}, 0, 0)$.

(ii) The vector $\hat{a}_3^T \vec{T} \hat{b}_2$ can be simplified as

$$\hat{a}_3^T \vec{T} \hat{b}_2 = ((\hat{a}_3, T_x \hat{b}_2), (\hat{a}_3, T_y \hat{b}_2), (\hat{a}_3, T_z \hat{b}_2)) = (-\left\|\hat{a}_3^{\min}\right\| \left\|T_x \hat{b}_2\right\|, 0, 0)$$

where $\hat{a}_3^{\min}$ is the unit vector antiparallel to $T_x \hat{b}_2$ and perpendicular to $T_y \hat{b}_2$ and $T_z \hat{b}_2$. Since $\hat{a}_3^{\min}$ is the unit vector, $\left\|\hat{a}_3^{\min}\right\| = 1$. Since the matrix $T_x^T T_x$ has two equal largest eigenvalue $\lambda_x^{\max}$, so $b_2^{\max}$ is another corresponding unit eigenvector. Then $\left\|T_x \hat{b}_2\right\|^2 = (\hat{b}_2, T_x^T T_x \hat{b}_2) = (\hat{b}_2, \lambda_x^{\max} \hat{b}_2^{\max})$.

If $\hat{b}_2$ is the unit vector along $\hat{b}_2^{\max}$ then $\left\|T_x \hat{b}_2\right\|^2 = \lambda_x^{\max}$.

Thus $\hat{a}_3^T \vec{T} \hat{b}_2 = (-\sqrt{\lambda_x^{\max}}, 0, 0)$.

(iii) The vector $\widehat{b}_3^T \vec{T} \widehat{b}_2$ can be simplified as

$$\widehat{b}_3^T \vec{T} \widehat{b}_2 = ((\widehat{b}_3, T_x \widehat{b}_2), (\widehat{b}_3, T_y \widehat{b}_2), (\widehat{b}_3, T_z \widehat{b}_2)) = (-\left\|\widehat{b}_3^{\min}\right\| \left\|T_y \widehat{b}_2\right\|, 0, 0)$$

$$= (-\sqrt{\lambda_x^{\max}}, 0, 0)$$

where $\widehat{b}_3^{\min}$ is the unit vector antiparallel to $T_x \widehat{b}_2$ and perpendicular to $T_y \widehat{b}_2$ and $T_z \widehat{b}_2$.

(iv) The vector $\widehat{b}_3^T \vec{T} \widehat{a}_2$ can be simplified as

$$\widehat{b}_3^T \vec{T} \widehat{a}_2 = ((\widehat{b}_3, T_x \widehat{a}_2), (\widehat{b}_3, T_y \widehat{a}_2), (\widehat{b}_3, T_z \widehat{a}_2)) = (-\left\|\widehat{b}_3^{\min}\right\| \left\|T_x \widehat{a}_2\right\|, 0, 0)$$

$$= (-\sqrt{\lambda_x^{\max}}, 0, 0)$$

where $\widehat{b}_3^{\min}$ is the unit vector antiparallel to $T_x \widehat{a}_2$ and perpendicular to $T_y \widehat{a}_2$ and $T_z \widehat{a}_2$.

We have

$$
\begin{aligned}
\left\langle B_M^{(4)} \right\rangle_\rho &= \max_{\widehat{a}_1, \widehat{b}_1} [(\widehat{a}_1, (\sqrt{\lambda_x^{\max}}, 0, 0)) + (\widehat{a}_1, (\sqrt{\lambda_x^{\max}}, 0, 0)) \\
&\qquad + (\widehat{b}_1, (\sqrt{\lambda_x^{\max}}, 0, 0)) + (\widehat{b}_1, (\sqrt{\lambda_x^{\max}}, 0, 0))] \\
&= 2 \left\| \widehat{a}_1^{\max} \right\| \left\| (\sqrt{\lambda_x^{\max}}, 0, 0) \right\| + 2 \left\| \widehat{b}_1^{\max} \right\| \left\| (\sqrt{\lambda_x^{\max}}, 0, 0) \right\| \\
&= 4 \sqrt{\lambda_x^{\max}}
\end{aligned}
$$

where $\widehat{a}_1^{\max}$ and $\widehat{b}_1^{\max}$ is the unit vector along $(\sqrt{\lambda_x^{\max}}, 0, 0)$.

