# QUANTUM CIRCUIS and SIMPLE QUANTUM ALGORITHMS

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2016

# Part I

# Seminal quantum algorithms

# SEMINAL QUANTUM ALGORITHMS

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February 6, 2016

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- The fastest classical algorithm to factor an $m$-bit integer that is product of two primes has exponential complexity $\mathcal{O}(e^{cm^{1/3}(\lg m)^{2/3}})$, where $e$ is the basis of natural logarithms.
- The fact that there is not known classical algorithm to factor integers is playing very important role in cryptography - for making secure encryptions and secure digital signatures.

## LARGEST PRIME

On February 3, 2016 C. Cooper from university Missouri announced a new (Mersene) prime

$$2^{74207181}$$

that has 5 millions more digits as previously known largest prime.

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Trapdoor information: $p, q, d$ (decryption exponent)

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**Plaintext** $w$
**Encryption:** cryptotext $c = w^e \bmod n$
**Decryption:** plaintext $w = c^d \bmod n$

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**Decrypt the cryptotext:**

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**encrypted using the RSA cryptosystem with 129 digit number, called also RSA129**

n: 114 381 625 757 888 867 669 235 779 976 146 612 010 218 296 721 242 362 562 561 842 935 706 935 245 733 897 830 597 123 513 958 705 058 989 075 147 599 290 026 879 543 541.

and with $e = 9007$.

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The problem was solved in 1994 by first factorizing n into one 64-bit prime and one 65-bit prime, and then computing the plaintext

THE MAGIC WORDS ARE SQUEMISH OSSIFRAGE

# FROM RSA CRYPTOSYSTEM to RSA SIGNATURES

The idea of RSA cryptosystem is simple.
Public key: modulus $n = pq$ and encryption exponent $e$.
Secret key: decryption exponent $d$ and primes $p, q$

Encryption of a message $w$: $c = w^e$
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and $w = s^e$ can then be verification of such a signature.

# WHY ARE RSA ENCRYPTIONS and SIGNATURES SECURE?

RSA encryptions and signatures are considered as secure because there is not known a methods that could be able factorize in RSA used moduly on current and in near future forseeable classical supercomputers.

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**Proof.** Let $a \neq \pm 1$ be such that $a^2 \equiv 1 \pmod{n}$. Since

$$a^2 - 1 = (a+1)(a-1),$$

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By using Euclid's algorithm to compute

$$gcd(a+1, n) \text{ and } gcd(a-1, n)$$

we can find, in $O(\lg n)$ steps, a prime factor of $n$.

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Period is the smallest integer $r$ such that

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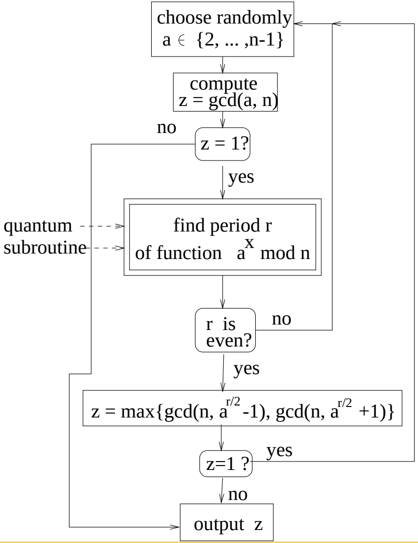
2. *Compute* $gcd(a,n)$. *If* $gcd(a,n) \neq 1$ *we have a factor.*
3. *Find period* $r$ *of function* $a^k \bmod n$.
4. *If* $r$ *is odd or* $a^{r/2} \equiv \pm 1 \pmod{n}$,*then go to step 1; otherwise stop.*

If this algorithm stops, then $a^{r/2}$ is a non-trivial solution of the equation

$$x^2 \equiv 1 \pmod{n}.$$

# A GENERAL SCHEME for Shor's ALGORITHM

The following flow diagram shows the general scheme of Shor's quantum factorization algorithm



# SHOR's FACTORIZATION ALGORITHM

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2. By measuring the last register the state collapses into the state

$$\frac{1}{\sqrt{A+1}}\sum_{j=0}^{A}|n,a,q,jr+l,y\rangle \text{ or, shortly } \frac{1}{\sqrt{A+1}}\sum_{j=0}^{A}|jr+l\rangle,$$

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5. By measuring the resulting state we get $c = \frac{jq}{r}$ and if $gcd(j,r) = 1$, what is very likely, then from $c$ and $q$ we can determine the period $r$.

