The death and rebirth of classical cryptography in a quantum world

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Outline

1. Pre-Quantum Cryptograpghy
   - Public Key Cryptography
   - RSA
2. Quantum Attacks on Classical Cryptosystems
   - Solving Hard Problems by Quantum Computers
   - Death of Classical Public Key Cryptography
   - Need for QKD
3. Quantum Cryptography
   - Quantum Key Distribution (QKD)
   - Other Quantum Cryptography Algorithms
4. Post-Quantum Cryptography
   - Rebirth of Classical Cryptography

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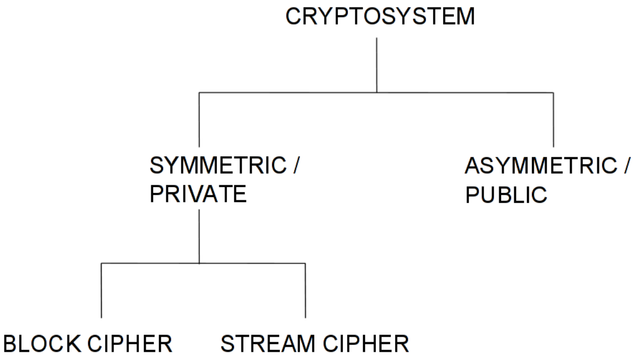
The Crypto World



PKC: Origin and History

Timeline

- 1976: The Idea - Whitfield Diffie and Martin Hellman
- 1976: Diffie and Hellman Key Exchange algorithm
- 1978: Rivest, Shamir and Adleman invented RSA

Actual Timeline (?) [announced in 1997]

- 1970: The Idea - James H. Ellis (British intelligence)
- 1973: Clifford Cocks developed RSA algorithm
- 1974: Malcom Williamson built Diffie-Hellman scheme

Public Key Framework

Goal: Alice and Bob communicate securely, avoiding Charles

Alice (receiver)	KEY GEN: Construct *related pair* of keys (public and private)
	KEY DIST: Publish public key and keep private key secret
Bob (sender)	GET KEY: Obtain an authentic Public Key of Alice
	ENCRYPT: Use it to encrypt message and send to Alice
Alice (receiver)	GET CIPHER: Obtain the ciphertext sent by Bob
	DECRYPT: Use Private Key to decrypt the ciphertext

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Example: RSA Cryptosystem

Key Gen	Choose two *large* primes p and q
	Compute the product $N = pq$
	Compute Euler's Totient function $\phi(N) = (p-1)(q-1)$
	Choose positive integer e such that $\gcd(e, \phi(N)) = 1$
	Compute d such that $ed \equiv 1 \pmod{\phi(N)}$
Key Dist	Public Key $= \langle N, e \rangle$ and Private Key $= \langle N, d \rangle$
Encryption	Message M produces Ciphertext $C = M^e \bmod N$
Decryption	Ciphertext C produces Message $M = C^d \bmod N$

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- Now he has to find the decryption exponent d which is e^{-1} in $\mathbb{Z}_{\phi(N)}$.
- One can check that $13 \times 395413 \equiv 1 \pmod{571152}$.
- Hence, the RSA parameters for Bob are
    - public key: $(13, 572681)$, and
    - private key: $(395413, 572681)$.

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- To encrypt a plaintext $m = 12345$, Alice uses Bob's public key $(13, 572681)$, and calculates $c = 12345^{13} \bmod 572681 = 536754$ and sends c to Bob.

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- To encrypt a plaintext $m = 12345$, Alice uses Bob's public key $(13, 572681)$, and calculates $c = 12345^{13} \bmod 572681 = 536754$ and sends c to Bob.
- To decrypt $c = 536754$, Bob calculates $536754^{395413} \bmod 572681 = 12345 = m$.

