NONLOCALITY ARGUMENTS

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Why to study?

Quantum nonlocality (!) is fascinating as:

- **It relates quantum theory with special relativity,**
- **It is an invaluable resource in many of those information processing tasks where quantum theory has got an edge over classical theory.**

What we mean by Quantum nonlocality?

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It gives accurate predictions for statistical distributions of outcomes obtained in a real experiment. However, it does not tell which outcome will be observed in a particular measurement-experiment unless the state undergoing the measurement is an eigenstate of the observable being measured.

What we mean by Quantum nonlocality?

Like any other theory, quantum theory too has some postulates (The List is not complete) :

- Associated with every quantum mechanical system S, there is a Hilbert space $\mathcal{H}_S$, whose dimension depends on the nature of the degree of freedom being considered for the system.
- **State of a system: Unit Vectors $|\psi\rangle \in \mathcal{H}_S$**

- **An observable $\hat{\mathcal{A}}$ for the system S is associated with a self-adjoint operator, acting on $\mathcal{H}_S$**
$$\hat{\mathcal{A}} = \sum_i a_i |\alpha_i\rangle\langle\alpha_i|$$
- **Possible measurement results of $\hat{\mathcal{A}}$ are the eigenvalues of $\hat{\mathcal{A}}$**
- **Outcome probability: $p(a_i) = |\langle\psi|\alpha_i\rangle|^2$ (Born's rule)**

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- **Possible results of measurement are 'up' with probability** $\frac{1}{2}$ **and 'down' with probability** $\frac{1}{2}$**.**
- **But, in a particular run, what would be the result of measurement; quantum theory cannot predict.**

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But the situation becomes subtler when we consider different measurements on two (or more) correlated quantum systems.

Bell scenario

A typical Bell experiment involves two spatially separated observers, Alice and Bob, who share a physical system consisting of two subsystems. They can perform measurements on the subsystems in their possessions and collect statistics to calculate the joint probabilities $p(a, b|AB, P)$. Here A and B denotes the observables chosen respectively by Alice and Bob; a and b are the corresponding outcomes. Each pair of subsystems is prepared by an agreed-upon reproducible procedure P (which in quantum mechanics is represented by quantum state for the pair of subsystems).

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- **The prediction for the observed joint probability by this model**

$$\int_\Lambda p(a, b|A, B, P, \lambda) p_P(\lambda)\ d\lambda = \mathrm{Prob}(a, b|A, B, P)(\mathrm{say}). \quad (1)$$

R W Spekkens *Phys. Rev. A* **71**, 052108 (2005)
E G Cavalcanti, H M Wiseman *Found. Phys.* **42**, *1329* (2012)

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- This implies that the outcome of Bob's measurement (and similarly also of Alice's measurement) is determined by A, B, P and λ only. Thus, for a deterministic ontological model

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- **A model is said to satisfy locality iff**

$$\begin{aligned} p(a|A, B, P, \lambda) &= p(a|A, P, \lambda)\ \forall\ a, A, B, \\ p(b|A, B, P, \lambda) &= p(b|B, P, \lambda)\ \forall\ b, A, B. \end{aligned} \tag{3}$$

H M Wiseman, J. Phys. A: Math. Theor. 47 424001 (31pp) (2014)

- **By Baye's theorem of conditional probability:**

$$p(a, b|A, B, P, \lambda) = p(a|A, B, P, \lambda)\ p(b|A, a, B, P, \lambda). \tag{4}$$

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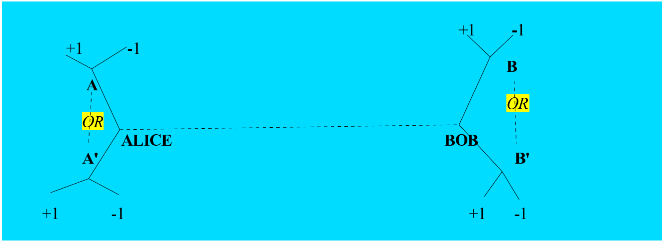
- **Thus in a local-deterministic model, Eq. (1), takes the following form**

$$\mathrm{Prob}(a, b|A, B, P) = \int_\Lambda p(a|A, P, \lambda)\ p(b|B, P, \lambda) p_P(\lambda)\ d\lambda. \tag{6}$$

Bell's nonlocality argument



Consider a physical system consisting of two subsystems shared between Alice and Bob. The two observers (Alice and Bob) have access to one subsystem each. For each pair of subsystems, the choices of observables and their respective outcomes occur in regions which are space-like separated from each other. Assume that Alice can run the experiments of measuring any one (freely chosen) of the two $\{+1,-1\}$-valued random variables A and $A^{'}$ corresponding to her subsystem whereas Bob can run the experiments of measuring any one (chosen freely) of the two $\{+1,-1\}$-valued random variables B and $B^{'}$ corresponding to the subsystem in his possession.