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Discrete Fourier Transform maps a vector $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})^T$ into the vector $DFT(\mathbf{a}) = A_n\mathbf{a}$, where $A_n$ is an $n \times n$ matrix such that $A_n[i,j] = \frac{1}{\sqrt{n}}\omega^{ij}$ for $0 \leq i, j < n$ and $\omega = e^{2\pi i/n}$ is the $n$**th root of unity**.

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The matrix $A_n$ has therefore the form

$$A_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{(n-1)} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{(n-1)} & \omega^{2(n-1)} & \dots & \omega^{(n-1)^2} \end{pmatrix}.$$

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The Inverse Discrete Fourier Transform is the mapping

$$DFT^{-1}(\mathbf{a}) = A_n^{-1}\mathbf{a},$$

where

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where for any $c \in \{0, \ldots, q-1\}$

$$\bar{f}(c) = \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} e^{2\pi iac/q} f(a), \tag{1}$$

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$$\bar{f}(c) = \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} e^{2\pi iac/q} f(a), \tag{1}$$

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$$F_q = \frac{1}{\sqrt{q}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{(q-1)} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(q-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{(q-1)} & \omega^{2(q-1)} & \dots & \omega^{(q-1)^2} \end{pmatrix},$$

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$$\alpha_c = \begin{cases} \frac{1}{\sqrt{r}} e^{2\pi i l_y c/q}, & \text{if } c \text{ is a multiple of } \frac{q}{r}; \\ 0, & \text{otherwise;} \end{cases}$$

Therefore

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The key point now is that the trouble-making offset $l_y$ appears now in the phase factor $e^{2\pi i l_y j/r}$ and has influence neither on the probabilities nor on values obtained by the measurement.

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In the general case, i.e., if

$$A \neq \frac{q}{r} - 1,$$

there is only a more sophisticated computation of the resulting probabilities and a more sophisticated way to determine $r$ (using a continuous fraction method to extract the period from its approximation).

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- To factor an integer $n$ Shor's algorithm uses $\mathcal{O}(\lg^3 n)$ steps and success probability is guaranteed to be at least $\Omega(\frac{1}{\lg \lg n})$.
- An analysis of Shor's algorithm therefore shows that by running the algorithm $\mathcal{O}(\lg \lg n)$ times, therefore in total in $\mathcal{O}(\lg^3 n \lg \lg n)$ times we have very high success probability.

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The exponent in th $l$-th factor of the above decomposition can be written as follows

$$\exp(\frac{\pi i(2^{m-1}x_{m-1} + 2^{m-2}x_{m-2} + \dots + 2x_1 + x_0)}{2^{l-1}})$$

$$= \exp(\frac{\pi i(2^{l-1}x_{l-1} + 2^{l-2}x_{l-2} + \dots + 2x_1 + x_0)}{2^{l-1}})$$

$$= (-1)^{x_{l-1}} exp(\frac{\pi ix_{l-2}}{2}) \dots exp(\frac{\pi ix_1}{2^{l-2}}) exp(\frac{\pi ix_0}{2^{l-1}})$$

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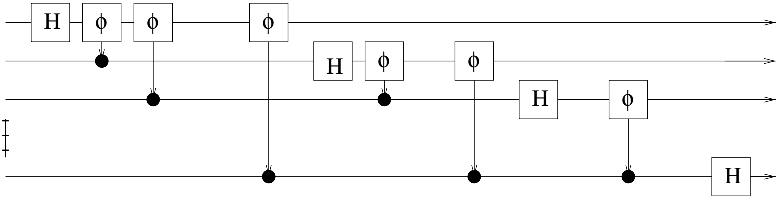


It has therefoe $O(n^2)$ gates.