$Case-II$: Here we consider the symmetric matrix $T_y^T T_y$ which has two equal largest eigenvalues $\lambda_y^{\max}$.

In this case, we will show $\left\langle B_M^{(5)} \right\rangle_\rho = 4\sqrt{\lambda_x^{\max}}$.

(i) The vector $\widehat{a}_3^T \vec{T} \widehat{a}_2$ can be simplified as

$$\widehat{a}_3^T \vec{T} \widehat{a}_2 = ((\widehat{a}_3, T_x \widehat{a}_2), (\widehat{a}_3, T_y \widehat{a}_2), (\widehat{a}_3, T_z \widehat{a}_2)) = (0, \left\|\widehat{a}_3^{\max}\right\| \left\|T_y \widehat{a}_2\right\|, 0)$$

where $\widehat{a}_3^{\max}$ is the unit vector along $T_y \widehat{a}_2$ and perpendicular to $T_x \widehat{a}_2$ and $T_z \widehat{a}_2$. Again $\left\|T_y \widehat{a}_2\right\|^2 = (T_y \widehat{a}_2, T_y \widehat{a}_2) = (\widehat{a}_2, T_y^T T_y \widehat{a}_2)$.

If $\lambda_y^{\max}$ is the largest eigenvalue of the symmetric matrix $T_y^T T_y$ and $\widehat{a}_2^{\max}$ is the corresponding unit vector then $\left\|T_x \widehat{a}_2\right\|^2 = (\widehat{a}_2, T_y^T T_y \widehat{a}_2) = (\widehat{a}_2, \lambda_y^{\max} \widehat{a}_2^{\max})$. If $\widehat{a}_2$ is the unit vector along $\widehat{a}_2^{\max}$ then $\left\|T_y \widehat{a}_2\right\|^2 = \lambda_y^{\max}$. Thus $\widehat{a}_3^T \vec{T} \widehat{a}_2 = (0, \sqrt{\lambda_y^{\max}}, 0)$.

(ii) The vector $\widehat{a}_3^T \vec{T} \widehat{b}_2$ can be simplified as

$$\widehat{a}_3^T \vec{T} \widehat{b}_2 = ((\widehat{a}_3, T_x \widehat{b}_2), (\widehat{a}_3, T_y \widehat{b}_2), (\widehat{a}_3, T_z \widehat{b}_2)) = (0, -\left\|\widehat{a}_3^{\min}\right\| \left\|T_y \widehat{b}_2\right\|, 0)$$

where $\widehat{a}_3^{\min}$ is the unit vector antiparallel to $T_y \widehat{b}_2$ and perpendicular to $T_x \widehat{b}_2$ and $T_z \widehat{b}_2$. Since the matrix $T_x^T T_x$ has two equal largest eigenvalue $\lambda_y^{\max}$, so let us consider $b_2^{\max}$ be another corresponding unit eigenvector.

Then $\left\|T_y \widehat{b}_2\right\|^2 = (\widehat{b}_2, T_y^T T_y \widehat{b}_2) = (\widehat{b}_2, \lambda_y^{\max} \widehat{b}_2^{\max})$.

If $\widehat{b}_2$ is the unit vector along $\widehat{b}_2^{\max}$ then $\left\|T_x \widehat{b}_2\right\|^2 = \lambda_y^{\max}$.

Thus $\widehat{a}_3^T \vec{T} \widehat{b}_2 = (0, -\sqrt{\lambda_x^{\max}}, 0)$.