Correctness and Security of RSA

Correctness depends on EULER FERMAT theorem

$$a^{\phi(n)} \equiv 1 \pmod{n} \quad \text{if} \quad \gcd(n, a) = 1$$

Security depends on FACTORIZATION problem

Obtain factors p, q given product $N = pq$

Factoring Challenge

What	Digits	Who	When
7141075053842	13	Carissan (Machine à Congruences)	1919
9999000099990001	16	Lehmer (Bicycle Sieve)	1926
$2^{93}+1$	28	Lehmer (Gear Sieve)	1932
RSA-129	129	600 volunteers all over the world (MPQS)	1994
RSA-130	130	Lenstra and group (GNFS)	1996
RSA-140	140	Montgomery, Leyland, Dodson, Zimmermann, Lenstra (GNFS)	1999
RSA-155	155	Muffet, Leyland, Montgomery, Dodson, Morain, Guillerm,Marchand, Lenstra, Zimmermann, Gilchrist, Aardal, Putnam (GNFS)	1999
$2^{953}+1$	158	Bahr, Boehm, Franke, Kleinjung (GNFS)	2002
RSA-160	160	Bundesamt für Sicherheit in der Informationstechnik (BSI) Researchers (GNFS)	2002
RSA-576	174	Franke, Kleinjung, Montgomery, te Riele, Bahr, Leclair, Leyland, Wackerbarth (GNFS)	2003
$11^{281}+1$	176	Aoki, Kida, Shimoyama and Ueda (GNFS)	2005
RSA-640	193	Bahr, Boehm, Franke, Kleinjung (GNFS)	2005
RSA-200	200	Bahr, Boehm, Franke, Kleinjung (GNFS)	2005

Best: RSA-768 (232 digits) factored by several researchers in 2010 (over 2 years)

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Period finding problem

Let

$$f : \{0, 1, 2, \ldots, M-1\} \to \{0, 1, 2, \ldots, M-1\}$$

be a periodic function of period r, meaning that

$$f(x) = f(x+r) \quad \forall x \in \{0, 1, 2, \ldots, M-1\}$$

and the values $f(x), f(x+1), f(x+2), \ldots, f(x+r-1)$ are all distinct.

For simplicity, one can assume that $M = 2^m$ that $r \leq M/2$.
Finding the unknown period is a hard problem in classical computing.

Quantum Algorithm for finding period

4. **Create the quantum state** $\frac{1}{\sqrt{M}} \sum_x |x\rangle |f(x)\rangle$.
5. **Measure the last** m **bits of the state**: for an output $y = f(x_0)$ with the smallest possible x_0, the residual state is

$$\frac{1}{\sqrt{[\frac{M}{r}]}} \sum_{t=0}^{[\frac{M}{r}]-1} |x_0 + tr\rangle |f(x_0)\rangle.$$

6. Ignore the last n bits and apply the Fourier transform to the first m bits to get

$$\frac{1}{\sqrt{M}} \frac{1}{\sqrt{[\frac{M}{r}]}} \sum_s \sum_{t=0}^{[\frac{M}{r}]-1} \omega^{(x_0+tr)\cdot s} |s\rangle.$$

Quantum Algorithm for finding period (contd...)

1. Measurement gives an integer s with probability

$$\frac{1}{M} \cdot \frac{1}{[\frac{M}{r}]} |\omega^{x_0 s}|^2 \left| \sum_{t=0}^{[\frac{M}{r}]-1} \omega^{(x_0+tr)\cdot s} \right|^2 = \frac{1}{M} \cdot \frac{1}{[\frac{M}{r}]} \left| \sum_{t=0}^{[\frac{M}{r}]-1} \omega^{trs} \right|^2 .$$

2. This probability is higher, the closer the unit vector ω^{rs} is to the positive real axis, or the closer rs/M is to some integer c.
3. Known value $s/M \approx$ unknown value c/r.
   This information suffices to determine r.

Order finding problem

For $a \in \mathbb{Z}_N^*$, the order of $a \in \mathbb{Z}_N^*$ (or the order of a modulo N) is the *smallest* positive integer r such that

$$a^r \equiv 1(\mod N).$$

The order finding problem is to find the order of an element a, given an integer $N \geq 2$ and an element $a \in \mathbb{Z}_N^*$.

Classically this problem is hard. But, quantum period finding can be used to solve order finding.