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**Given is:** An (efficiently computable) function $f : G \to R$, where $G$ is a group and $R$ is a finite set.

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# QUANTUM SEARCHING
# in
# UNORDERED SETS

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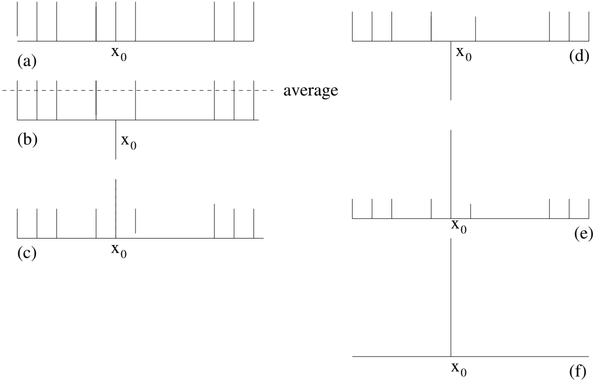
**Grover's quantum algorithm** exists that needs $\mathcal{O}(\sqrt{N})$ steps.

Here is the basic idea of the algorithm - "cooking" a solution.



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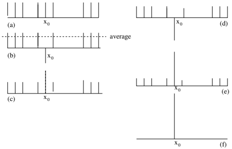

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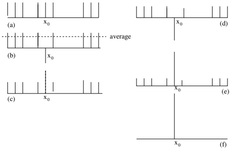


The figure above demonstrates some steps of the Grover algorithm.

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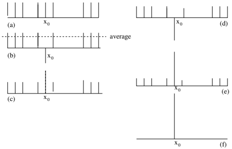


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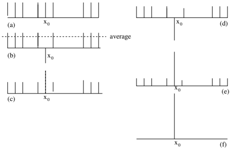


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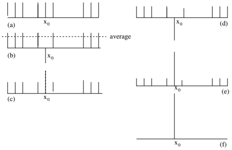


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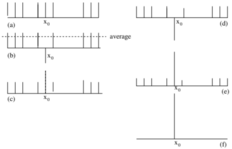


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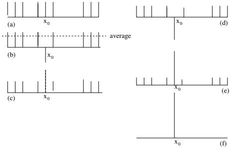


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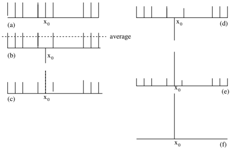


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- Figure (e), shows the result after another inversion about the average. In case this process iterates for a proper number of steps we get that the amplitude at state $|x_0\rangle$ is (almost) 1 and amplitudes at other states are (almost) 0. A measurement in such a situation produces $x_0$ as the classical outcome.

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Inversion about the average is the unitary transformation

$$D_n : \sum_{i=0}^{2^n-1} a_i|\phi_i\rangle \to \sum_{i=0}^{2^n-1} (2E - a_i)|\phi_i\rangle,$$

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where $E$ is the average of $\{a_i \,|\, 0 \le i < 2^n\}$. This important transformation can be performed by the matrix

$$-H_n V_0^n H_n = D_n = \begin{pmatrix} -1 + \frac{2}{2^n} & \frac{2}{2^n} & \dots & \frac{2}{2^n} \\ \frac{2}{2^n} & -1 + \frac{2}{2^n} & \ddots & \frac{2}{2^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{2^n} & \frac{2}{2^n} & \dots & -1 + \frac{2}{2^n} \end{pmatrix}.$$

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The matrix $D_n$ is clearly unitary and it can be shown to have the form $D_n = -H_n V_0^n H_n$, where

$$V_0^n[i,j] = 0 \text{ if } i \neq j,\, V_0^n[1,1] = -1 \text{ and } V_0^n[i,i] = 1 \text{ if } 1 < i \leq n.$$

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**Example:** Let $a_i = a$ if $i \neq x_0$ and $a_{x_0} = -a$. Then

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Therefore, the value of $2E - a_i$ is smaller than $a$ if $i \neq i_0$, and increases otherwise - if $i = i_0$.