(iii) The vector $\overset{\frown}{b}_3^T \vec{T} \overset{\frown}{b}_2$ can be simplified as

$$\overset{\frown}{b}_3^T \vec{T} \overset{\frown}{b}_2 = ((\overset{\frown}{b}_3, T_x \overset{\frown}{b}_2), (\overset{\frown}{b}_3, T_y \overset{\frown}{b}_2), (\overset{\frown}{b}_3, T_z \overset{\frown}{b}_2)) = (0, -\left\| \overset{\frown}{b}_3^{\min} \right\| \left\| T_y \overset{\frown}{b}_2 \right\|, 0)$$

$$= (0, -\sqrt{\lambda_y^{\max}}, 0)$$

where $\overset{\frown}{b}_3^{\min}$ is the unit vector antiparallel to $T_y \widehat{a}_2$ and perpendicular to $T_x \overset{\frown}{b}_2$ and $T_z \overset{\frown}{b}_2$.

(iv) The vector $\overset{\frown}{b}_3^T \vec{T} \widehat{a}_2$ can be simplified as

$$\overset{\frown}{b}_3^T \vec{T} \widehat{a}_2 = ((\overset{\frown}{b}_3, T_x \widehat{a}_2), (\overset{\frown}{b}_3, T_y \widehat{a}_2), (\overset{\frown}{b}_3, T_z \widehat{a}_2)) = (0, -\left\| \overset{\frown}{b}_3^{\min} \right\| \left\| T_y \widehat{a}_2 \right\|, 0)$$

$$= (0, -\sqrt{\lambda_y^{\max}}, 0)$$

where $\overset{\frown}{b}_3^{\min}$ is the unit vector antiparallel to $T_y \widehat{a}_2$ and perpendicular to $T_x \widehat{a}_2$ and $T_z \widehat{a}_2$.

We have

$$\left\langle B_M^{(5)} \right\rangle_\rho = \max_{\widehat{a}_1, \widehat{b}_1} [(\widehat{a}_1, (0, \sqrt{\lambda_y^{\max}}, 0)) + (\widehat{a}_1, (0, \sqrt{\lambda_y^{\max}}, 0))$$

$$+ (\widehat{b}_1, (0, \sqrt{\lambda_y^{\max}}, 0)) + (\widehat{b}_1, (0, \sqrt{\lambda_y^{\max}}, 0))]$$

$$= 2 \left\| \widehat{a}_1^{\max} \right\| \left\| (0, \sqrt{\lambda_y^{\max}}, 0) \right\| + 2 \left\| \widehat{b}_1^{\max} \right\| \left\| (0, \sqrt{\lambda_y^{\max}}, 0) \right\|$$

$$= 4 \sqrt{\lambda_y^{\max}}$$

where $\widehat{a}_1^{\max}$ and $\widehat{b}_1^{\max}$ is the unit vector along $(0, \sqrt{\lambda_y^{\max}}, 0)$.

$Case-III$: Here we consider the symmetric matrix $T_z^T T_z$ which has two equal largest eigenvalues $\lambda_z^{\max}$. In this case, we obtain

$$\begin{aligned}
\left\langle B_M^{(6)} \right\rangle_\rho &= \max_{\widehat{a}_1, \widehat{b}_1} [(\widehat{a}_1, (0,0,\sqrt{\lambda_z^{\max}})) + (\widehat{a}_1, (0,0,\sqrt{\lambda_z^{\max}})) \\
&\quad + (\widehat{b}_1, (0,0,\sqrt{\lambda_z^{\max}})) + (\widehat{b}_1, (0,0,\sqrt{\lambda_z^{\max}}))] \\
&= 2\left\|\widehat{a}_1^{\max}\right\| \left\|(0,0,\sqrt{\lambda_z^{\max}})\right\| + 2\left\|\widehat{b}_1^{\max}\right\| \left\|(0,0,\sqrt{\lambda_z^{\max}})\right\| \\
&= 4\sqrt{\lambda_z^{\max}}
\end{aligned}$$

where $\widehat{a}_1^{\max}$ is the unit vector parallel to $(0,0,\sqrt{\lambda_z^{\max}})$ and perpendicular to $(\sqrt{\lambda_x^{\max}},0,0)$; $\widehat{b}_1^{\max}$ is the unit vector antiparallel to $(0,0,\sqrt{\lambda_z^{\max}})$ and perpendicular to $(\sqrt{\lambda_x^{\max}},0,0)$.