Reducing factoring to order finding

- Suppose that the random choice of a is in $\mathbb{Z}_N^*$ (which is very likely), and that the order r of a is even.
- N divides $a^r - 1 = (a^{r/2} + 1)(a^{r/2} - 1)$.
- N cannot divide $a^{r/2} - 1$, otherwise $r/2 < r$ would have been the order.
- If $N \nmid a^{r/2} + 1$ (lucky case), $gcd(N, a^{r/2} - 1)$ gives a non-trivial factor of N.

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- So far, the largest number factored by a quantum computer is 56153, using 4 qubits in an NMR system (Chinese group, PRL 2012).

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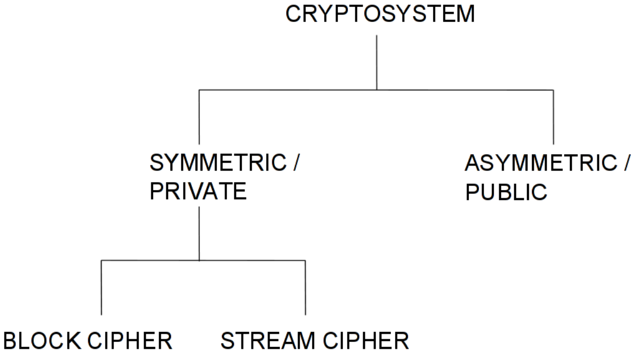
- Shor's algorithm for discrete logarithm can be generalized to find hidden subgroups in abelian groups.

Need for Quantum Key Distribution



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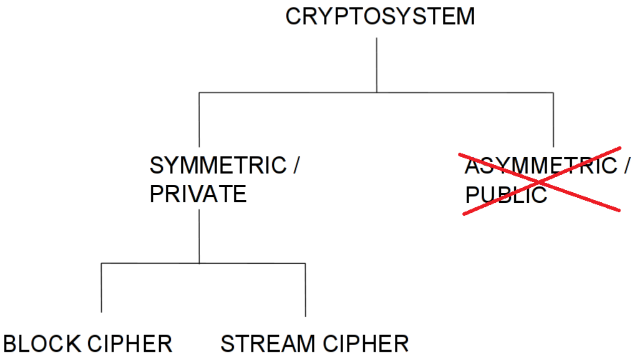

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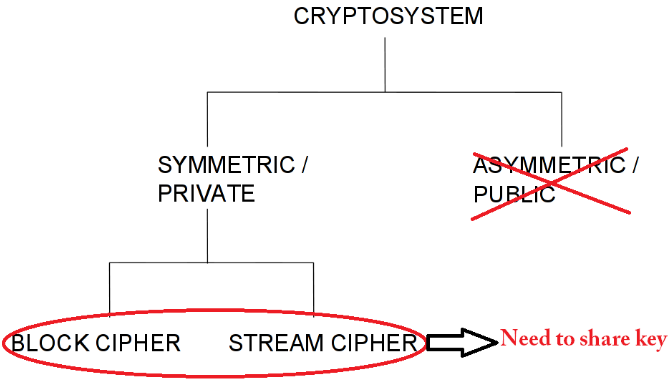

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BB84 Protocol

Uses two conjugate bases $+ = \{\uparrow, \rightarrow\}$ and $\times = \{\nearrow, \nwarrow\}$ to establish a secret key between two parties at a distance.

Alice's bit	0	1	1	0	1	0	0	1
Alice's basis	+	+	X	+	X	X	X	+
Alice's polarization	$\uparrow$	$\rightarrow$	$\nwarrow$	$\uparrow$	$\nwarrow$	$\nearrow$	$\nearrow$	$\rightarrow$
Bob's basis	+	X	X	X	+	X	+	+
Bob's measurement	$\uparrow$	$\nearrow$	$\nwarrow$	$\nearrow$	$\rightarrow$	$\nearrow$	$\rightarrow$	$\rightarrow$
Public discussion								
Shared Secret key	0		1			0		1

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  - Fully Device Independent (FDI) QKD [Vazirani and Vidick, PRL, 2014]

Non-QKD Quantum Crypto

- Quantum commitment
- Quantum SMC
- Position-based quantum cryptography

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Thank You

Questions / Comments ?

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