Assignment1: Show that in a local-deterministic (local-realistic) model

$$|\langle AB\rangle + \langle AB'\rangle + \langle A'B\rangle - \langle A'B'\rangle|_{LRT} \leq 2$$

This inequality is famously known as Bell's inequality.

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- **You will also need to assume**

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- In case of difficulty, please go through Rev. Mod. Phys. 86, 839 (2014) (page 3, para 2 of its arxiv version: arXiv 1303.2849v2)

The assumption of free will

Measurement independence is the property that the distribution of the underlying variable is independent of the measurement settings, i.e.,

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i.e., the measurement settings can be chosen freely (independent of the underlying variable λ)

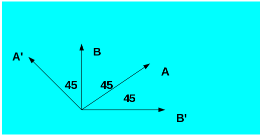
M J W Hall *Phys. Rev. Lett.* **105**, 250404 (2010)

QM violates BI

For the purpose of showing that the above inequality may get violated in quantum mechanics, consider a system of two spin-1/2 particles in the state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\Psi_z\rangle_A \otimes |\Psi_{-z}\rangle_B - |\Psi_{-z}\rangle_A \otimes |\Psi_z\rangle_B)$$

The observables A and A' for particle 1 and B and B' for particle 2 are chosen as spin observables in the following way



It can be shown that (please show):

$$\langle\Psi|\boldsymbol{\sigma.n_i} \otimes \boldsymbol{\sigma.n_j}|\Psi\rangle = -\boldsymbol{n_i.n_j}$$

Thus

$$|\langle AB\rangle + \langle AB'\rangle + \langle A'B\rangle - \langle A'B'\rangle|_{QM} = 2\sqrt{2}$$

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This is what we mean by quantum nonlocality.

Can this nonlocality be used to signal?

(Bell's inequality from operational assumptions)

- **No-signalling (signal locality) is said to be satisfied iff**

$$\begin{aligned} p(a|A,B,P) &= p(a|A,P)\ \forall\ a,A,B, \\ p(b|A,B,P) &= p(b|B,P)\ \forall\ b,A,B. \end{aligned} \tag{7}$$

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- Bohm's model is an example of a model that violates locality but not signal locality.

Bell's inequality from operational assumptions

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E G Cavalcanti, H M Wiseman *Found. Phys.* **42**, *1329* (2012)

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- **We now show: predictability $\wedge$ signal locality $\Rightarrow$ factorizability relation**

E. G. Cavalcanti, Howard M. Wiseman *Found. Phys.* **42**, 1329 (2012).

Bell's inequality from operational assumptions

The assumption of predictability implies:

$$p(a, b|A, B, P, \lambda) = p(a, b|A, B, P). \tag{10}$$

This is because $p(a, b|A, B, P) \in \{0, 1\}$ hence conditioning it on further variable(s) cannot alter it.

According to Baye's theorem,

$$p(a, b|A, B, P) = p(a|A, B, P)p(b|A, a, B, P). \tag{11}$$

The assumption of predictability implies that $b = f(A, B, P)$ (i.e., b is specified by specifying A, B, and P) and hence

$$p(b|A, a, B, P) = p(b|A, B, P). \tag{12}$$

BI from $\cdots$

Putting for $p(b|A, a, B, P)$ from Eq. (12) into Eq. (11), we get

$$p(a, b|A, B, P) = p(a|A, B, P)p(b|A, B, P) \tag{13}$$

From the assumption of signal locality

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Putting this into Eq. (10), we get

$$p(a, b|A, B, P, \lambda) = p(a|A, P)p(b|B, P). \tag{15}$$

By conditioning the RHS of the above equation on λ, we get the desired factorizability relation.