Thus, the maximum expectation value of the Mermin operator with respect to the state ρ is given by

$$\left\langle B_M^{\max} \right\rangle_\rho = \max\left\{ \left\langle B_M^{(4)} \right\rangle_\rho , \left\langle B_M^{(5)} \right\rangle_\rho , \left\langle B_M^{(6)} \right\rangle_\rho \right\}$$

$$= \max\left\{ 4\sqrt{\lambda_x^{\max}}, 4\sqrt{\lambda_y^{\max}}, 4\sqrt{\lambda_z^{\max}} \right\}$$

The Mermin inequality is violated if

$$\left\langle B_M^{\max} \right\rangle_\rho > 2 \Rightarrow \max\left\{ 4\sqrt{\lambda_x^{\max}}, 4\sqrt{\lambda_y^{\max}}, 4\sqrt{\lambda_z^{\max}} \right\} > 2$$

Result-3: If ${T_x}^T T_x$ has two equal largest eigenvalue $\lambda_x^{\max}$ and ${T_y}^T T_y$, and ${T_z}^T T_z$ have unique largest eigenvalue $\lambda_y^{\max}, \lambda_z^{\max}$ respectively then Mermin's inequality is violated if

$$\left\langle B_M^{\max} \right\rangle_\rho = \max\left\{4\sqrt{\lambda_x^{\max}}, 2\sqrt{\lambda_y^{\max}}, 2\sqrt{\lambda_z^{\max}}\right\} > 2$$

Result-4: If ${T_y}^T T_y$ has two equal largest eigenvalue $\lambda_y^{\max}$ and ${T_x}^T T_x$, and ${T_z}^T T_z$ have unique largest eigenvalues $\lambda_x^{\max}, \lambda_z^{\max}$ respectively then Mermin's inequality is violated if

$$\left\langle B_M^{\max} \right\rangle_\rho = \max\left\{2\sqrt{\lambda_x^{\max}}, 4\sqrt{\lambda_y^{\max}}, 2\sqrt{\lambda_z^{\max}}\right\} > 2$$

Result-5: If ${T_z}^T T_z$ has two equal largest eigenvalue $\lambda_z^{\max}$ and ${T_x}^T T_x$, and ${T_y}^T T_y$ have unique largest eigenvalue $\lambda_x^{\max}, \lambda_y^{\max}$ respectively then Mermin's inequality is violated if

$$\left\langle B_M^{\max} \right\rangle_\rho = \max\left\{2\sqrt{\lambda_x^{\max}}, 2\sqrt{\lambda_y^{\max}}, 4\sqrt{\lambda_z^{\max}}\right\} > 2$$

Result-6: If ${T_x}^T T_x$ and ${T_y}^T T_y$ has two equal largest eigenvalues $\lambda_x^{\max}, \lambda_y^{\max}$ respectively and ${T_z}^T T_z$ has unique largest eigenvalue $\lambda_z^{\max}$, then Mermin's inequality is violated if

$$\left\langle B_M^{\max} \right\rangle_\rho = \max\left\{4\sqrt{\lambda_x^{\max}}, 4\sqrt{\lambda_y^{\max}}, 2\sqrt{\lambda_z^{\max}}\right\} > 2$$

Result-7: If ${T_x}^T T_x$ and ${T_z}^T T_z$ has two equal largest eigenvalues $\lambda_x^{\max}, \lambda_z^{\max}$ respectively and ${T_y}^T T_y$ has unique largest eigenvalue $\lambda_y^{\max}$, then Mermin's inequality is violated if

$$\left\langle B_M^{\max} \right\rangle_\rho = \max\left\{4\sqrt{\lambda_x^{\max}}, 2\sqrt{\lambda_y^{\max}}, 4\sqrt{\lambda_z^{\max}}\right\} > 2$$