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and iterate $\lfloor \frac{\pi}{4}\sqrt{2^n} \rfloor$ times the transformation (so called Grover's iterate):

$$-\underbrace{H_n V_0^n H_n V_f}\,|\phi\rangle \rightarrow |\phi\rangle.$$

Afterwords, measure the register to get some $x_1$ (hopefully $x_0$) and check whether $f(x_1) = 1$. If not, repeat the procedure.

It has been shown that the above algorithm is optimal for finding the solution with probability $> \frac{1}{2}$.

In the case that there are $t$ solutions, repeat the above iteration

$$\left\lfloor \frac{\pi}{4}\sqrt{\frac{2^n}{t}} \right\rfloor \quad \text{times.}$$

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Hence

$$j = \frac{\pi}{4\theta} - \frac{1}{2} + \frac{m\pi}{2\theta}$$

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The aim now is to find such an $j$ which maximizes $k_j$ and minimizes $l_j$. For doing that we need to take $j$ such that $\cos((2j+1)\theta) = 0$, that is $(2j+1)\theta = (2m+1)\frac{\pi}{2}$.

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# QUANTUM SEARCH in ORDERED LISTS

A related problem to that of a search in an unordered list is a search in an ordered list of $n$ items.

- The best upper bound known today is $\frac{3}{4}\lg n$.
- The best lower bound known today is $\frac{1}{12}\lg n - \mathcal{O}(1)$.

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- $\varepsilon$ can be at most $\frac{1}{2^{n^\alpha}}$ using $\mathcal{O}(n^{0.5+\alpha})$ queries.
- To achieve no error ($\varepsilon = 0$), $\theta(n)$ queries are needed.

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  It has been shown that also more generalized formulas of the type

  $$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots \forall x_k \exists y_k P(x_1, y_1, x_2, y_2, \ldots, x_k, y_k)$$

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   3. Measure the first register. If $y'$ is the outcome and $s_{y'} < s_y$, take as a new threshold the index $y'$.
3. Return as the output the last threshold $y$.

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Using $\mathrm{QFT}_{p-1}$ twice, on the third and fourth sub-register of the state $|x, g, \mathbf{0}, \mathbf{0}, \mathbf{0}\rangle$, we get

$$|\phi\rangle = \frac{1}{p-1}\sum_{a=0}^{p-2}\sum_{b=0}^{p-2}|x, g, a, b, \mathbf{0}\rangle,$$

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As the next step we apply $\mathrm{QFT}_{p-1}$ on $|\phi'\rangle$ twice, once to map each $a$ to each $c$ with the amplitude $\frac{1}{\sqrt{p-1}} e^{2\pi i ac/(p-1)}$ and once to map each $b$ to each $d$ with amplitude $\frac{1}{\sqrt{p-1}} e^{2\pi i bd/(p-1)}$.

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The resulting state will be:

$$|\phi_1\rangle = \frac{1}{(p-1)^2}\sum_{a,b,c,d=0}^{p-2} e^{\frac{2\pi i}{p-1}(ac+bd)}|c, d, g^a x^{-b} \bmod p\rangle.$$

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By substituting $a = k + rb + j_b(p-1)$ we get as the probability

$$Pr = \left| \frac{1}{(p-1)^2} \sum_{b=0}^{p-2} e^{\frac{2\pi i}{p-1}(kc+cj_b(p-1)+b(d+rc))} \right|^2 = \left| \frac{1}{(p-1)^2} e^{\frac{2\pi ikc}{p-1}} \sum_{b=0}^{p-2} e^{\frac{2\pi i}{p-1}(b(d+rc))} \right|^2$$

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The above probability $Pr$ is therefore 0 if

$$d + rc \not\equiv 0 \bmod (p-1)$$

because, as in the previous algorithm, in such a case the sum in the above expression is over a set of complex numbers equally spaced around the unit circle.