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Randomness Certification by Bell's Theorem

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Randomness Certification by Bell's Theorem

- **Consider a situation where BI is violated but no-signalling is satisfied (!).**
- **Such a situation guarantees the presence of true randomness.**

This is important as

- Randomness is a valuable resource for various important tasks.
- For these tasks, the genuineness of the used randomness is of primary concern.
- Traditionally there have been two approaches to generate random numbers:
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S.Pironio et al., *Nature* **464** 1021 (2010)

Bell inequality as measure of entanglement

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- The Hardy's argument of "nonlocality without inequality" has been considered to be the "best version of Bell's theorem".

N.D. Mermin, *Am. J.Phys.* **62**, 880 (1994)

Hardy's nonlocality argument

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- **Consider now the following four conditions**:

$$\begin{aligned} \mathrm{Prob}(+1,+1|A,\ B,\ P) &> 0, & (16)\\ \mathrm{Prob}(-1,+1|A^{'},B,\ P) &= 0, & (17)\\ \mathrm{Prob}(+1,-1|A,\ B^{'},\ P) &= 0, & (18)\\ \mathrm{Prob}(+1,+1|A^{'},\ B^{'},\ P) &= 0. & (19) \end{aligned}$$

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- **The above four conditions together form the basis of Hardy's nonlocality argument.**
- **The first condition says that in an experiment in which Alice chooses to measure the observable A and Bob chooses the observable B, the probability that both will get $+1$ as measurement outcomes is nonzero.** Other conditions can be analyzed similarly.
- **The Hardy's nonlocality argument makes use of the fact that these four conditions cannot be fulfilled simultaneously in the framework of a local-realistic theory, but they can be in quantum mechanics.**

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- **However, for $\lambda \in \Lambda'$, this condition implies $p(-1|A^{'}P,\lambda) = 0$ or equivalently for these λ's, $p(+1|A^{'}P,\lambda) = 1$.**
- **Similar reasoning for the third Hardy's condition provides** $p(+1|B^{'}P,\lambda) = 1$ for $\lambda \in \Lambda'$.

Hardy's condition and local-realistic $\cdots$

Thus a local- realistic model will predict for the last probability in the Hardy's condition as follows:

$$\begin{aligned} \mathrm{Prob}(+1,+1|A^{'},\ B^{'},\ P) &= \int_{\Lambda} p(+1|A^{'},P,\lambda)\ p(+1|B^{'},P,\lambda)p(\lambda)\ d\lambda \\ &\geq \int_{\Lambda'} p(+1|A^{'},P,\lambda)\ p(+1|B^{'},P,\lambda)p(\lambda)\ d\lambda \\ &= \int_{\Lambda'} p(\lambda)\ d\lambda > 0. \end{aligned} \tag{21}$$

Every pure nonmaximally entangled state of two-qubits exhibits Hardy's nonlocality

Any pure nonmaximally entangled state $|\psi\rangle$ of two spin-1/2 particles can be written as

$$|\psi\rangle = a|v_1\rangle \otimes |v_2\rangle + b|u_1\rangle \otimes |v_2\rangle + c|v_1\rangle \otimes |u_2\rangle \, (abc \neq 0)$$

for a proper choice of orthonormal basis $\{|u_i\rangle, |v_i\rangle\}$ for i-th particle, $i = 1, 2$; $|u_1\rangle, |u_2\rangle$ need not bear any relationship with each other.

S. Goldstein, *Phys. Rev. Lett.* **72** (1994)

Assignment2: Check that this state satisfies all the four Hardy's conditions for the following choice of observables:

$$
\begin{aligned}
A &= |w_1^\perp\rangle\langle w_1^\perp| - |w_1\rangle\langle w_1|, \\
A' &= |u_1\rangle\langle u_1| - |v_1\rangle\langle v_1|, \\
B &= |w_2^\perp\rangle\langle w_2^\perp| - |w_2\rangle\langle w_2|, \\
B' &= |u_2\rangle\langle u_2| - |v_2\rangle\langle v_2|
\end{aligned}
$$

where

$$
\begin{aligned}
|w_1\rangle &= \frac{a|v_1\rangle + b|u_1\rangle}{\sqrt{|a|^2 + |b|^2}}, \\
|w_2\rangle &= \frac{a|v_2\rangle + c|u_2\rangle}{\sqrt{|a|^2 + |c|^2}}.
\end{aligned}
$$

Thank You