Result-8: If ${T_y}^T T_y$ and ${T_z}^T T_z$ has two equal largest eigenvalues $\lambda_y^{\max}, \lambda_z^{\max}$ respectively and ${T_x}^T T_x$ has unique largest eigenvalue $\lambda_x^{\max}$, then Mermin's inequality is violated if

$$\left\langle B_M^{\max} \right\rangle_\rho = \max\left\{2\sqrt{\lambda_x^{\max}}, 4\sqrt{\lambda_y^{\max}}, 4\sqrt{\lambda_z^{\max}}\right\} > 2$$

Example-1

Generalised GHZ is given by $|\psi_{GHZ}\rangle = \alpha|000\rangle + \beta|111\rangle$.

The matrices $T_x^T T_x, T_y^T T_y$ are given by

$$T_x^T T = T_y^T T_y = \begin{pmatrix} 4\alpha^2\beta^2 & 0 & 0 \\ 0 & 4\alpha^2\beta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix $T_z^T T_z$ is a zero matrix.

The largest eigenvalues of $T_x^T T_x, T_y^T T_y$ are given by

$\lambda_x^{\max} = \lambda_y^{\max} = 4\alpha^2\beta^2$.

The maximum expectation value of the Mermin operator with respect to the state $|\psi_{GHZ}\rangle$ is given by $\left\langle B_M^{\max} \right\rangle_{|\psi_{GHZ}\rangle\langle\psi_{GHZ}|} = 8\alpha\beta$.

Therefore, the state $|\psi_{GHZ}\rangle$ violates Mermin's inequality is

$2\alpha\beta > \dfrac{1}{2}$. The same result has been found by Scarani and Gisin.

[V. Scarani, and N. Gisin, J. Phys. A 34, 6043 (2001).]

Example-2

Let us consider a pure state $|\psi\rangle_{W,S} = \sqrt{1-p}\,|W\rangle + \sqrt{p}\,|000\rangle.$

where $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$

The matrices $T_x^T T_x, T_y^T T_y$ are given by

$$T_x^T T_x = \begin{pmatrix} \frac{4}{9}(1-p)^2 & 0 & \frac{4}{3\sqrt{3}}\sqrt{p}(1-p)^{\frac{3}{2}} \\ 0 & 0 & 0 \\ \frac{4}{3\sqrt{3}}\sqrt{p}(1-p)^{\frac{3}{2}} & 0 & \frac{4}{9}(1-p)(1+2p) \end{pmatrix},$$

$$T_y^T T_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{9}(1-p)^2 & 0 \\ 0 & 0 & \frac{4}{9}(1-p)^2 \end{pmatrix}$$

$$T_z^{\,T} T_z = \begin{pmatrix} \frac{4}{9}(1-p)(1+2p) & 0 & f \\ 0 & \frac{4}{9}(1-p)^2 & 0 \\ f & 0 & g \end{pmatrix}$$

$$\text{where } f = \frac{4}{3\sqrt{3}}\sqrt{p}\,(1-p)^{\frac{3}{2}} + \frac{2}{\sqrt{3}}\sqrt{p}\sqrt{1-p}\,(2p-1)$$

$$g = \frac{4}{3}p(1-p) + (2p-1)^2$$

The largest eigenvalues of $T_x^T T_x, T_y^T T_y$ and $T_z^T T_z$ are given by

$$\lambda_x^{\max} = (1-p)(\frac{4}{9}+\frac{2p}{9}+\frac{2}{9}\sqrt{12p-3p^2}), \lambda_y^{\max} = \frac{4}{9}(1-p)^2$$

$$\lambda_z^{\max} = \frac{1}{18}\sqrt{256p^4-640p^3+672p^2-232p+25}+\frac{13}{18}+\frac{8}{9}p^2-\frac{10}{9}p$$