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then the above sum does not depend on $b$ and it is equal to

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The square of its absolute value, the probability, is therefore $\frac{1}{(p-1)^2}$.

Consequence: the measurements on the first and second register provide a (random) $c < p-1$ and a $d$ such that

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If $gcd(c, p-1) = 1$, $r$ can now be obtained as a unique solution of the above congruence equation.

The above probability $Pr$ is therefore 0 if

$$d + rc \not\equiv 0 \text{ mod } (p-1)$$

because, as in the previous algorithm, in such a case the sum in the above expression is over a set of complex numbers equally spaced around the unit circle.

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If $gcd(c, p-1) = 1$, $r$ can now be obtained as a unique solution of the above congruence equation.

The number of computations needed to be performed, in order to get the probability close to 1 for finding $r$, is polynomial in $\lg\lg p$.

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- Shor's result have been generalized to show that a large range of cryptosystems, including elliptic curve cryptosystems, would be vulnerable to attacks using quantum computers.

# FOURIER TRANSFORM on ABELIAN GROUPS

We show now basics how the concept of Fourier Transform is defined on any finite Abelian group.

# CHARACTERS on ABELIAN GROUPS

Let $G$ be an Abelian additive group and $|G| = n$. A character $\chi$ of $G$ is any morphism $\chi : G \to \mathbf{C}/0$. That means that for any $g_1, g_2 \in G$ it holds:

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Example 1 Any cyclic group of $n$ elements is isomorphic to the group $\mathbf{Z}_n$ and all its characters have the form, for some $y \in \mathbf{Z}_n$:

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Example 2 In the additive group $\mathbf{F_2}^m$, of all binary strings of length $m$, all characters have the form, for some binary $m$-bit strings $x$ and $y$:

$$\chi_y(x) = (-1)^{x \cdot y},$$

where $x \cdot y = \sum_{i=1}^{m} x_i y_i \bmod 2$

# ORTHOGONALITY of CHARACTERS

Any function $f : G \to \mathbf{C}$ on an Abelian group $G = \{g_1, \dots, g_n\}$ can be specified by the vector $(f(g_1), \dots, f(g_n))$, and if the scalar product of two functions is defined in the standard way as

$$\langle f|g\rangle = \sum_{i=1}^{n} f^*(g_i)h(g_i),$$

then for any characters $\chi_1$ and $\chi_2$ on $G$ it holds

$$\langle \chi_i|\chi_j\rangle = \begin{cases} 0, & \text{if } i \neq j \\ n, & \text{if } i = j \end{cases}$$

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Therefore, the functions $\{B_i = \frac{1}{\sqrt{n}}\chi_i\}$ form an orthonormal basis on the set of all functions $f : G \to \mathbf{C}$.

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Any function $f : G \rightarrow \mathbf{C}$ has a unique representation with respect to the basis $\{B_i = \frac{1}{\sqrt{n}}\chi_i\}_{i=1}^n$,

$$f = \hat{\mathrm{f}}_1 B_1 + \ldots + \hat{\mathrm{f}}_n B_n$$

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Therefore in $\mathbf{Z}_n$ the Fourier transform has the form

$$\hat{\mathrm{f}}(x) = \frac{1}{\sqrt{n}} \sum_{y \in \mathbf{Z}_n} e^{-\frac{2\pi ixy}{n}} f(y)$$

and in $\mathbf{F}_2^m$ the Fourier transform has the form

$$\hat{\mathrm{f}}(x) = \frac{1}{\sqrt{2^m}} \sum_{y \in \mathbf{F}_2^m} (-1)^{x \cdot y} f(y).$$