The maximum expectation value of the Mermin operator with respect to the state $|\psi\rangle_{W,S}$ is given by

$$\langle B_M^{\max}\rangle_{|\psi\rangle_{W,S}\langle\psi|} = 4\sqrt{\lambda_y^{\max}}\ , \ 0 \le p \le 0.43$$

$$= 2\sqrt{\lambda_x^{\max}}\ , \ 0.43 \le p \le 0.45$$

$$= 2\sqrt{\lambda_z^{\max}}\ , 0.45 \le p \le 1$$

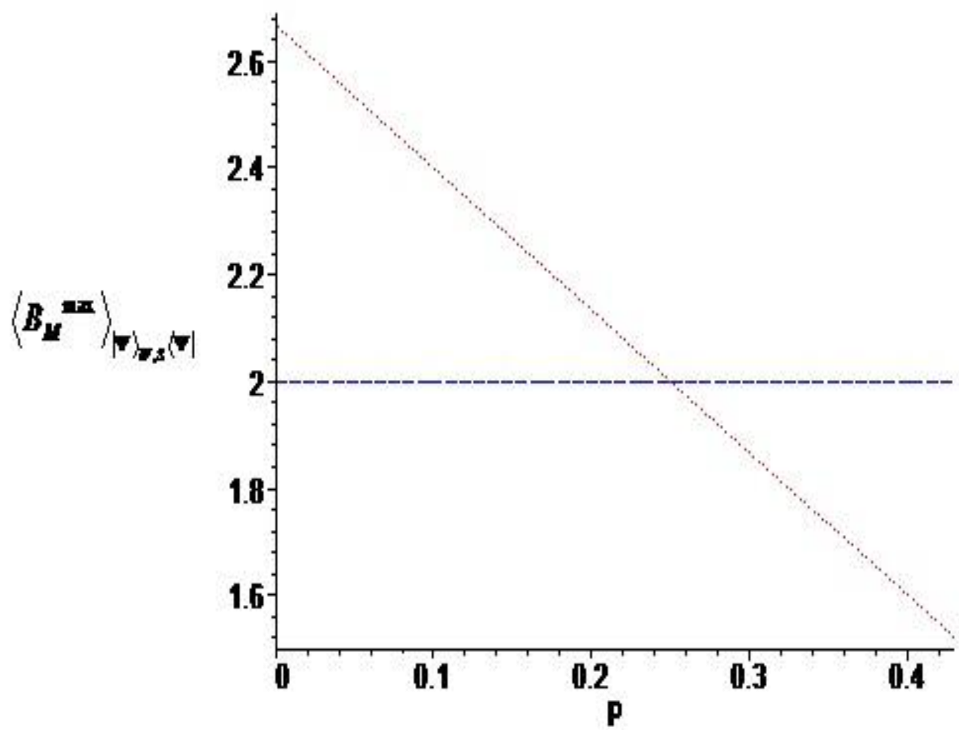

*Fig.*1(*a*)

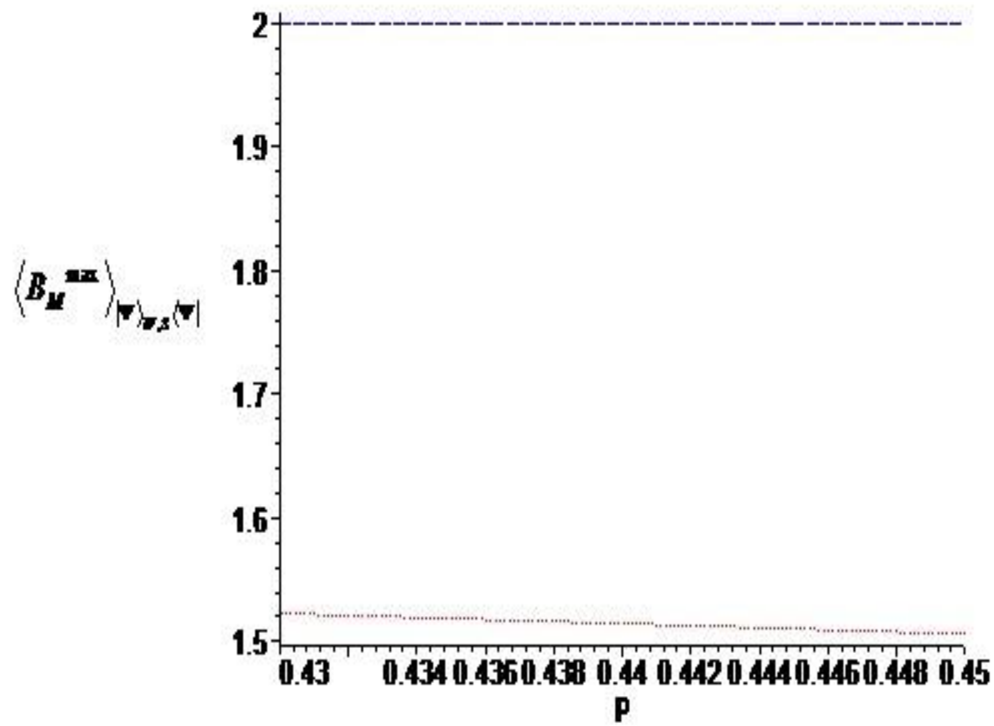

Fig.1(b)

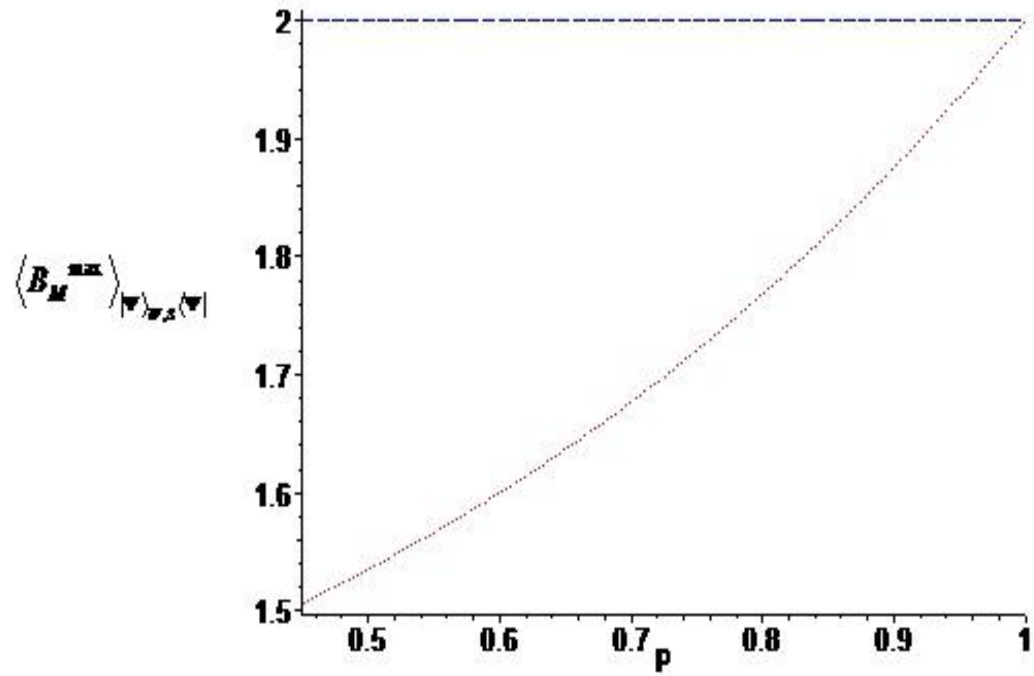

*Fig.*1(*c*)

The state $\left|\psi\right\rangle_{W,S}$ violates Mermin inequality when $0 \leq p \leq 0.25$.

This result was obtained by Chi et.al.

D. P. Chi, K. Jeong, T. Kim, K. Lee, and S. Lee, Phys. Rev. A 81, 044302 (2010).

THANK YOU